# Superstring Perturbation Theory Revisited

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I want to give a couple of examples of what I mean in saying that the infrared behavior of string theory is the same as that of a corresponding field theory. Let us consider a Feynman diagram. A very simple question of infrared behavior is to consider what happens when a single propagator goes on shell.

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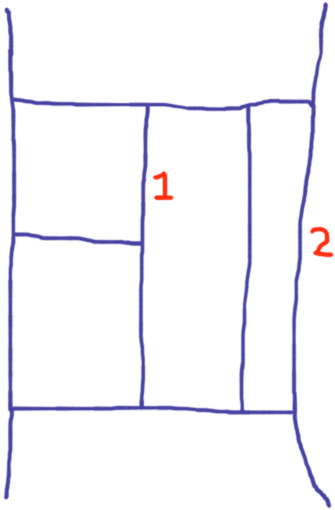
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Let us assume our particles are massless so the propagator is $1/k^2$. In $D$ noncompact dimensions, the infrared behavior when the momentum in a single generic propagator goes to zero is

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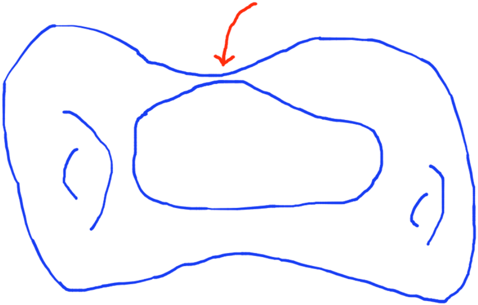
and this converges if $D > 2$. (For an exceptional internal line, such as the one labeled 2 in the diagram, the infrared behavior when a single momentum goes to zero is worse, because this forces other propagators to go on shell. In the case shown in the sketch, the condition to avoid a divergence is actually $D > 4$.)

All this has a close analog in string theory. First of all, a nonseparating line that goes to zero momentum is analogous to a nonseparating degeneration of a Riemann surface.



A degeneration of a Riemann surface – separating or not – can be described by an equation

$$xy = q,$$

where $x$ is a local parameter on one side, $y$ is one on the other, and $q$ measures the narrowness of the neck – or, by a conformal transformation, the length of the tube separating the two sides.

The contribution of a massless string state propagating through the neck is

$$\int d^D k \int |d^2 q| q^{L_0 - 1} \overline{q}^{\overline{L}_0 - 1} = \int d^D k \int |d^2 q| |q \overline{q}|^{k^2/2 - 1}$$

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where I use $L_0 = \overline{L}_0 = k^2/2$. Instead of doing the integral, let us introduce the analog of the Schwinger parameter by $q = \exp(-(t + is))$ where $s$ is an angle and $t$ plays the same role as the Schwinger parameter of field theory. The integral over $s$ just gives a factor of $2\pi$, giving

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(Note that I indicated the upper limit of the $t$ integral but not the lower limit, which is affected by modular invariance.)

This agrees perfectly with field theory even before doing the $k$ or $t$ integral, bearing in mind that the Schwinger representation of the Feynman propagator is

$$\frac{1}{k^2} = \int_0^\infty \mathrm{d}t \exp(-tk^2).$$

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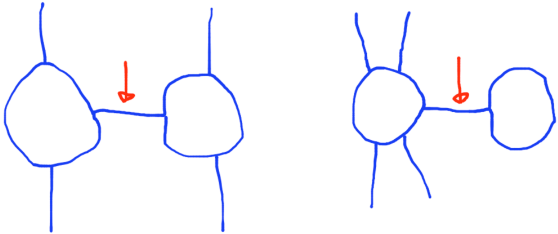

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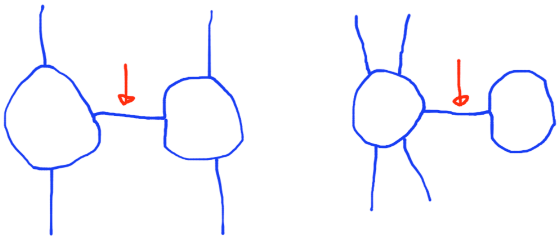

The difference is that in the second case the external lines are all on one side.

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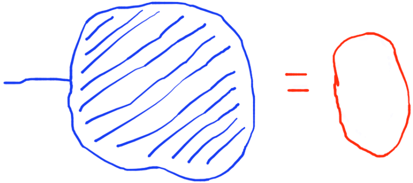

We have to impose this condition for all massless scalars. However, it is non-trivial only for the ones that are invariant under all (local or global) symmetries.

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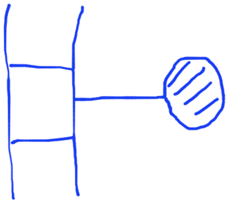

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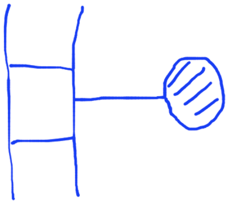

and evaluate the $S$-matrix by summing the others.

All this is relevant to perturbative string theory, since whenever we do have a perturbative string theory, there is always at least one massless neutral scalar field that might have a tadpole, namely the dilaton. So perturbative string theory will only make sense if the dilaton tadpole vanishes (along with other tadpoles, if there are more massless scalars).

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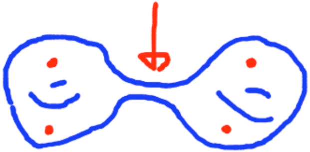

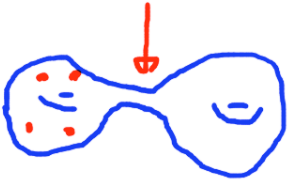

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Instead of talking more about what doesn't work in general, let us discuss what does work.

Instead of talking more about what doesn't work in general, let us discuss what does work. First of all, there is a natural measure on supermoduli space, which I will call $\widetilde{\mathcal{M}}_{g,n}$. This was constructed in the 1980's via conformal field theory (in varied approaches by G. Moore, P. C. Nelson, and J. Polchinski; E. & H. Verlinde; L. Alvarez-Gaumé, C. Gomez, P. C. Nelson, G. Sierra, and C. Vafa; and D'Hoker and Phong) by adapting the analogous formulas for the bosonic string.

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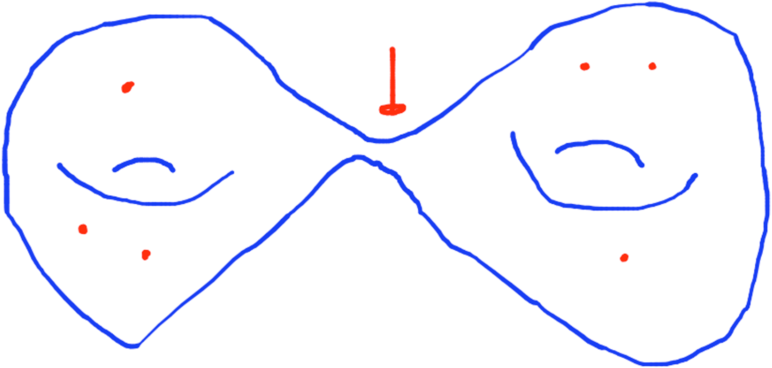
Supermoduli space is not compact – or if we take its Deligne-Mumford compactification, then the function we want to integrate has singularities – precisely because of the infrared effects that we have been talking about.



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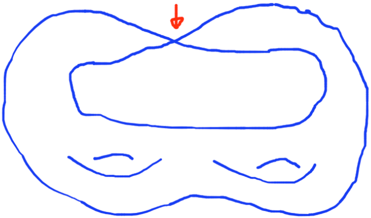
Although supermoduli space is very subtle, if one asks precisely the questions whose answers one needs, those particular questions tend to have simple answers. For instance, although a sum over spin structures (independent of the integration over supermoduli) does not make sense in general, a very small piece of it makes sense when a node develops



and this leads to the GSO projection on the physical states that propagate through the node.

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For the super case, the gluing of local parameters $x, \theta$ to $y, \psi$ is by an almost equally simple formula

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Importantly, the gluing depends in both cases on only one bosonic parameter $\varepsilon$ or $q$. In the super case, there are no odd moduli for the gluing.

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You probably all know the basic idea of fermionic integration by parts, which is that for an odd variable $\alpha$ and any function $f(\alpha)$, one has

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Indeed the Berezin integral

$$\int \mathrm{d}\alpha \cdot 1 = 0, \int \mathrm{d}\alpha \cdot \alpha = 1$$

is defined to make this true.

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Once has has the right definitions, the supermanifold version of Stokes's theorem says just what one would expect:

$$\int_X \mathrm{d}\Lambda = \int_{\partial X} \Lambda.$$

Here $\mathrm{d}\Lambda$ is the analog of a "volume form" and $\Lambda$ is the analog of a "form of codimension 1."

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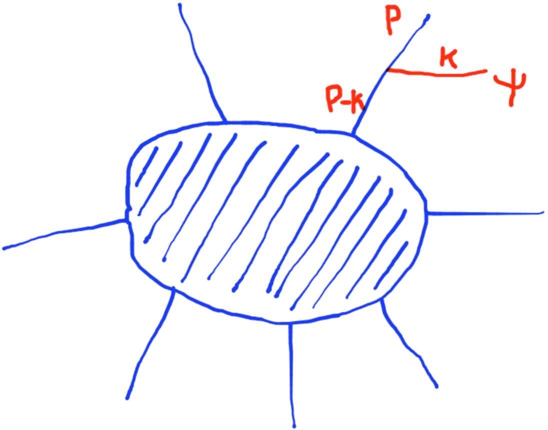
We consider a scattering amplitude involving a soft gravitino. We take its wavefunction to be $\Psi_{I\alpha} = \exp(ik \cdot x)\eta_{I\alpha}$ where $I$ is a vector index and $\alpha$ is a spinor index. A matrix element for emission of a soft gravitino has singular terms where the gravitino is attached to an external leg:



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Now if we set the gravitino polarization vector-spinor $\eta_{I\alpha}$ to be $k_I\zeta_\alpha$ (for some spinor $\zeta_\alpha$), then the whole amplitude must vanish. This is a special case of the decoupling of states $\{Q, W\}$ for any $W$.

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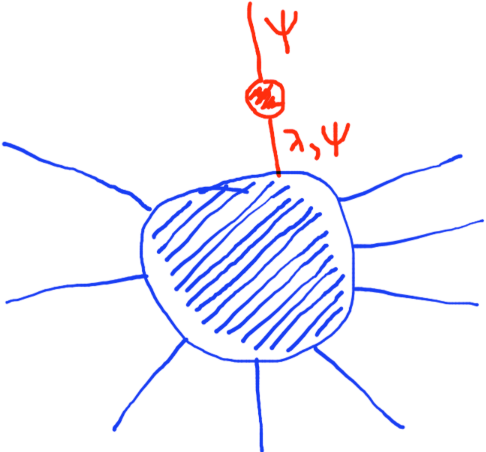
This type of argument may be familiar from field theory. It works the same way in string theory, except that we have to know that the massless tadpoles vanish (or none of the amplitudes are defined). However, in either field theory or string theory, I have left something out so far. Potentially, the supersymmetric Ward identity can contain another term if the coupling of a soft gravitino has a singular contribution like this:



This happens if loops generate a term in the effective action that is of the form $\overline{\Psi}_I \Gamma^I \lambda$, with some previously massless fermion $\lambda$, or a term $\overline{\Psi}_I \Gamma^{IJ} \Psi_J$.

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This happens if loops generate a term in the effective action that is of the form $\overline{\Psi}_I \Gamma^I \lambda$, with some previously massless fermion $\lambda$, or a term $\overline{\Psi}_I \Gamma^{IJ} \Psi_J$. In the first case, supersymmetry is spontaneously broken, with $\lambda$ as a Goldstone fermion; in the second case, we land in AdS space with unbroken supersymmetry. (An example of the first type was described by Dine, Ichinose, Seiberg and by Atick, Dixon, Sen in the 1980's. No example of the second type seems to be known in any dimension, and I wonder if there is a general no go theorem.)

In many classes of string vacua, it is straightforward to prove that $\overline{\Psi}_I \Gamma^I \lambda$ and $\overline{\Psi}_I \Gamma^{IJ} \Psi_J$ terms are not generated by loops.

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So all we need in order to land in a happy place is an extension of this type of reasoning to show that the massless tadpoles vanish.

So all we need in order to land in a happy place is an extension of this type of reasoning to show that the massless tadpoles vanish. Though this is expected to follow from spacetime supersymmetry, I believe that the type of argument I have given is not quite powerful enough to prove it.

Given the experience from the old literature (see for example E. Martinec (1986), Atick, Moore and Sen (1988)), one expects that what one should do is to make a similar argument but with $k$ set to 0 at the beginning.

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If these are the only nonzero boundary contributions, then again we get the supersymmetric Ward identity, much as before.

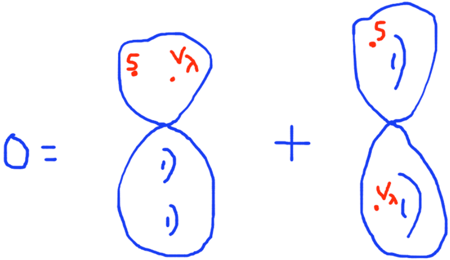
The other contributions that might appear (because they involve on-shell momentum flowing through the node) correspond to supersymmetry breaking (or a cosmological constant) or a massless tadpole. We'll draw them in a moment, in a slightly simpler situation.

To finally address the question of whether there is a massless tadpole, let us replace the product $V_1 \cdots V_n$ with a single vertex operator $V_\lambda$ of a massless fermion that is a superpartner of a scalar $\phi$ whose tadpole we want to understand. The relation

$$0 = \int_{\partial\Gamma} \langle S V_\lambda \rangle$$

is now simple because $\partial\Gamma$ has only two types of components.

The relation is explicitly then



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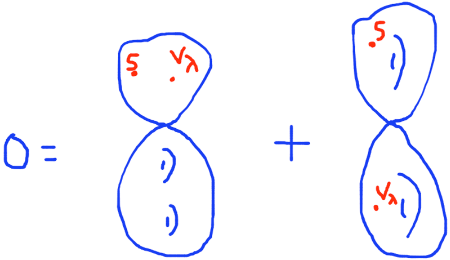


The first term is the dilaton tadpole, and the second may appear precisely when supersymmetry is spontaneously broken (or a cosmological constant is being generated).

When one can show that the gravitino cannot gain a mass in perturbation theory – for instance in $\mathbb{R}^{10}$ – this relation should (when combined with what was discovered in the 80's and a few details that we haven't had time for today) – remove the very slight unclarity that has surrounded superstring perturbation theory.