Constraining theories with higher spin symmetry

Juan Maldacena
Institute for Advanced Study

Strings 2012
Munich

Based on: 1112.1016 and 1204.3882 by J. M. and A. Zhiboedov.
• Elementary particles can have spin.

• Even massless particles can have spin.

• Interactions of massless particles with spin are very highly constrained.
  
  Spin 1 = Yang Mills
  Spin 2 = Gravity
  Spin s>2 (higher spin) = No interacting theory in asymptotically flat space

• Coleman Mandula theorem: The flat space S-matrix cannot have any extra spacetime symmetries beyond the (super) poincare group. Needs an S-matrix.

• Yes go: Vasiliev: Constructed interacting theories with massless higher spin fields in AdS$_4$. 
• $\text{AdS}_4 \rightarrow$ dual to $\text{CFT}_3$
• Massless fields with spin $s \geq 1 \rightarrow$ conserved currents of spin $s$ on the boundary.
• Conjectured $\text{CFT}_3$ dual: $N$ free fields in the singlet sector

\[
\dot{j}_{\mu_1 \cdots \mu_s} = \sum_{i=1}^{N} \varphi^i \partial_{\mu_1} \cdots \partial_{\mu_s} \varphi^i
\]

• This corresponds to the massless spin $s$ fields in the bulk.
Interacting theory in the bulk.

$$G_N \sim 1/N$$

Free large $N$ theory on the boundary.
• What are the CFT’s with higher spin symmetry (with higher spin currents) ?
• We will answer this question here:
• They are essentially free field theories
• This is the analog of the Coleman Mandula theorem for CFT’s, which do not have an S-matrix. Or the Coleman Mandula theorem for AdS.
• We will also constrain theories where the higher spin symmetry is “slightly broken” = broken by 1/N effects.
Why is higher spin symmetry interesting?

• If it describes just free theories, why do we care?

• It captures the gauge invariant symmetries of free gauge theories. Interactions $\rightarrow$ breaking the symmetry...
Spontaneously broken symmetry

• The most interesting aspect is when it is broken !.

• Recall: massive spin 1 (weakly coupled) \(\rightarrow\) Higgs mechanism.

• In weakly coupled string theory we have massive particles of spin \(s > 2\). Can it be viewed as a sort of spontaneously broken higher spin symmetry? In flat space \(\rightarrow\) not clear. In AdS, we can controllably higgs an infinite set of higher spin symmetries.

• How unique is string theory? Is it just the weakly coupled theory of massive higher spin particles. (weakly coupled strings).

• Emergence of a local bulk in AdS is a process in classical string theory. How is it be governed by the breaking of this symmetry?
Back to the unbroken case
Assumptions

• We have a CFT obeying all the usual assumptions: Locality, OPE, existence of the stress tensor with a finite two point function, etc.
• If our starting point is AdS $\rightarrow$ Assume it defines a CFT on the AdS boundary.
• The theory is unitary
• We have a conserved current of spin, $s>2$.
• We are in $d=3$
• (We have only one conserved current of spin 2.)
Conclusions

• There is an infinite number of higher spin currents, with even spin, appearing in the OPE of two stress tensors.

• All correlators of these currents have two possible forms:

  1) Those of $N$ free bosons in the singlet sector

  $$j_{\mu_1 \cdots \mu_s} = \sum_{i=1}^{N} \varphi_i \leftrightarrow \partial_{\mu_1} \cdots \leftrightarrow \partial_{\mu_s} \varphi_i$$

  2) Those of $N$ free fermions in the singlet sector
Idea of the method

• We do not have the algebra of symmetries, we need to find it.
• This is contained in three point functions of conserved currents.
• Use conformal symmetry to constrain the three point function of conserved currents up to a few constants.
• Use the existence of an extra higher spin charge to derive relations between different three point functions.
• These determine all three point functions and fix the symmetry algebra.
• Using this big algebra, fix all other correlators.
Plan

• Unitarity bounds, higher spin currents.
• Simple argument for small dimension operators
• Outline of the full argument
Unitarity bounds

• Scalar operator: $\Delta \geq \frac{1}{2}$ (in $d=3$)
• Spin $s$ $\hat{J}_{\mu_1 \cdots \mu_s}$ (Symmetric traceless indices)
• Bound: Twist $= \Delta - s \geq 1$.
• If the twist $=1$, the we have a conserved current. $\partial_{\mu_1} \hat{J}_{\mu_1 \cdots \mu_s} = 0$
Charges

We consider minus components only:

\[ j_s \equiv j_- \ldots \ldots \]

\[ Q_{\mu_2 \ldots \mu_s} = \int_{\text{Space}} j_{0 \mu_2 \ldots \mu_s} \]

\[ Q_s \equiv \int_{x^+ = \text{const}} dx^- dy j_s \quad \text{All minus components!} \]

Spin s-1, Twist = 0
Removing operators in the twist gap

• Scalars with $1 > \Delta \geq \frac{1}{2}$
• Assume we have a current of spin four.
• The charge acting on the operator can only give (same twist $\rightarrow$ only scalars)

\[ [Q_4, \Phi] = \partial^3 \Phi \]
\[ \partial \equiv \partial_{-} \]

• Charge conservation on the four point function implies (in Fourier space)

\[ \sum_{i=1}^{4} k_i^3 = 0 \]
\[ \sum_{i=1}^{4} k_i = 0 \]

Of course we also have:
• This implies that the momenta are equal in pairs $\rightarrow$ the four point function factorizes into a product of two point functions.

$$\langle 4pt \rangle = \langle \Phi(1)\Phi(2) \rangle \langle \Phi(3)\Phi(4) \rangle + \text{permutations}$$

• We can now look at the OPE as $1 \rightarrow 2$ , and we see that the stress tensor can appear only if $\Delta=\frac{1}{2}$ .

• So we have a free field !

• Intuition: Transformation = momentum dependent translation $\rightarrow$ momenta need to be equal in pairs. Same reason we get the Coleman Mandula theorem !
Twist one

• Now we have:

\[
[Q_s, j_{s'}] = \sum_{s''} c_{s,s',s''} \partial^{s+s'-s''-1} j_{s''}
\]

• Sum over $S''$ has finite range

• Some $c$’s are non-zero, e.g. $[Q_s, j_2] = \partial j_s + \cdots$
Three point functions of three conserved currents are constrained to only three possible structures:

\[ \langle j_{s_1} j_{s_2} j_{s_3} \rangle = A \langle j_{s_1} j_{s_2} j_{s_3} \rangle_{\text{bosons}} + B \langle j_{s_1} j_{s_2} j_{s_3} \rangle_{\text{fermions}} + C \langle j_{s_1} j_{s_2} j_{s_3} \rangle_{\text{odd}} \]

- Bosons
- Fermions
- Odd (involves the epsilon symbol).

- We have more than one because we have spin
- The theory is not necessarily a superposition of free bosons and free fermions (think of s=2 !)
Brute Force method

• Acting with the higher spin charge, and writing the most general action of this higher spin charge we get a linear combination of the rough form

\[ 0 = \sum c_\partial^{\text{powers}} \langle j j j \rangle \]

Coefficients in Transformation law

• The three point functions are constrained to three possible forms by conformal symmetry \( \to \) lead to a large number of equations that typically fix many of the relative coefficients of various terms.

• The equations separate into three sets, one for the bosons part, one for the fermion part and one for the odd part.
• In this way one constrains the transformation laws.
Outline of a more elegant method

• Consider the light-like OPE of two stress tensors

\[
\overline{j_2 j_2}_b = \lim_{y_{12} \to 0} y_{12} \lim_{x_{12}^+ \to 0} j_2(x_1) j_2(x_2) = \partial_1^2 \partial_2^2 B(x_1 x_2)
\]

• This defines a quasi-bilocal operator \( B \).

• The three point functions simplify a lot in this limit, while still giving strong constraints.

• (Similar to the OPE in deep inelastic scattering)
• Given that a higher spin current exists.
• One considers the charge conservation identity for
  \[ Q_s \quad \text{on} \quad \langle j_2 j_2 j_s \rangle \]
• We know a term involving \( \langle j_2 j_2 j_2 \rangle \) is nonzero.
• This implies that currents with spins: 4, ..., 2 s -2 , exist in the right hand side of the OPE of two stress tensors.
• Repeating the argument, we get an infinite number of even spin currents (since 2 s -2 > s if s > 2)
• We now consider the action of all these $Q_s$ with even spin on the OPE of two stress tensors.

• One can then show that these charges acting on the quasi-bilocal $B$ has the form

$$[Q_s, B(x_1, x_2)] = (\partial_1^{s-1} + \partial_2^{s-1}) B(x_1, x_2)$$
• Consider a correlator
\[ \langle B(x_1, x_2) \cdots B(x_{2n-1}, x_{2n}) \rangle \]
• The \( Q_s \) charge conservation identities imply that these correlation functions factorize into two point functions of free fields.
• Relative normalizations fixed by the Ward identities of the stress tensor which is in \( B \).
• Same as correlators of (with an analytic continuation of \( N \rightarrow \tilde{N} \))
\[
B(x_1, x_2) = \sum_{i=1}^{N} : \varphi^i(x_1) \varphi^i(x_2) :
\]
• \( B \) is a true bilocal.
• Here we assumed that B is non-zero. If it is zero, then we can take a second possible light-cone limit and isolate a new quasi-bilocal which we interpret as coming from a theory of free fermions.

• If there is a single spin two conserved current, then we either have one case or the other.
Quantization of $\tilde{\mathcal{N}}$

- We can show that the single remaining parameter, call it $\tilde{\mathcal{N}}$, is an integer.
- It is simpler for the free fermion theory
- It has a twist two scalar operator

$$\tilde{j}_0 = \sum_i \psi^i \psi^i$$

- Consider the two point function of $(\tilde{j}_0)^q$

$$\langle (\tilde{j}_0)^q (\tilde{j}_0)^q \rangle \propto \tilde{\mathcal{N}} (\tilde{\mathcal{N}} - 1) \cdots (\tilde{\mathcal{N}} - q + 1)$$

- If $\tilde{\mathcal{N}}$ is not an integer some of these are negative.
- So $\tilde{\mathcal{N}}=\mathcal{N}$
Conclusions

• Thus, we have proven the conclusion of our statement.
• $N$ is quantized $\rightarrow$ Coupling constant of Vasiliev-like theories is quantized!

• Generalizations:
  - More than one conserved spin two current $\rightarrow$ expect the product of free theories (we did the case of two)
  - Higher dimension.
Slightly broken higher spin symmetry

• Vasiliev theory + boundary conditions that break the higher spin symmetry \( \rightarrow \) Dual to the large N Wilson Fischer fixed point...

Polyakov, Klebanov
Giombi, Yin
Giombi, Minwalla, Prakash, Trivedi, Wadia, Yin
Chang, Minwalla, Sharma, Yin
Almost conserved higher spin currents

• There are interesting theories where the conserved currents are conserved up to $1/N$ corrections.
• Vasiliev’s theory with boundary conditions that break the higher spin symmetry
• $N$ fields coupled to an $O(N)$ Chern-Simons gauge field at level $k$.

$$k \int Tr[AdA + \frac{2}{3}A^3]$$

• ‘t Hooft-like coupling

$$\lambda = \frac{N}{k}$$

Giombi, Minwalla, Prakash, Trivedi, Wadia, Yin
Aharony, Gur-Ari, Yacoby
Fermions + Chern Simons

- Spectrum of ``single trace'' operators same as in the free case.
- Violation of current conservation: (2pt fns set to 1 )

\[ \partial_{\mu} j_{\mu}^{\pm} = \frac{\alpha}{\sqrt{N}} (\tilde{j}_0 \partial j_2 - \frac{5}{2} \partial \tilde{j}_0 j_2) \]

- Insert this into correlation functions

\[ \sum_i \int_{S_i} \langle j^n_{\pm} j_2(x_1) j_2(x_2) j_2(x_3) \rangle \sim \frac{a_1}{\sqrt{N}} \int d^3x \langle [\partial \tilde{j}_0 j_2 - \frac{2}{5} \tilde{j}_0 \partial j_2] (x) j_2(x_1) j_2(x_2) j_2(x_3) \rangle. \]

Usual charges

Use factorization
• We had three series of solutions: Bosons, fermions and odd ones.
• Here the extra term mimics the contribution like the one we would have for $j_0$ in the boson and odd solutions. (But we do not have such operator)
• Conclusion: All three point functions are

\[
\langle 3pt \rangle = \frac{1}{\sqrt{\tilde{N}}} \left[ \frac{1}{1 + \tilde{\lambda}^2} (\text{Fermions}) + \frac{\tilde{\lambda}}{1 + \tilde{\lambda}^2} (\text{Odd}) + \frac{\tilde{\lambda}^2}{1 + \tilde{\lambda}^2} (\text{Bosons}) \right]
\]

• Two parameter family of solutions

\[
\tilde{\lambda} = \lambda + c_3 \lambda^3 + \cdots
\]

\[
\tilde{N} = N f(\lambda)
\]

• From this analysis, we do not know the relation to the microscopic parameters \( N, k \).

• Direct computation:

\[
\tilde{\lambda} = \tan \left( \frac{\pi \lambda}{2} \right)
\]

\[
\tilde{N} = N \frac{\sin \pi \lambda}{\pi \lambda}
\]

Aharony, Gur-Ari, Yacoby
\[ \langle 3pt \rangle = \frac{1}{\sqrt{\tilde{N}}} \left[ \frac{1}{1 + \tilde{\lambda}^2} \text{(Fermions)} + \frac{\tilde{\lambda}}{1 + \tilde{\lambda}^2} \text{(Odd)} + \frac{\tilde{\lambda}^2}{1 + \tilde{\lambda}^2} \text{(Bosons)} \right] \]

- As \( \tilde{\lambda} \to \infty \) we get the large \( N \) limit of the Wilson Fischer fixed point.
- The operator \( \tilde{j}_0 \) becomes the operator \( (\phi_i \phi_i)_{WF} \) which has dimension two (as opposed to the free field value of one). It also becomes parity even.
Three dimensional bosonization

\[
\lambda_B = \frac{1}{\lambda_F}
\]

Bosons $U(N)_k \leftrightarrow$ Fermions $U(k)_N$

Giombi, Minwalla, Prakash, Trivedi, Wadia, Yin JM, Zhiboedov
Aharony, Gur-Ari, Yacoby
• Higher point functions → could be done in principle, but seems messy..
Conclusions

• Proved the analog of Coleman Mandula for CFT’s. Higher spin symmetry $\rightarrow$ Free theories.

• Used it to constrain Vasiliev-like theories. Quantization of the coupling.

• A similar method constrains theories with a higher spin symmetry violated at order $1/N$. 
Future

• It is interesting to consider theories which have other ‘‘single trace’’ operators (twist 3) that can appear in the right hand side of the divergence of the currents.
• These are Vasiliev theories + matter.
• We get this when the boundary theory has adjoint matter.
• What are the constraints on ‘‘matter’’ theory added to a system with higher spin symmetry?.
• Can we extend the analysis to the case of single trace breaking of the higher spin symmetry?
• Of course, this will be an alternative way of doing usual perturbation theory. One advantage is that one deals only with gauge invariant quantities.
• But it could teach us how the higher spin symmetries are broken in string theory.
Vielen Dank!