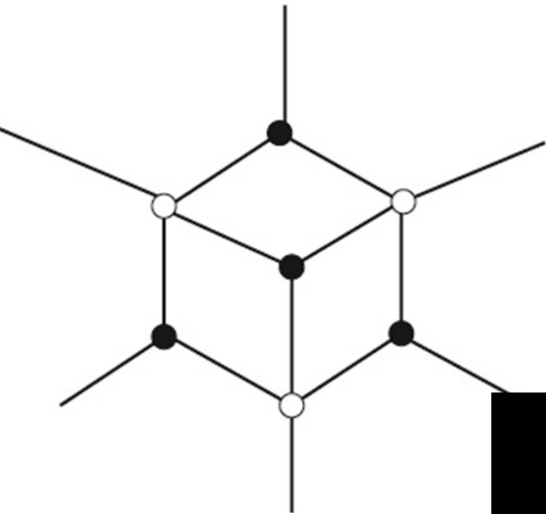
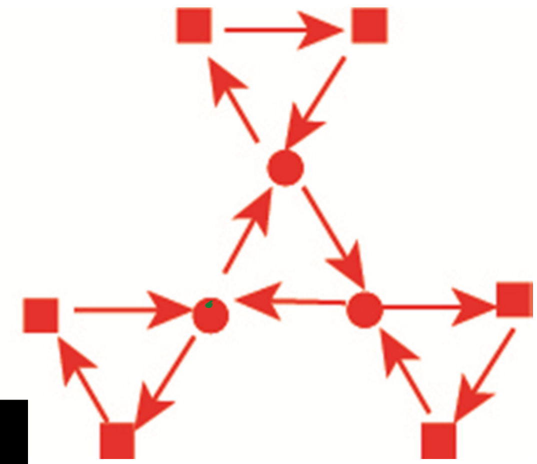




Networks

Quivers

3-manifolds

Masahito Yamazaki

(Princeton, PCTS)

Strings 2012, Munich

based on papers

<u>M. Y.</u>

arXiv:1203.5784 [JHEP]
“Quivers, YBE and 3-manifolds”

<u>Y. Terashima + M. Y.</u>

arXiv:1203.5792 [PRL]
“Emergent 3-manifolds from 4d superconformal indices”

<u>D. Xie + M. Y.</u>

arXiv:1207.0811
“Network and Seiberg Duality”

Overview and Summary

Two known relations

along the lines of
[Alday-Giaotto-Tachikawa] ('09)

6d (2,0) on mfd M $\rightsquigarrow$ SCFT $T[M]$

"3d/3d"

3-mfd

$$Z_{3d\ \mathcal{N}=2\ T[M]}[S^3_b] = Z_{3d\ SL(2)\ CS\ (\text{level } t \sim 1/b^2)}[M]$$

[Dimofte-Gukov-Hollands] [Drukker-Gaiotto-Gomis] [Hosomichi-Lee-Park] ('10)
[(Nagao-)Terashima-Y] [Dimofte-Gukov-(Gaiotto)]
[Cecotti-Cordova-Vafa] ('11),
[Gabella-Dimofte-Xie-Yamazaki] (in progress)

$\longrightarrow$ Dimofte's Talk

"4d/2d"

2-mfd

$$I_{4d\ \mathcal{N}=2\ T[\Sigma]}[S^3 \times S^1] = Z_{2d\,\mathrm{TQFT}}[\Sigma]$$

superconformal index

[Gadde, Pomoni, Rastelli, Razamat '09],
[Gadde, Rastelli, Razamat, Yan '11],....

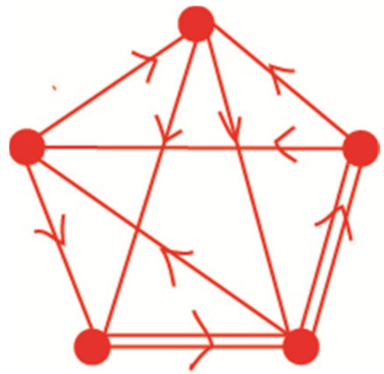
$\longrightarrow$ Rastelli's Talk

$$I_{4d\ \mathcal{N}=2 \atop T[\Sigma]}[S^3 \times S^1] = Z_{2d\ \mathrm{TQFT}}[\Sigma]$$

$$Z_{3d\ \mathcal{N}=2 \atop T[M]}[S^3] = Z_{3d\ \mathrm{CS}_{SL(2)}}[M]$$

4d N=1 SCFTs, the old and the new

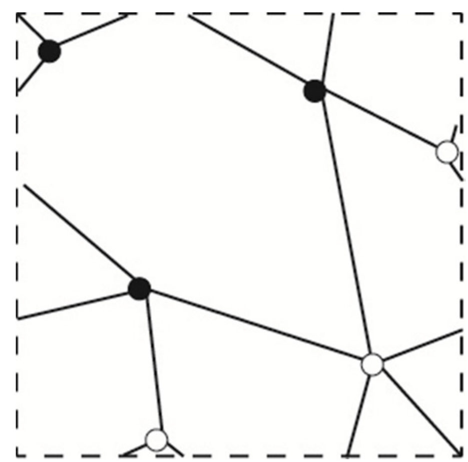
① 4d $\mathcal{N}=1$ quiver gauge theories



---From toric CY 3-fold + D3

---dual to $AdS_5 \times$ (Sasaki-Einstein)

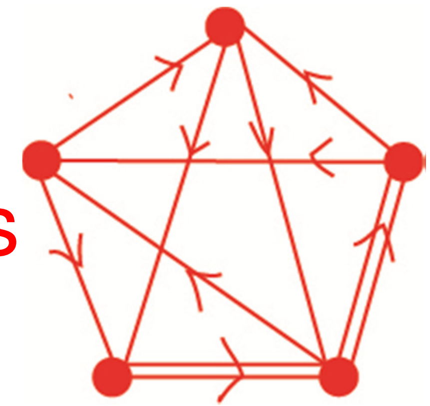
---From a bipartite network on T^2



[Douglas-Moore] (‘96) ,…….
O(100) papers on “brane tilings”
and their gravity duals
[Hanany-Kennaway],
[Franco-Hanany-Kennaway-Vegh-Wecht] (‘05),…

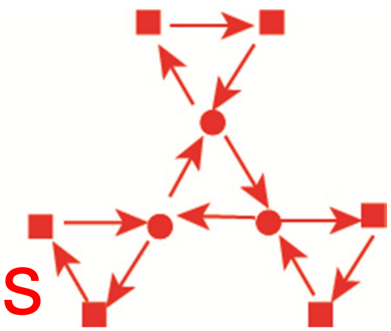
1 4d $\mathcal{N}=1$ quiver gauge theories

---From NS5 on Σ + D5



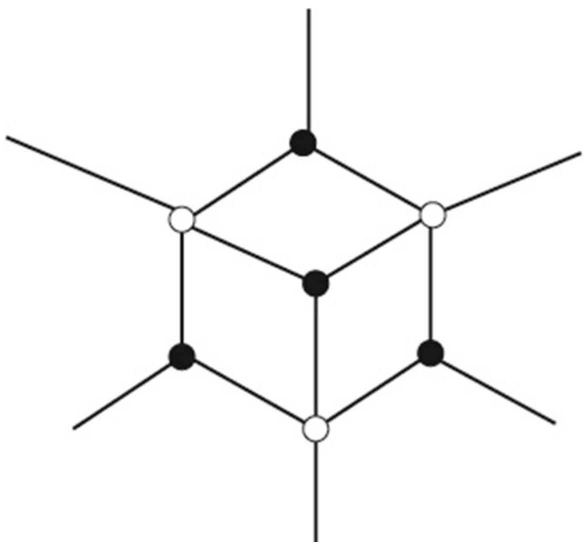
[Hanany-Zaffaroni] ('98) ,……..
[Feng-He-Kennaway-Vafa] ('06),
[Imamura] ('06), [Imamura-Isono-Kimura-Y] ('07),
[Y] ('08)

② 4d $\mathcal{N}=1$ quiver gauge theories



---From a cell of $(Gr_{k,n})_{\geq 0}$
---From a permutation
---From a **planar** bipartite network

[Xie-Y] [Franco] (July ’12)
cf. Arkani-Hamed’s talk



"theories of class N"

network

① 4d $\mathcal{N}=1$ quiver gauge theories

---From a bipartite network on T^2

brane realization

② 4d $\mathcal{N}=1$ quiver gauge theories

---From a planar bipartite network

R^2

no brane realization

Main Results

[Y], [Terashima-Y] ('12)

$$I_{4d\ \mathcal{N}=1}\left[S^3 \times S^1\right] = Z_{2d\ \text{spin}}\left[T^2\ R^2\right]$$

quiver SCFT

system

① ②

[Y], [Terashima-Y] ('12)

$$I_{4d\ \mathcal{N}=1}[S^3 \times S^1] = Z_{2d\ \text{spin}}[T^2 \times R^2]$$

quiver SCFT

system

$$Z_{3d\ \mathcal{N}=2}[S^3_b] \text{ "=" } Z_{3d\ CS}^{SL(2)}[M]$$

quiver SCFT

[Y], [Terashima-Y] ('12)

$$I_{4d\ \mathcal{N}=1}\left[S^3 \times S^1\right] = Z_{2d\ \underset{\text{system}}{\text{spin}}}\left[\frac{T^2}{R^2}\right]$$

quiver SCFT

$\Big\downarrow$ dim. reduction + Higgsing

$$Z_{3d\ \mathcal{N}=2}\left[S^3_b\right] = Z_{3d\ \underset{CS}{SL(2)}}\left[M\right]$$

quiver SCFT

[Y], [Terashima-Y] ('12)

$$I_{4d\ \mathcal{N}=1}[S^3 \times S^1] = Z_{2d\ \text{spin system}}\left[\begin{matrix} T^2 \\ R^2 \end{matrix}\right]$$

quiver SCFT

dim. reduction + Higgsing

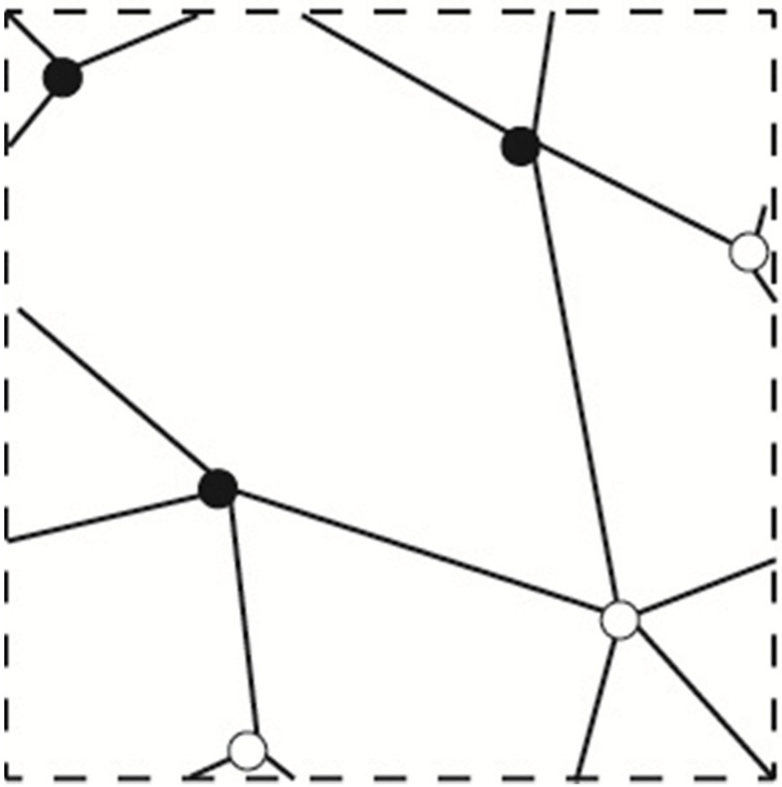
dimensional oxidation!

$$Z_{3d\ \mathcal{N}=2}[S^3_b] = Z_{3d\ SL(2)\ \text{CS}}[M]$$

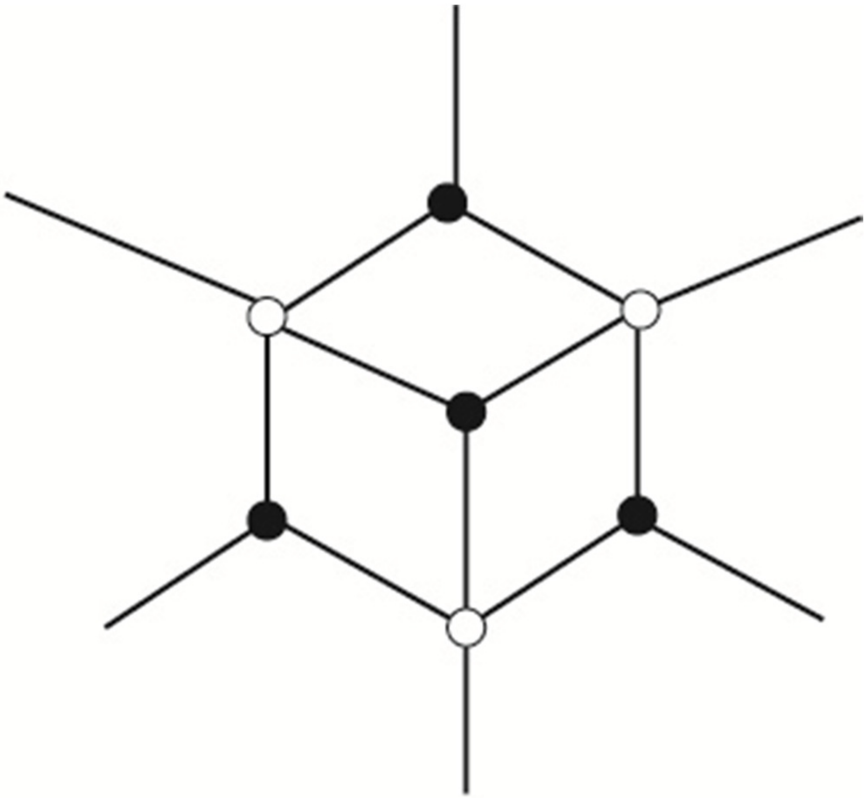
quiver SCFT

4d N=1 quiver SCFTs and Superconformal Indices

We begin with a bipartite network

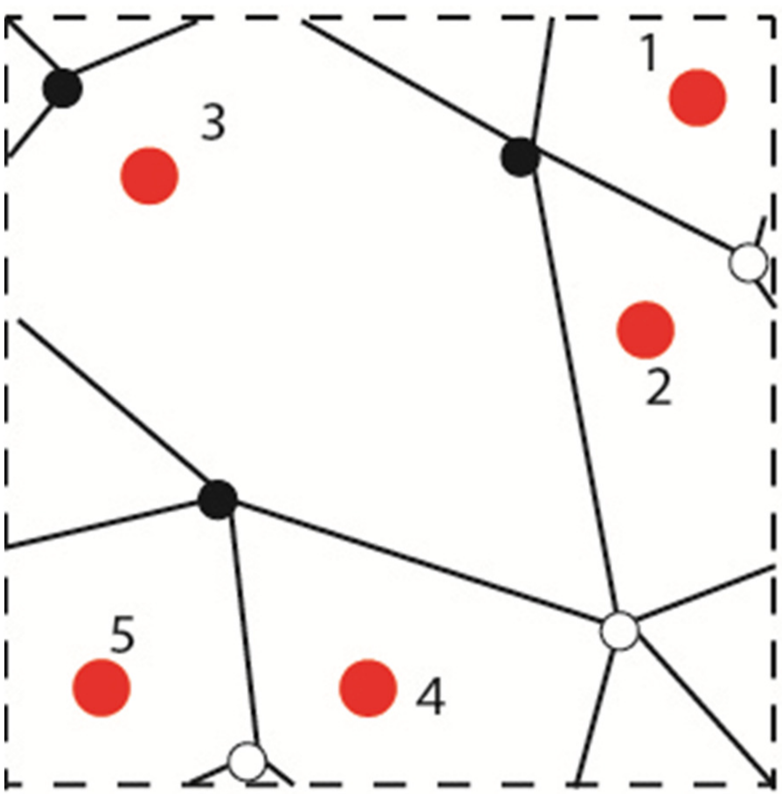


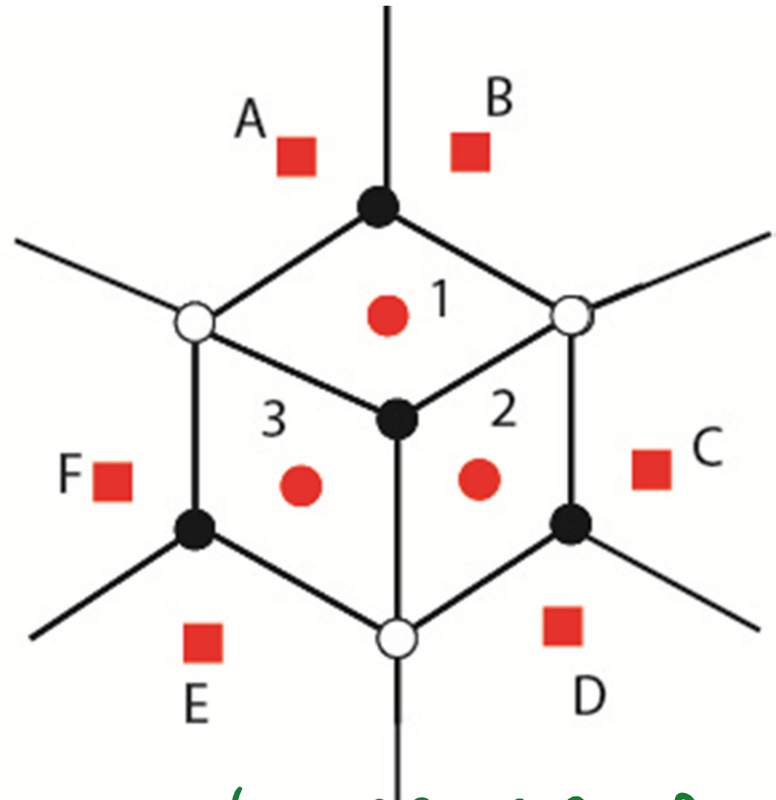
torus



planar

Each external/internal face gives a gauge/global SU(N) symmetry [N=2 hereafter]

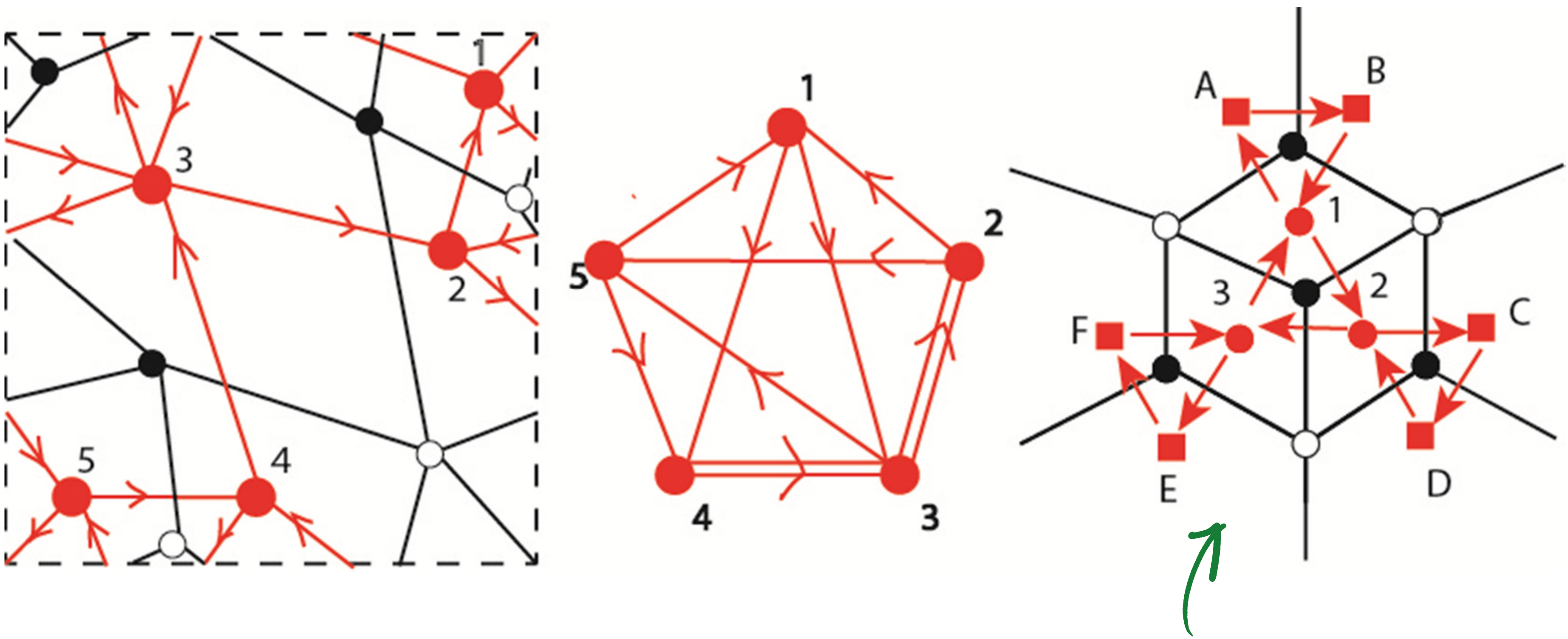




● SU(2) gauge sym.

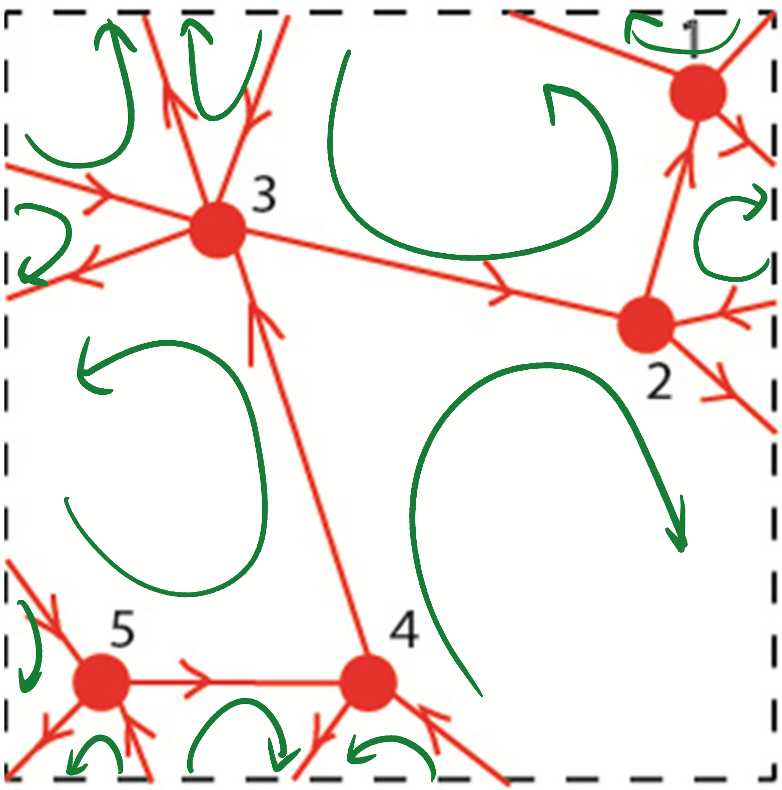
■ " global "

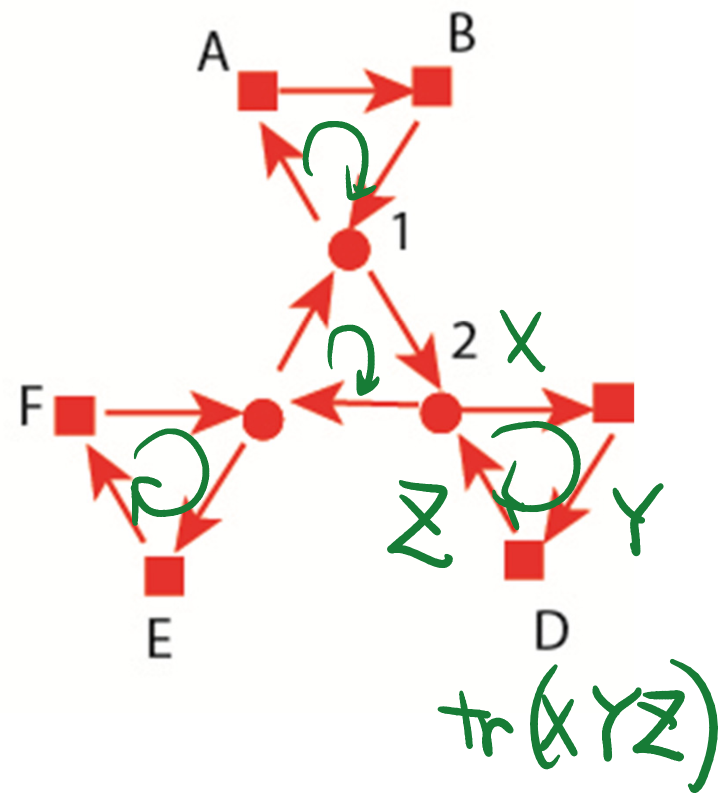
The dual graph gives a quiver diagram
(i.e. an edge = bifundamental)



rule for planar case: on the boundary no
bifundamentals for white vertices

a superpotential term for each small loop in the quiver diagram.





4d superconformal index $(S^1 \times S^3)$

[Kinney Maldacena Minwalla Raju '05]
[Romelsberger '05]

$$I(p,q,u) = \mathrm{Tr}\left[(-1)^F p^{\frac{E+j_2}{3}+j_1} q^{\frac{E+j_2}{3}-j_1} u^F\right]$$

4d superconformal index $(S^1 \times S^3)$

[Kinney Maldacena Minwalla Raju '05]
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$$I(p,q,u) = \mathrm{Tr}\left[(-1)^F p^{\frac{E+j_2}{3}+j_1} q^{\frac{E+j_2}{3}-j_1} u^F\right]$$

conf. dim.

spins

flavor charge

4d superconformal index

$$I(p,q,u) = \mathrm{Tr}\left[(-1)^F p^{\frac{E+j_2}{3}+j_1} q^{\frac{E+j_2}{3}-j_1} u^F\right]$$

$$= \int \underbrace{\left(\prod_v d\sigma_v\right)}_{\text{Cartan of gauge group}} \underbrace{\underbrace{\left(\prod_v I_v^{\text{vect}}(e^{i\sigma_v})\right)}_{\text{vector mult.}} \underbrace{\left(\prod_{e=(v,w)} I_e^{\text{chiral}}(e^{i\sigma_v}, e^{i\sigma_w})\right)}_{\text{chiral mult.}}}_{\text{1-loop}}$$

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4d superconformal index

$$I(p,q,u) = \mathrm{Tr}\left[(-1)^F p^{\frac{E+j_2}{3}+j_1} q^{\frac{E+j_2}{3}-j_1} u^F\right]$$

$$= \int \Bigg(\prod_{\underset{\text{vertex}}{v}} d\sigma_v\Bigg) \Bigg(\prod_v \underbrace{I_v^{\mathrm{vect}}}_{\text{vertex}}(e^{i\sigma_v})\Bigg) \Bigg(\prod_{e=(v,w)} \underbrace{I_e^{\mathrm{chiral}}}_{\text{edge}}(e^{i\sigma_v}, e^{i\sigma_w})\Bigg)$$

Vertex vertex edge

4d superconformal index

Spectral parameter

$$I(p,q,u) = \mathrm{Tr}\left[(-1)^F p^{\frac{E+j_2}{3}+j_1} q^{\frac{E+j_2}{3}-j_1} u^F\right]$$

$$= \int \left(\prod_v d\sigma_v\right) \left(\prod_v I_v^{\text{vect}}(e^{i\sigma_v})\right) \left(\prod_{e=(v,w)} I_e^{\text{chiral}}(e^{i\sigma_v}, e^{i\sigma_w})\right)$$

Vertex
||
U(1) spin

vertex

edge

interaction
(Boltzmann weight)

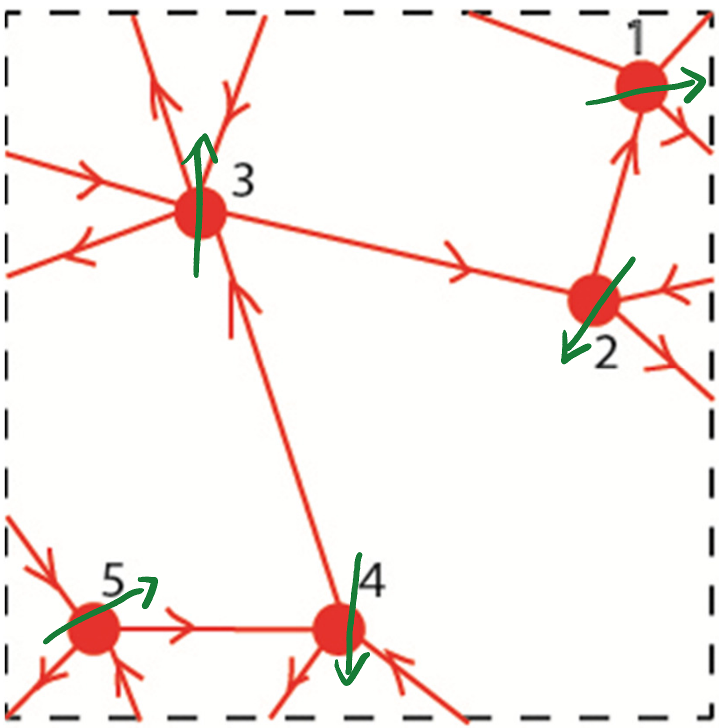
2/7/22

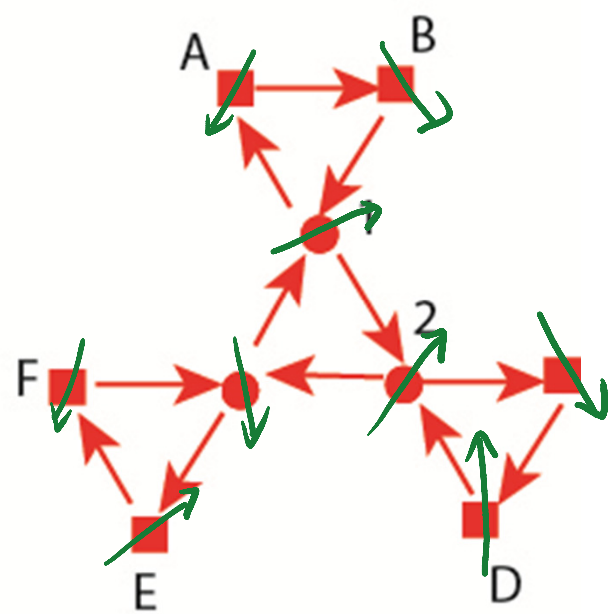
[Y], [Terashima-Y] ('12)

$$I_{4d\ \mathcal{N}=1}\left[S^3\times S^1\right] = Z_{2d\ \text{spin}}\left[\begin{matrix}T^2\\ R^2\end{matrix}\right]$$

quiver SCFT

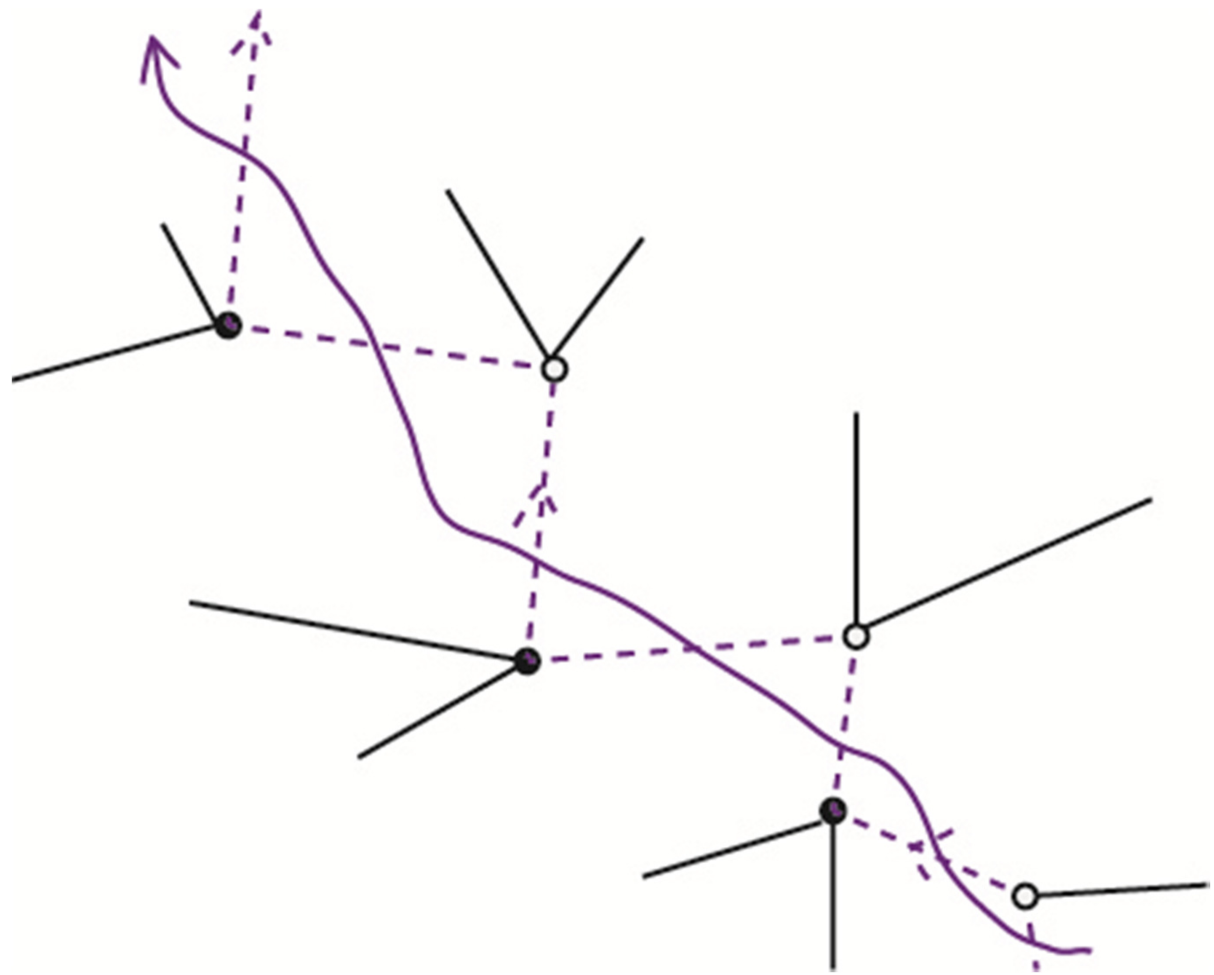
spin system





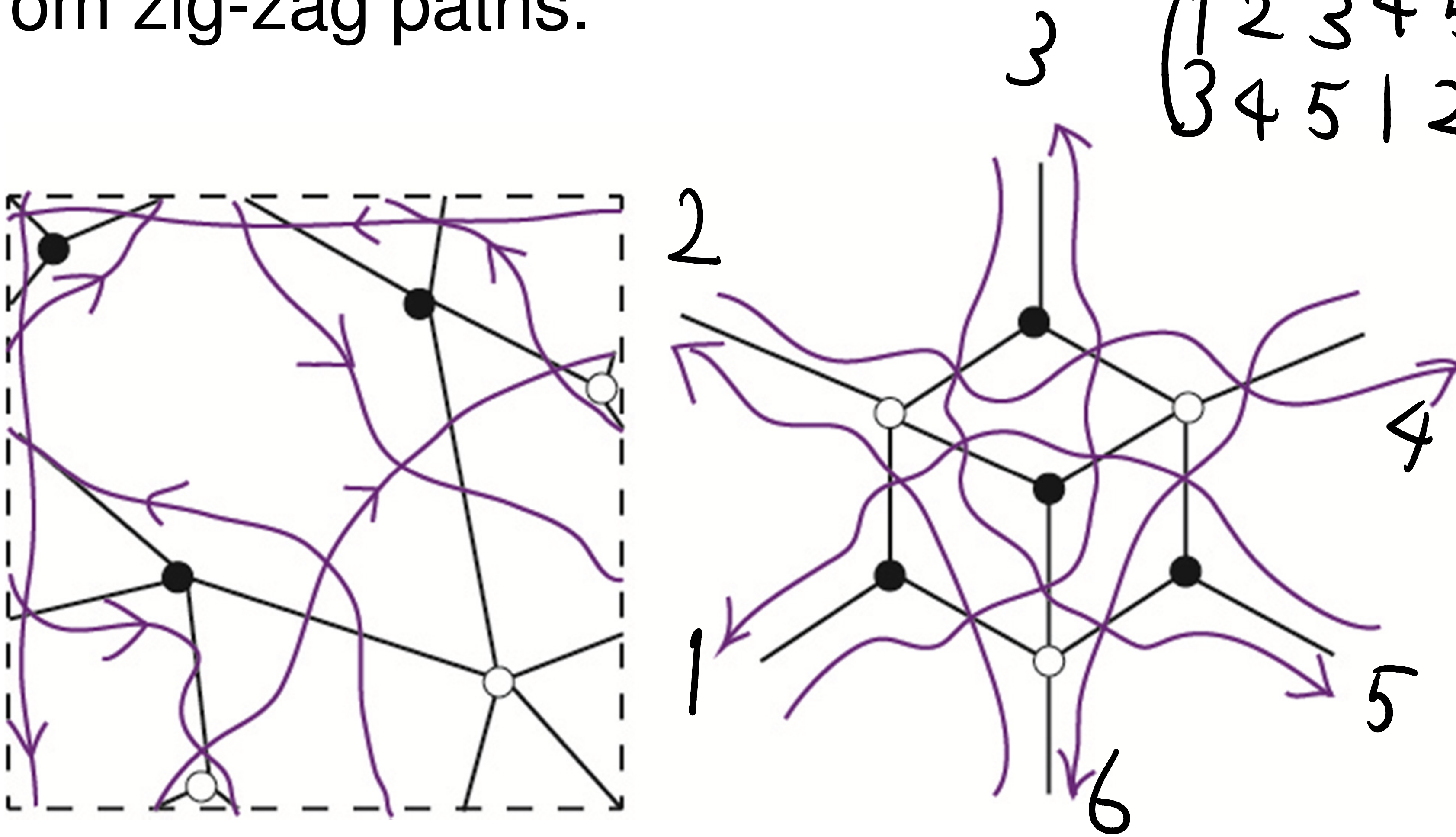
Integrability

The network is characterized by zig-zag paths.



The bipartite graph could be reconstructed from zig-zag paths.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \end{pmatrix}$$



[Thurston] ('04), [Hanany-Vegh] ('05), [Goncharov-Kenyon] ('11) ….

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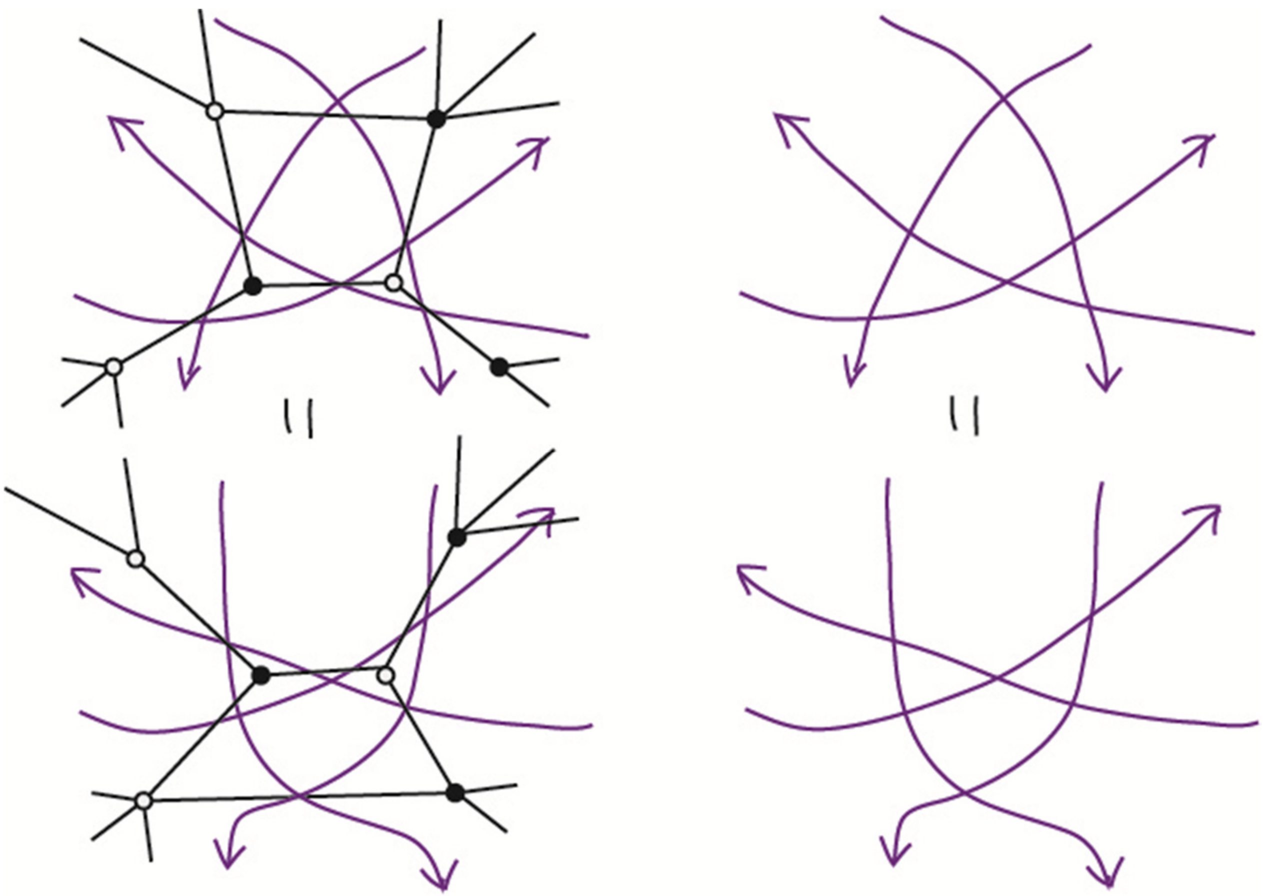
Seiberg duality = double Yang-Baxter move

Our index coincides with the model by
[Bazhanov-Sergeev] ('10 '11) …. cf. [Baxter] ('78)



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Reduction to 3d

and 3-manifolds

[MY], [Terashima-MY] (‘12)

Reduction to 3d

[Dolan-Spiridonov-Vartanov]
[Gadde-Yan] [Imamura] ('11)

$$I_{4d}[S^3 \times S^1_\beta](p, q, u)$$

$\Gamma(p, q; x)$

elliptic gamma function

$\beta \to 0$

$$p = e^{-\beta(1+\eta)}, \quad q = e^{-\beta(1-\eta)}$$

$$u = e^{-\beta\mu} \leftarrow \theta$$

R-charge

$$Z_{3d}[S^3_b](\theta)$$

[Hama-Hosomichi-Lee] ('10)

$$b = \sqrt{\frac{1+\eta}{1-\eta}}$$

$S_b(x)$

quantum dilog function

Reduction to 3d

$$
\begin{array}{l}
I_{4d}^{SU(2)}[S^3 \times S^1](p, q, u) \\
\downarrow \\
Z_{3d}^{SU(2)}[S_b^3](\theta) \\
\downarrow \qquad\qquad \text{Higgsing} \\
Z_{3d}^{U(1)}[S_b^3](\theta)
\end{array}
$$

[Y] ('12)

Reduction to 3d and 2d

$$I_{4d}[S^3 \times S^1](p,q,u)$$

$$\downarrow$$

$$Z_{3d}^{SU(2)}[S^3_b](\theta)$$

$$\downarrow \quad \text{Higgsing}$$

$$Z_{3d}^{U(1)}[S^3_b](\theta)$$

[Y] ('12)

$$\downarrow \; b \to 0$$

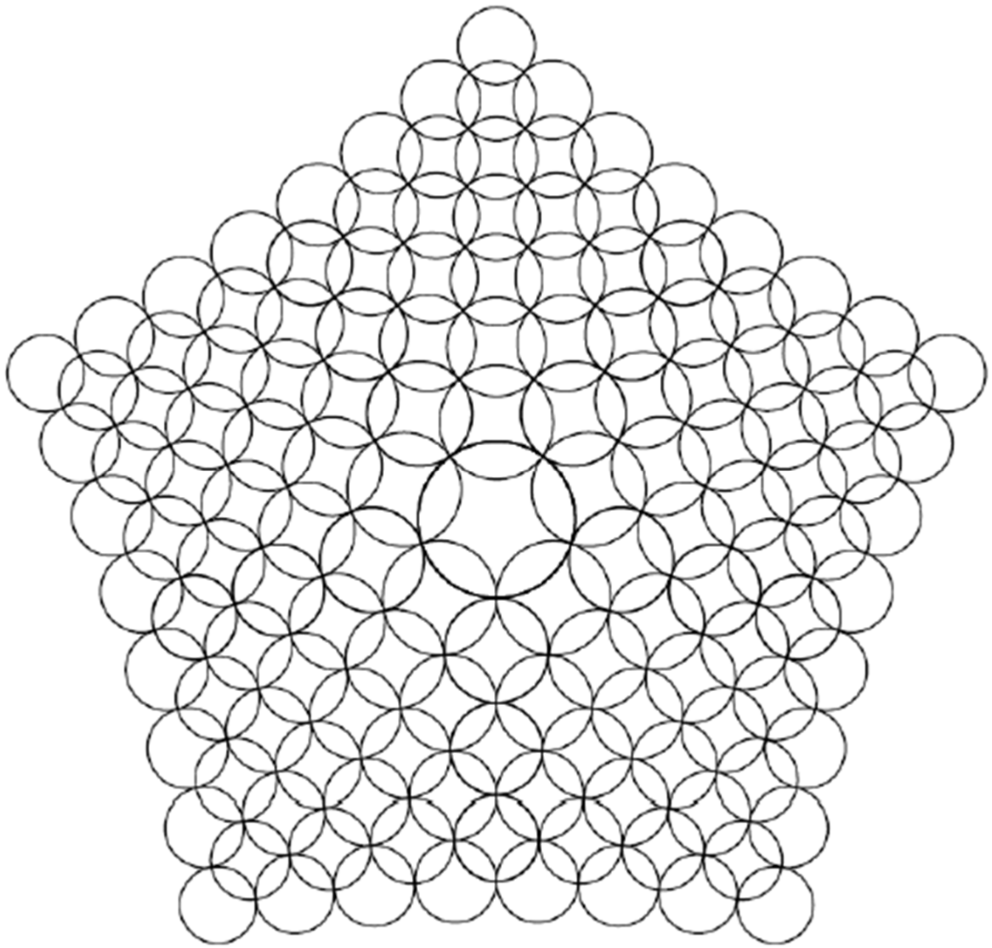
cf. [Nekrasov-Shatashvili] ('11)

2d (2,2)

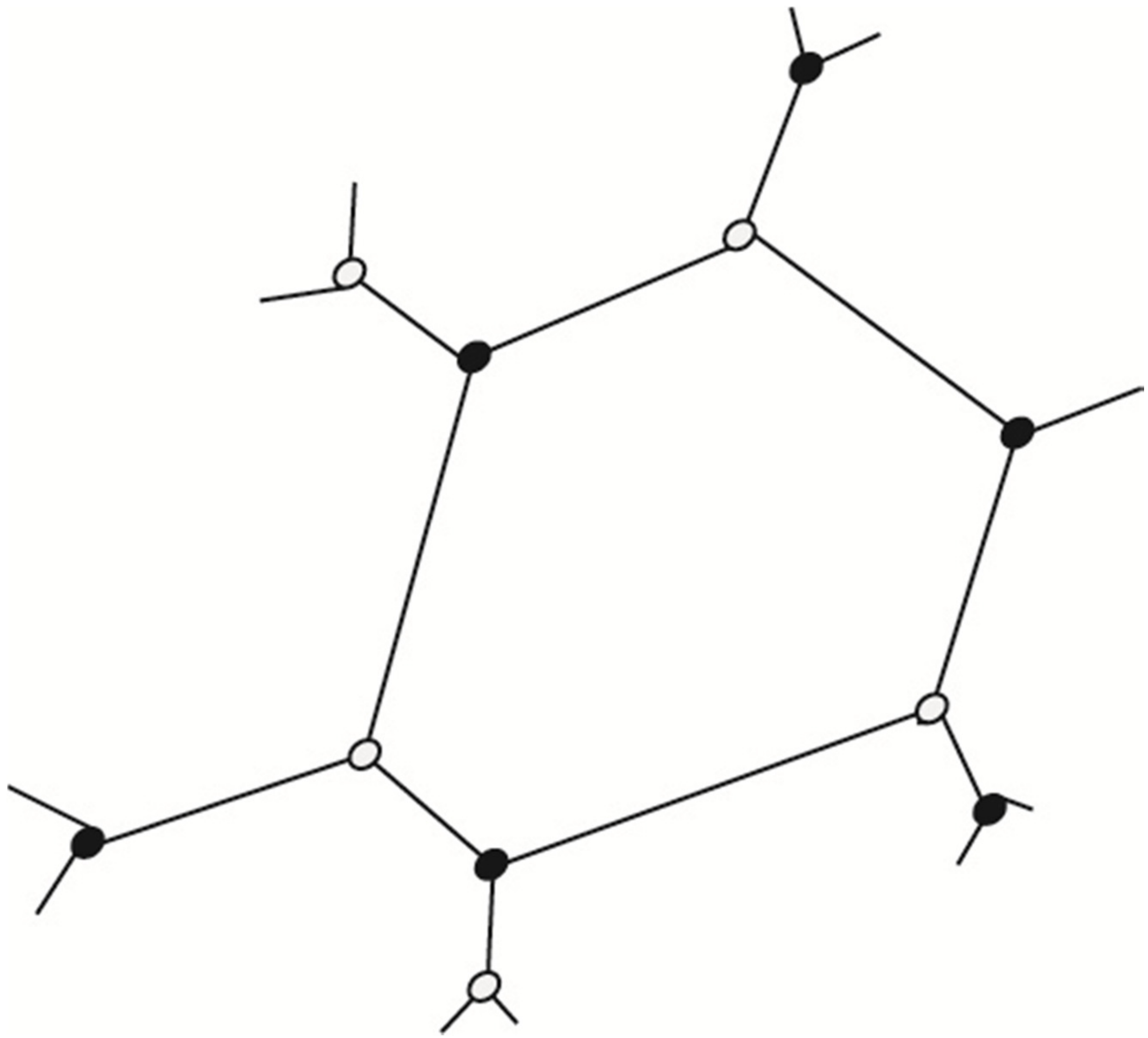
Vac. $\exp\left[\frac{\partial}{\partial \sigma} \mathcal{W}_{2d}(\sigma; \theta)\right] = 1$

twisted superpotential

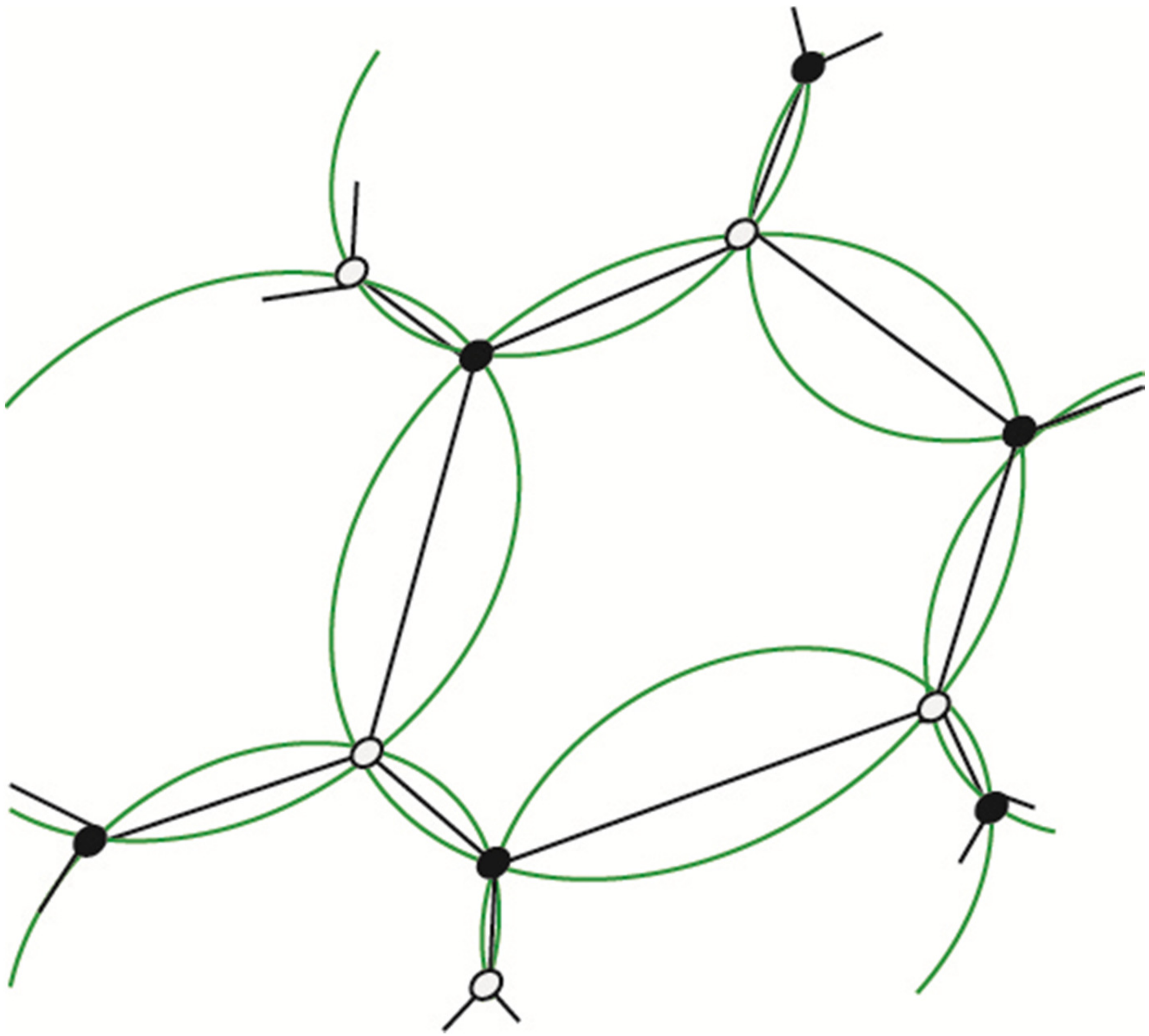
“circle pattern”



“circle pattern”



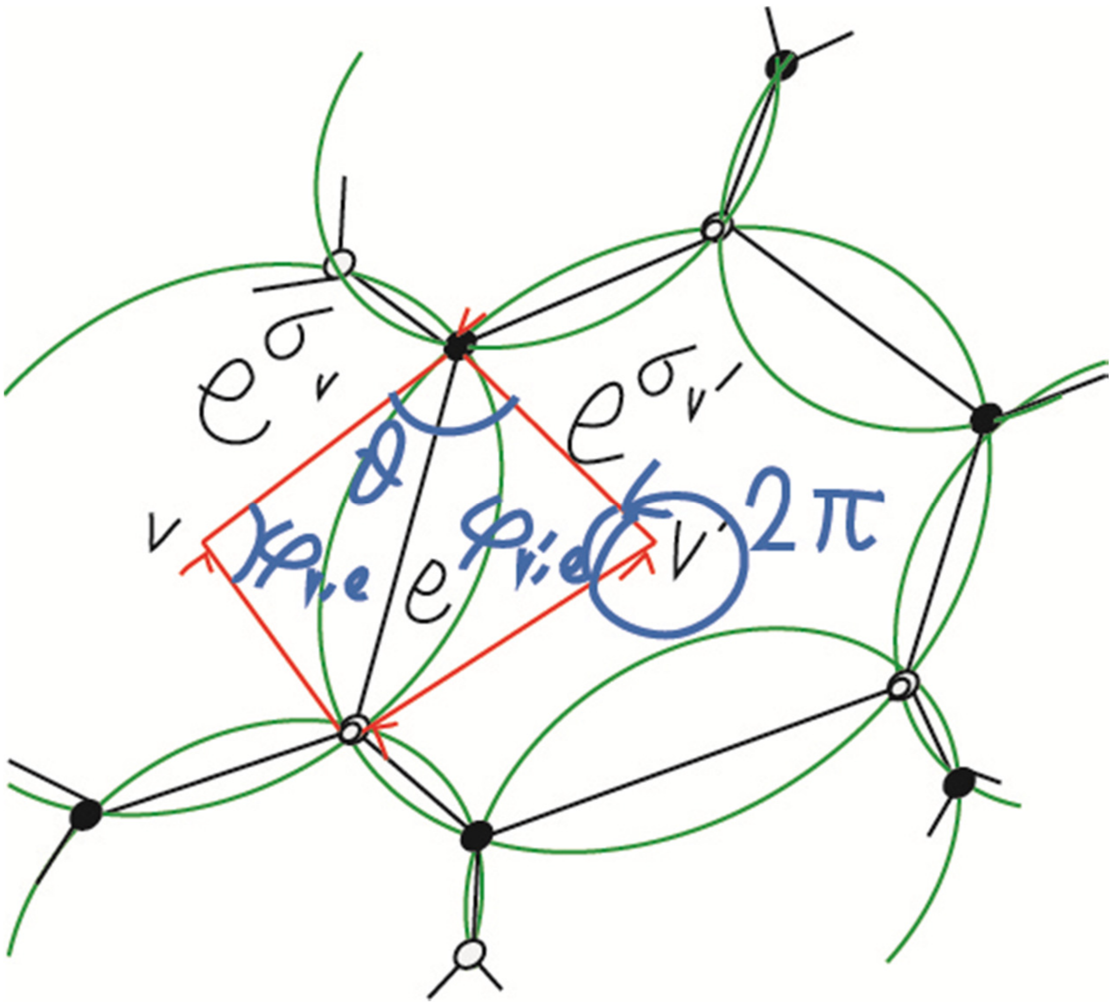
“circle pattern”



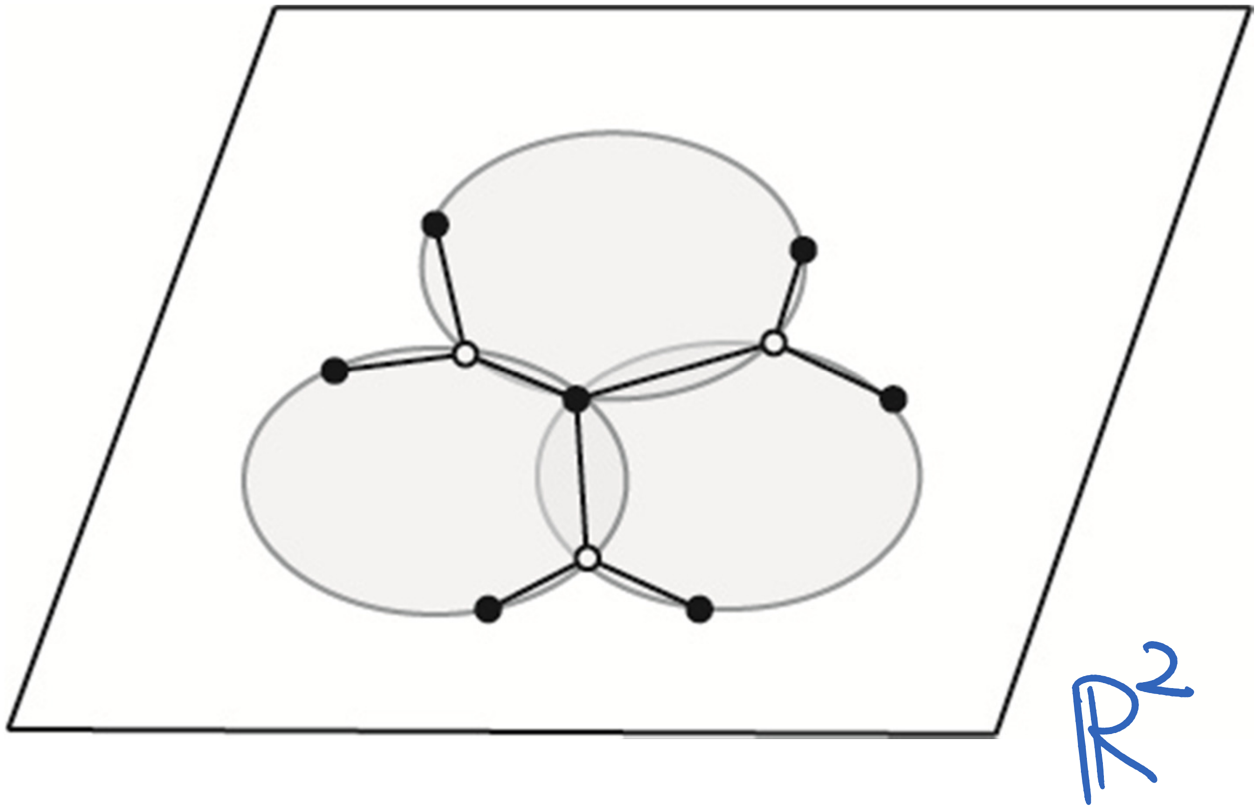
$$\exp\left[\frac{\partial}{\partial\sigma} W_{2d}(\sigma;\theta)\right] = 1 \Rightarrow \sum_{e\in v} \varphi_{v,e} = 2\pi$$

[Bobenko-Springborn] ('02)

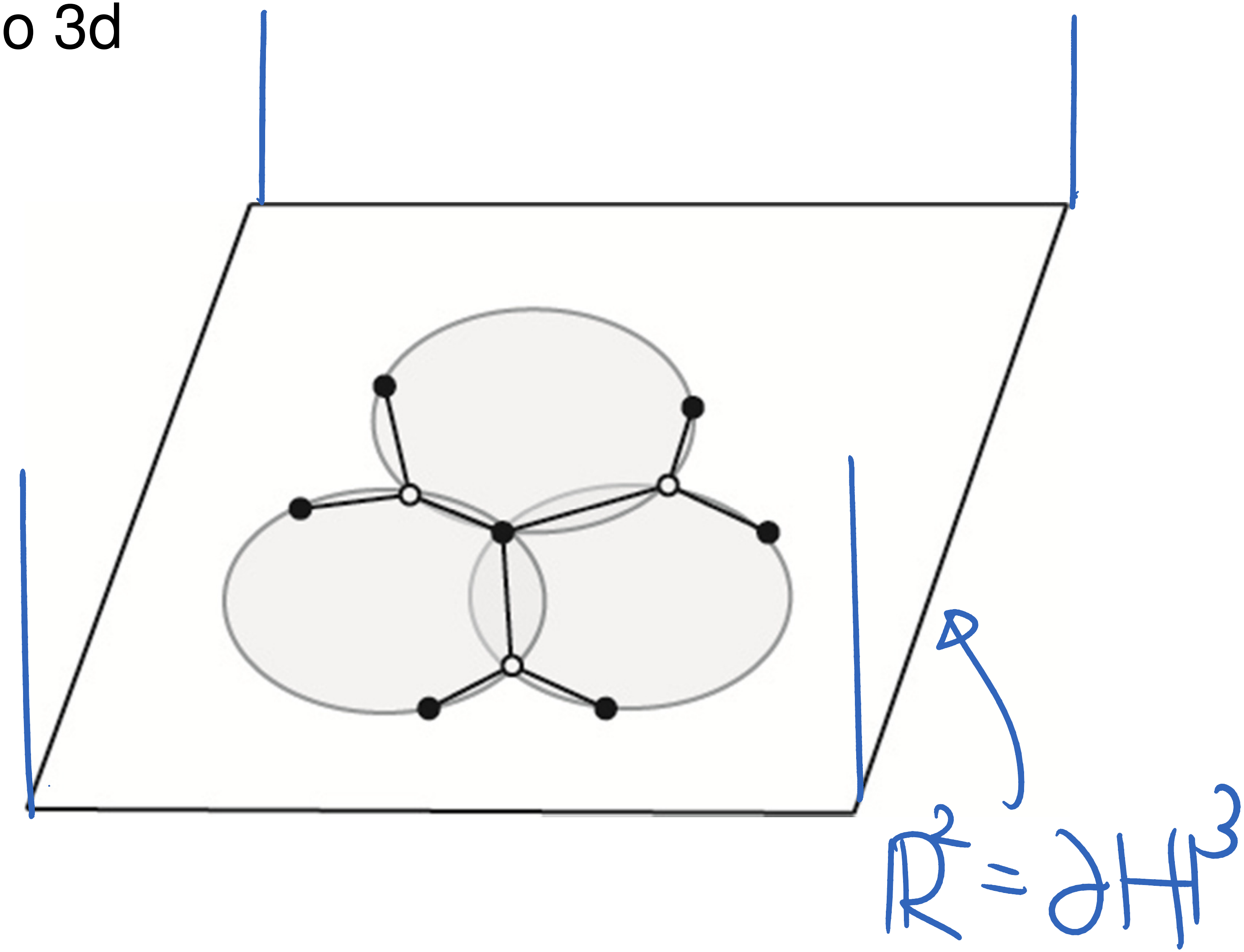
[Bazhanov-Mangazeev-Sergeev] ('07)



Lift to 3d

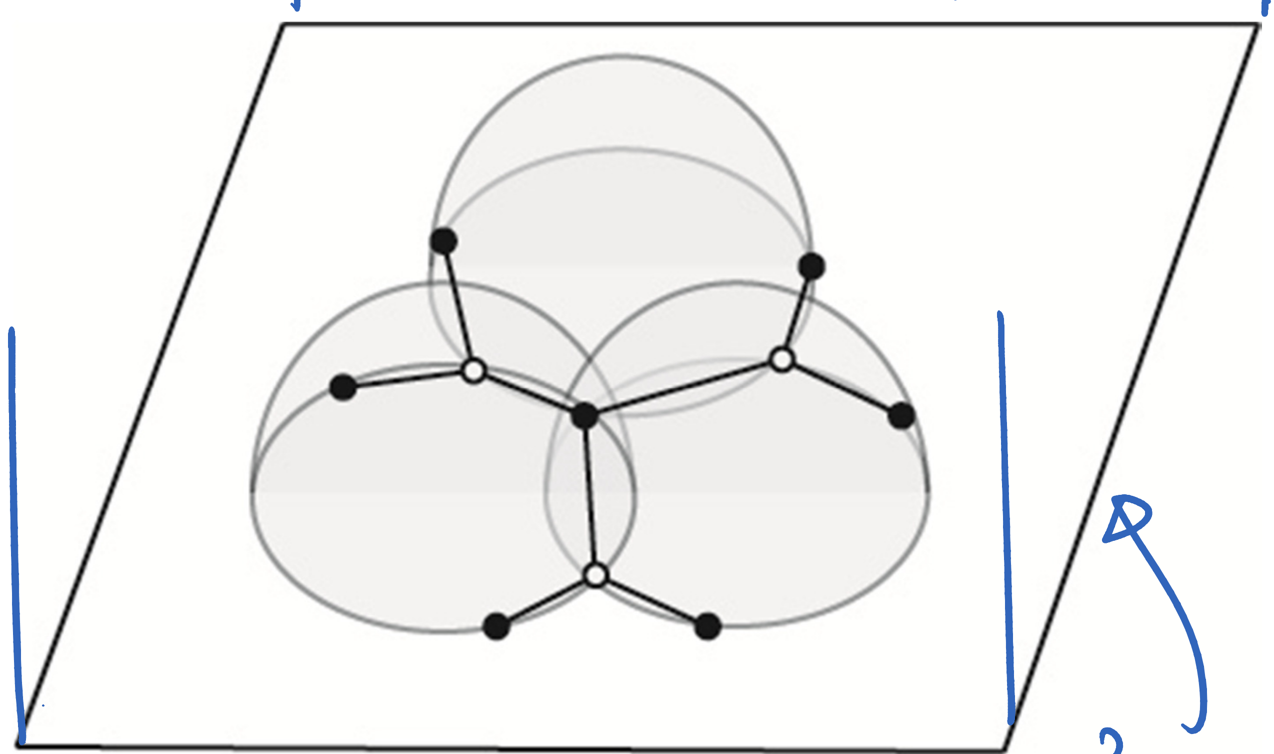


Lift to 3d



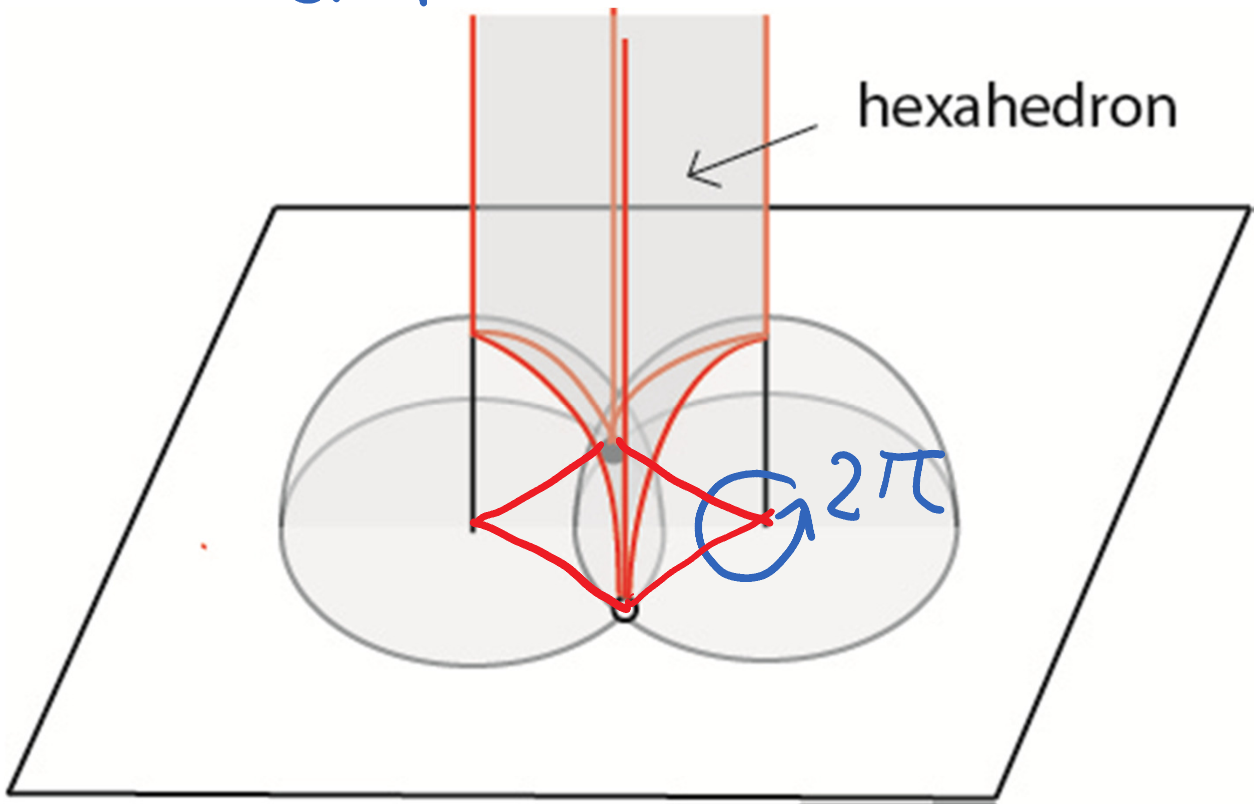
Lift to 3d

M : complement in $\mathbb{H}^3$



$\mathbb{R}^2 = \partial\mathbb{H}^3$

$$\mathcal{W}_{2d}(\sigma)\Big|_{\text{crit}} = \mathrm{Vol}(M) = \mathcal{U}(\text{hexahedron})$$



vacuum equation

= gluing of hexahedra

$$\text{``}Z_{3d\ \mathcal{N}=2}[S^3_b]\text{''} = \text{``}Z_{3d\ SL(2)\ CS}^{\,t}[M]$$

quiver SCFT

$t \sim 1/b^2$

$$\begin{array}{ccc}
\text{``}Z_{3d\ \mathcal{N}=2}[S^3_b]\text{''} & & \text{``}Z_{3d\ SL(2)\ CS}^{\,t}[M] \\
\Big\downarrow{\scriptstyle b\to 0} & & \Big\downarrow{\scriptstyle t\to\infty} \\
W_{2d}(\sigma) & \Longrightarrow & \text{classical hyperbolic geom of } M
\end{array}$$

Relation with topological strings
(for torus, isoradial networks)

[Y] ('12)

$$F_{\mathrm{top},0} = \int_{\mathbb{R}^2} d\vec{x}\, \overbrace{\mathcal{L}\left[W_{2d}(\delta:\theta)\big|_{\mathrm{crit}} \right]}^{\text{Ronkin function}}$$

on toric CY_3

Legendre transform w.r.t. θ

Based on [Kenyon] ('02)
[Kenyon-Okounkov-Sheffield]('06)
[Tilière] ('07)
[Ooguri-Y] ('10)

Summary // cell of $(\mathrm{Gr}_{k,n})_{\geq 0}$

Permutation π

toric data Δ

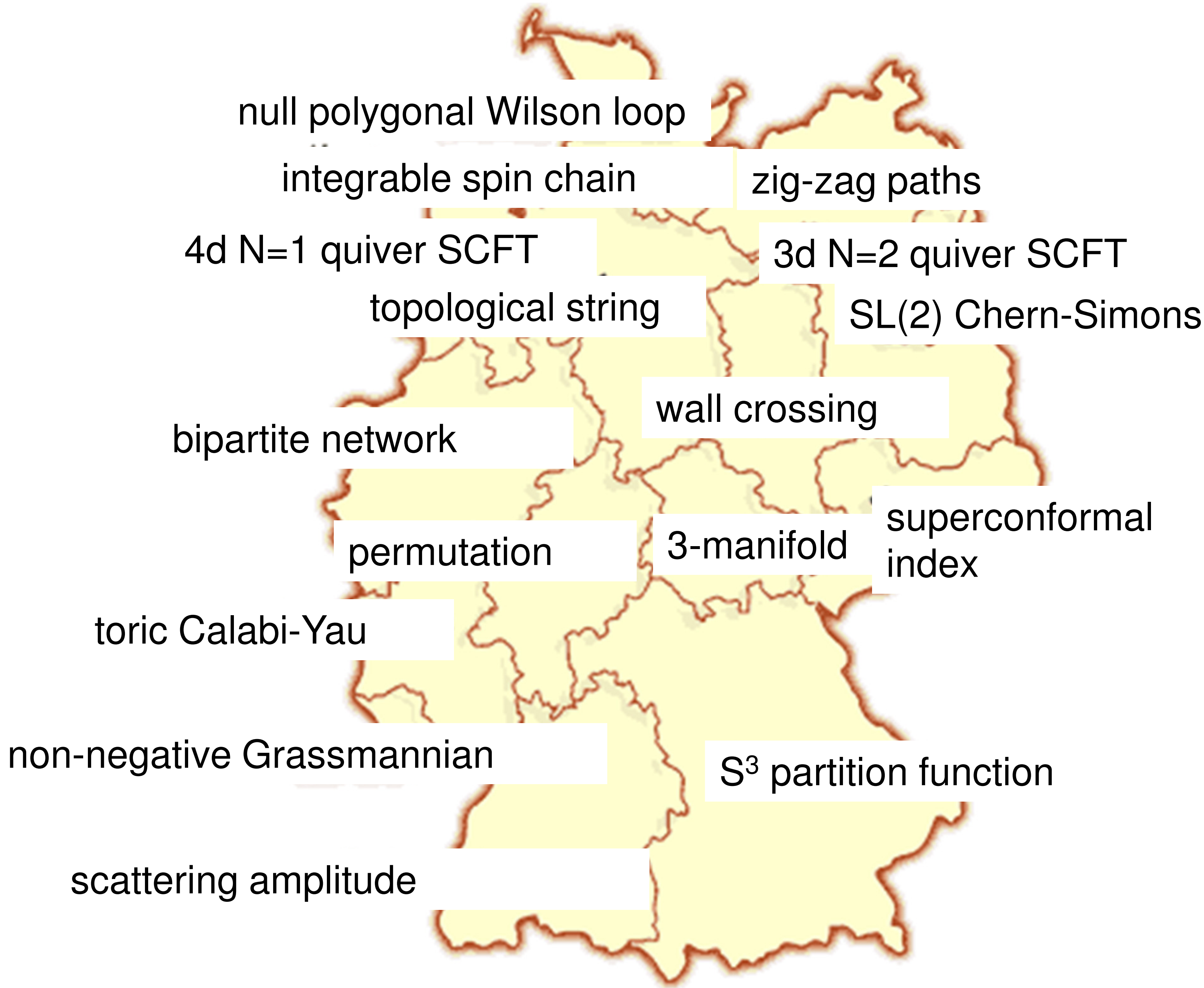
bipartite network N on $\left\{\begin{array}{l} \mathbb{R}^2 \\ T^2 \end{array}\right.$

4d N=1 SCFTs $T[N]$

“class N”

2d integrable spin chain $=$ 4d superconformal index

3d hyperbolic geometry “$=$” 3d S^3 partition function


null polygonal Wilson loop
integrable spin chain
zig-zag paths
4d N=1 quiver SCFT
3d N=2 quiver SCFT
topological string
SL(2) Chern-Simons
wall crossing
bipartite network
superconformal index
permutation
3-manifold
toric Calabi-Yau
non-negative Grassmannian
S3 partition function
scattering amplitude

Space of SCFTs?