Mass, quasi-local mass, and the flow of static metrics

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Quasilocal mass

- QLM "usefully" characterizes bounded n-dim submanifolds of Riemannian n-manifolds; or bounded n-dim spacelike submanifolds of $(n+1)$-spacetimes.
- Examples:
  - Brown-York mass: $\int_\Sigma (H_0 - H)\, dA$,
  - $H =$ mean curvature of Σ,
  - $H_0 =$ mean curvature of isometrically embedded copy of Σ in flat space.
  - Hawking mass: $\frac{1}{16\pi}\sqrt{A(\Sigma)}\left(1 - \int_\Sigma H^2 dA\right)$.
  - Bartnik mass.

Bartnik's quasi-local mass: time symmetric case

Definition:

- Consider a bounded domain B with compact boundary ∂B embedded isometrically in an
  - asymptotically flat,
  - scalar curvature $R \geq 0$
  - Riemannian manifold that admits no stable minimal sphere outside B.

  These are the *allowed extensions* of B.
- Compute ADM mass of each allowed extension.
- The infimum of the ADM mass over all allowed extensions is the quasi-local mass m_B of B.

Remarks on QLM Definition

Note that:

- $B \subseteq B' \;\Rightarrow\; m_{B'} \geq m_B$ (monotonicity).
- $m_B \geq 0$ (positivity).

Note also:

- Extensions need not be smooth.
- "Geometric boundary conditions": Induced metrics and mean curvatures must match across ∂B (but extrinsic curvature not required to match).
- That is, Bartnik permits distributional stress-energy on boundary.

The conjecture

- Question: When (if ever) is $m_B > 0$?
- Conjecture: The mass m_B is realized as the ADM mass of some extension of B (i.e., the infimum in Bartnik's definition is a minimum).
- Furthermore, the extension that realizes m_B is a solution of the static Einstein equations:

$$R^g_{ij} = k_n^2 \nabla_i u \nabla_j u \ .$$

- Bianchi identity implies $\Delta u = 0$.
- For $k_n^2 = \frac{n-1}{n-2}$ the static Einstein equations imply that the locally static spacetime metric

$$ds^2 = -e^{2u} dt^2 + e^{-\frac{2u}{n-2}} g_{ij} dx^i dx^j$$

is Ricci-flat.

Flow of Static Metrics (B List; PhD thesis under G Huisken)

$$\frac{\partial g_{ij}}{\partial \lambda} = -2\left(R_{ij} - k_n^2 \nabla_i u \nabla_j u\right)$$

$$\frac{\partial u}{\partial \lambda} = \Delta u$$

Fixed points yield locally static, Ricci-flat spacetime metrics

$$ds^2 = -e^{2u}dt^2 + e^{-\frac{2u}{n-2}} g_{ij} dx^i dx^j \ ,$$

where we have taken $k_n^2 = \frac{n-1}{n-2}$.

Static Metric Flow is Ricci Flow (+ Diffeo)

- Begin with metric

$$dS^2 = G_{\mu\nu} dx^\mu dx^\nu = e^{2k_n u} dt^2 + g_{ij} dx^i dx^j$$

with

$$\frac{\partial u}{\partial t} = 0 \ , \ \frac{\partial g_{ij}}{\partial t} = 0.$$

- Define vector field

$$X := -k_n \nabla^i u \frac{\partial}{\partial x^i}.$$

- Apply Hamilton-DeTurck flow

$$\frac{\partial G_{\mu\nu}}{\partial \lambda} = -2R^G_{\mu\nu} + \mathcal{L}_X G_{\mu\nu}.$$

- This yields the static metric flow equations of the last slide.

Aside: Ricci solitons from scalar fields

- Solutions (u, g_{ij}) of the static equations (= Euclidean-signature Einstein-free-scalar equations) produce local steady Ricci solitons

$$dS^2 = e^{2k_n u} dt^2 + g_{ij} dx^i dx^j \ , \ X = -k_n \nabla^i u \frac{\partial}{\partial x^i} \ .$$

- Local solutions (u, g_{ij}) of the (Euclidean-signature) free-scalar equations with cosmological term

$$R^g_{ij} = k_n^2 \nabla_i u \nabla_j u + \kappa g_{ij} \ ,$$

produce local scaling Ricci solitons

$$\begin{aligned} dS^2 &= e^{2k_n u} dt^2 + g_{ij} dx^i dx^j \ , \\ X &= -\kappa t \frac{\partial}{\partial t} - k_n \nabla^i u \frac{\partial}{\partial x^j} \ . \end{aligned}$$

Examples

- A local soliton from the Schwarzschild metric:

$$ds^2 = \left(1 - \frac{2m}{r}\right)^{\sqrt{2}} dt^2 + r\,(r-2m)\left(d\theta^2 + \sin^2\theta d\phi^2\right) \ .$$

- From the Buchdahl (Janis-Newman-Winnicour, etc) solution:

$$\begin{aligned} ds^2 &= \left(1 - \frac{2m}{r}\right)^{3\sqrt{1-\delta}/2} dt^2 + \left(1 - \frac{2m}{r}\right)^{\delta} d\tau^2 \\ &\quad + \left(1 - \frac{2m}{r}\right)^{-\delta} dr^2 + r^2 \left(1 - \frac{2m}{r}\right)^{1-\delta} \left(d\theta^2 + \sin^2\theta d\phi^2\right) \ . \end{aligned}$$

- A complete, expanding soliton:

$$ds^2 = \frac{e^{2r}}{\left(1 + e^{\sqrt{2}r}\right)} dt^2 + dr^2 + \left(1 + e^{\sqrt{2}r}\right)\left(d\theta^2 + \sinh^2\theta d\phi^2\right) \ .$$

List's flow: strategy

"If you can't solve a problem, then there is an easier problem you can solve: Find it!"—G Polya (source: Wikipedia), and probably a great many other thesis supervisors as well.

- Specialize to flow on $\mathbb{R}^n$: no inner boundary ∂B.
  - List (PhD thesis): Flow exists for some interval $\lambda \in [0, T)$.
  - Can extend beyond T unless $\lim_{\lambda \nearrow T} \sup_{x \in \mathbb{R}^n} |\mathrm{Riem}| = \infty$.
- Specialize to "strongly asymptotically flat" initial data.
  - List (PhD thesis): Flow preserves asymptotic flatness, mass remains constant.
- Specialize to rotational symmetry.
- Further specializations if necessary (initial data, dimension,...).

List's flow on complete manifolds (no boundary)

- $\frac{\partial}{\partial\lambda}|\nabla u|^2 = \Delta|\nabla u|^2 - 2|\nabla\nabla u|^2 - 4\left(|\nabla u|^2\right)^2$.
- Max principle: $|\nabla u|^2 < \frac{const}{1+\lambda}$. "Deviation from ($n$-dim) Ricci flow dies out."
- $\frac{\partial S}{\partial\lambda} = \Delta S + 2S_{ij}S^{ij} + 4\left(\Delta u\right)^2$,
  where $S_{ij} := R_{ij} - k_n^2\nabla_i u\nabla_j u$ and $S := R - k_n^2|\nabla u|^2$
- Minimum principle: $S = R - k_n^2|\nabla u|^2 \geq -\frac{const}{1+\lambda}$.
- Bad news: $R > 0$ is not preserved, but "almost":
  (i) $R \geq -\frac{const}{1+\lambda}$ and
  (ii) $R \geq k_n^2|\nabla u|^2 \geq 0$ is preserved.

Rotational Symmetry

- Symmetry: $ds^2 = f^2(\lambda, r)dr^2 + r^2 g(S^{n-1}, \mathrm{can})$, $u = u(\lambda, r)$.
- Flow equations acquire DeTurck (diffeomorphism) term, which can be solved for and eliminated.
- The $\frac{\partial g_{ij}}{\partial \lambda}$ equation reduces to a single PDE. System becomes:

$$
\begin{aligned}
\frac{\partial f}{\partial \lambda} &= \Delta f + X \cdot \nabla f - \frac{(n-2)}{r^2 f}\left(f^2 - 1\right) + k_n^2 f |\nabla u|^2 \ , \\
X &:= \left(\frac{n-2}{r} - \frac{n}{rf^2} - \frac{1}{f^3}\frac{\partial f}{\partial r}\right)\frac{\partial}{\partial r} \ , \\
\frac{\partial}{\partial \lambda}|\nabla u|^2 &= \Delta |\nabla u|^2 - 2|\nabla\nabla u|^2 - 4\left(|\nabla u|^2\right)^2 \ .
\end{aligned}
$$

Intuition from Ricci flow

- If $\lambda \in \mathbb{R}$, system is equivalent to Hamilton-DeTurck flow on $\mathbb{R}^{n+1}$ with $\mathbb{R} \times \mathrm{SO}(n)$ symmetry.
- If $t \in S^1$, system is equivalent to Hamilton-DeTurck flow on $S^1 \times \mathbb{R}^n$ with $\mathrm{U}(1) \times \mathrm{SO}(n)$ symmetry (perturbed "hot flat space").
- Good news: Hot flat space (with finite boundary rather than asymptotic flatness: $S^1 \times B^n$) attracts flows in numerical simulations [Headrick-Wiseman arxiv:hep-th/0606086].
- Good news: Ricci flow on $\mathbb{R}^{n+1}$ with $\mathrm{SO}(n+1)$ symmetry exhibits long-time existence *and convergence* for asymptotically flat initial data that does not contain any minimal sphere [Oliynyk-Woolgar arxiv:math/0607438].

Gutperle, Headrick, Minwalla, Schomerus (GHMS, 2002)

- Expanding Ricci soliton on $\mathbb{R}^2$ given by

$$ds^2 = \lambda \left(f^2(r) dr^2 + r^2 \zeta^2 d\theta^2 \right) \ , \ \zeta = const.$$

- Changing coordinates to $\rho = r\sqrt{\lambda}$, we get

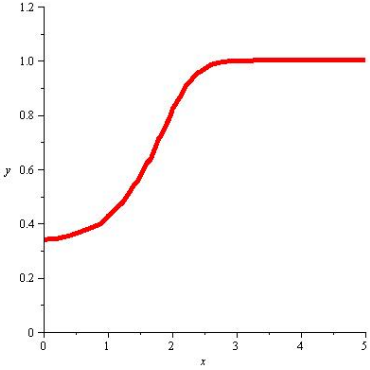
$$ds^2 = f^2(\rho/\sqrt{\lambda}) d\rho^2 + \rho^2 \zeta^2 d\theta^2 \ .$$

- $f(x)$ is given implicitly by

$$\left(\frac{1}{\zeta} - 1 \right) \exp \left(\frac{1}{\zeta} - 1 - \frac{x^2}{2\alpha'} \right) = \left(\frac{1}{f(x)} - 1 \right) \exp \left(\frac{1}{f(x)} - 1 \right)$$

- $f(x) \to \zeta$ for $x \searrow 0$.
- $f(x) \to 1$ for $x \nearrow \infty$.
- f is monotonic on $(0, \infty)$.

Graph of $f(x)$ for $\zeta = 1/3$



$$\begin{aligned} ds^2 &= \lambda\left(f^2(r)dr^2 + r^2\zeta^2 d\theta^2\right) \\ &= f^2(\rho/\sqrt{\lambda})d\rho^2 + \rho^2\zeta^2 d\theta^2 \end{aligned}$$

- Asymptotic deficit angle is 2D mass (hint: think of isoperimetric defict).
- At fixed λ, the $\rho \to \infty$ limit is flat cone of deficit angle $\delta = 2\pi(1 - \zeta)$.
- At fixed ρ, the $\lambda \to \infty$ limit is flat space.

Lessons Learned

- Mass (deficit angle) constant along flow.
  - Oliynyk-Woolgar; Dai-Ma: True in n dimensions.
  - List, PhD Thesis: Mass stays constant for List's flow, in any dimension.)
- Geometric limit is flat space; mass jumps to zero.
  - Oliynyk-Woolgar: True in n dimensions.
- Brown-York quasilocal mass evaporates smoothly; evaporation occurs primarily on distance scales $r < \sqrt{\lambda}$.
  - Oliynyk-Woolgar: True in n dimensions.
- Agrees with qualitative "rolling tachyon" predictions in string theory.

Rotationally symmetric List flow: quantities of interest

- Must ensure coordinates stay good:
  - $H = \frac{n-1}{rf}$ = mean curvature of orbits of $\mathrm{SO}(n)$. Coordinates good if no minimal sphere forms (at infinity; $f \to \infty$).
  - $X = \left[\frac{1}{f^3}\frac{\partial f}{\partial r} + \frac{(n-2)}{r}\left(1 - \frac{1}{f^2}\right)\right]\frac{\partial}{\partial r}$ = DeTurck vector field.
- Must bound the curvatures:
  - $\kappa_1 = \frac{1}{rf^3}\frac{\partial f}{\partial r}$ = sectional curvature in planes containing $\frac{\partial}{\partial r}$.
  - $\kappa_2 = \frac{1}{r^2}\left(1 - \frac{1}{f^2}\right)$ = sectional curvature in planes $\perp \frac{\partial}{\partial r}$.
  - $R = (n-1)\left[\frac{2}{rf^3}\frac{\partial f}{\partial r} + \frac{(n-2)}{r^2}\left(1 - \frac{1}{f^2}\right)\right]$

Getting started

- Estimate $w(\lambda, r) := f^2(\lambda, r) - 1$:

$$\frac{\partial w}{\partial \lambda} = \Delta w + Y \cdot \nabla w - \frac{2(n-2)}{r^2} w + 2k_n^2 f^2 |\nabla u|^2$$

- Lower bound, prevents degeneration of coordinates:

$$n > 2: \qquad w(\lambda, r) \geq \inf_r \{w(0, r)\} > -1$$
$$n = 2: \qquad w(\lambda, r) \geq \min \left\{ w_\infty, \inf_r \{w(0, r)\} \right\}$$

- Upper bound, using $|\nabla u|^2 \leq C_1/(1+\lambda)$:

$$w(\lambda, r) \leq C_2 (1+\lambda)^p \ , \ p = 2k_n^2 C_1$$

- Note: No minimal spheres form in *finite* time; can form in limit.

Estimates

(a) $const \leq f^2 \leq const \cdot (1+\lambda)^p$.

(b) $\frac{1}{r^2}\left(f^2 - 1\right) \geq -const \cdot (1+\lambda)^{p+1}$.

(c) $|\nabla u|^2 \leq \frac{const}{1+\lambda}$.

(d) $R(\lambda, x) \geq -\frac{const}{1+\lambda}$ "Sum of sectional curvatures $> -\infty$."

(e) $R(\lambda, x) \geq k_n^2 |\nabla u|^2$ $\forall (\lambda, x)$ if true $\forall (0, x)$.

Further estimates

If we *assume* that

(f) $\frac{1}{r}|\nabla u| \leq \alpha = const$,

then we also deduce that

(g) $\frac{1}{r^2}\left(f^2-1\right) \leq const \cdot \left[1+\alpha^2(1+\lambda)^{p+1}\right] \leq const \cdot e^{const\cdot\lambda}$,

(h) $\left|\frac{\partial}{\partial r}\left(|\nabla u|\right)\right| \leq const \cdot e^{const\cdot\lambda}$, and

(i) $\kappa_1 + \frac{2}{f^2(1+f)}\frac{1}{r^2}\left(f^2-1\right) \leq const \cdot e^{const\cdot\lambda}$,

- where $\kappa_1 := \frac{1}{rf^3}\frac{\partial f}{\partial r}$ = sectional curvature in planes tangent to $\frac{\partial}{\partial r}$.

Long time existence

(j) (b)+(i) $\Rightarrow \kappa_1 \leq const \cdot e^{const \cdot \lambda}$.

(k) (a)+(g) $\Rightarrow \kappa_2 \leq const \cdot e^{const \cdot \lambda}$.

(d) Recall $R \equiv (n-1)\left[2\kappa_1 + (n-2)\kappa_2\right] \geq -\frac{const}{1+\lambda}$.

(l) Thus (j)+(k)+(d) $\Rightarrow \kappa_1 \geq -const \cdot e^{const \cdot \lambda}$

(m) and $\kappa_2 \geq -const \cdot e^{const \cdot \lambda}$ when $n > 2$.

Proposition: If $\frac{1}{r}|\nabla u| \leq const$ for all $\lambda \geq 0$, then sectional curvatures do not diverge for any $\lambda \in [0, T]$ and any $T > 0$. Then [List, PhD thesis] the flow exists for all $\lambda \geq 0$.

Open questions

- Can we prove $\frac{1}{r}|\nabla u| \leq const$?
  - Yes! ...if $n = 2$.

Proposition: If $n = 2$ the flow exists for all $\lambda \geq 0$.

- Can we prove convergence of the flow to a limit, say flat space with $u = 0$?
  - No, not even close, but as Polya said: "If you can't solve a problem, then there is an easier problem you can solve: Find it!"
  - We did.

Closing remark: Stationary metrics

- A stationary metric is a Lorentzian metric with timelike Killing vector field which is not necessarily hypersurface-orthogonal.
- The stationary metric

$$ds^2 = -e^{2u}\left(dt + B_i dx^i\right)^2 + e^{-\frac{2u}{n-2}} g_{ij} dx^i dx^j$$

is Ricci flat if (u, B_k, g_{ij}) obey

$$\begin{aligned}
0 &:= \Delta u + \frac{1}{4} e^{2\left(\frac{n-1}{n-2}\right)u} |F|^2 \ , \\
0 &= -\nabla^j F_{ij} - 2\left(\frac{n-1}{n-2}\right) F_{ij} \nabla^j u \ , \\
0 &= -2R_{ij} + 2\left(\frac{n-1}{n-2}\right) \nabla_i u \nabla_j u \\
&\quad - e^{2\left(\frac{n-1}{n-2}\right)u} \left[F_{ik} F_j{}^k - \frac{1}{2(n-2)} g_{ij} |F|^2 \right] \ .
\end{aligned}$$

Flow of stationary metrics

The flow

$$\begin{aligned}
\frac{\partial u}{\partial \lambda} &:= \Delta u + \frac{1}{4} e^{2\left(\frac{n-1}{n-2}\right)u} |F|^2 \ , \\
\frac{\partial B_i}{\partial \lambda} &= -\nabla^j F_{ij} - 2\left(\frac{n-1}{n-2}\right) F_{ij} \nabla^j u \ , \\
\frac{\partial g_{ij}}{\partial \lambda} &= -2R_{ij} + 2\left(\frac{n-1}{n-2}\right) \nabla_i u \nabla_j u \\
&\quad - e^{2\left(\frac{n-1}{n-2}\right)u} \left[F_{ik} F_j{}^k - \frac{1}{2(n-2)} g_{ij} |F|^2 \right] \ .
\end{aligned}$$

- is as well-behaved as Yang-Mills-Ricci flow,
- contains List's flow as the $B_k = 0$ case,
- has fixed points that correspond to Ricci-flat stationary metrics,
- but generally does not appear to arise as Hamilton-DeTurck flow.

Is there a Hamilton-DeTurck flow whose fixed points are Ricci-flat stationary metrics?