# Geometric flows and holography

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# Introduction

The aim of this talk is two-fold:

1. I would like to argue for a connection between geometric flows and three dimensional quantum field theories using holography.
2. Discuss the necessary holographic tools needed to flesh out this connection.

# References

- The first part is based on on-going work with I. Bakas.
- The second part is based on

  KS, Balt van Rees, Phys.Rev.Lett. (2008), arXiv:0805.0150.

  KS, Balt van Rees, arXiv:0812.xxxx

# The main idea

The main idea is the following:

- Certain geometric flows can be embedded in Einstein's equations with negative cosmological constant in four dimensions.
- Solutions that are asymptotically $AdS_4$ encode quantum field theory (QFT) data for a QFT in three dimensions.
- Therefore, these geometric flows should be related to QFTs in three dimensions.

# Geometric flows and Asymptotically AdS spacetimes

There are two main examples of such connection:

1. Calabi flow and Robinson-Trautman spacetimes.
2. Normalized Ricci flow and certain perturbations of $AdS_4$ Schwarzschild black holes.

# Robinson-Trautman spacetimes

- The metric is given by

$$ds^2 = 2r^2 e^{\Phi(z,\bar{z};u)} dz d\bar{z} - 2du dr - F(r,u,z,\bar{z}) du^2$$

- The function $F$ is uniquely determined in terms of $\Phi$,

$$F = r\partial_u \Phi - \Delta\Phi - \frac{2m}{r} - \frac{\Lambda}{3} r^2$$

  where $\Lambda$ is related to the cosmological constant and $\Delta = e^{\Phi}\partial_z \partial_{\bar{z}}$.

- The function $\Phi(z,\bar{z};u)$ should solve the following Robinson-Trautman equation,

$$3m\partial_u \Phi + \Delta\Delta\Phi = 0.$$

# Calabi flow

- The Calabi flow is defined for a metric $g_{a\bar{b}}$ on a Kaehler manifold $M$ by the Calabi equation

$$\partial_u g_{a\bar{b}} = \frac{\partial^2 R}{\partial z^a \partial z^{\bar{b}}}$$

  where $R$ is the curvature scalar of $g$.

- For $M = S^2$ the Calabi equation becomes the Robinson-Trautman equation with $m = 2/3$.

# Schwarzschild AdS solution from Robinson-Trautman

- The Robinson-Trautman solutions are Asymptotically locally AdS solutions (AlAdS).
- A special solution of the Robinson-Trautman equation is

$$e^{\Phi_0(z,\bar{z})} = \frac{1}{(1+z\bar{z}/2)^2}$$

  This leads to the Schwarzschild $AdS_4$ solution.
- A certain class of perturbations of the Schwarzschild $AdS_4$ solution fall into the Robinson-Trautman metrics.
- The late time behavior, as the solution approaches the Schwarzschild $AdS_4$ solution, is computable at the non-linear level.

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# Normalized Ricci flow

- Recall that the Ricci flow equation is

$$\partial_u g_{ij} = -R_{ij}$$

  This flow does not preserve the spacetime volume.
- One can modify the flow to become volume preserving leading to the normalized Ricci flow. For metrics $ds^2 = 2e^{\Phi} dz d\bar{z}$ on $S^2$ this flow is governed by

$$\partial_u \Phi = \Delta \Phi + 1$$

# Normalized Ricci flow and large $AdS_4$ black holes

- The constant curvature metric provides a fixed point for the flow.
- The spectrum of axial perturbations as the flow approaches this fixed point can be computed analytically.
- Large $AdS_4$ black holes exhibit certain purely dissipative axial perturbations with exactly the same (imaginary) frequencies (computed now numerically) as in the normalized Ricci flow. [I. Bakas (2008)]

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# Geometric flows and AdS/CFT

- We have seen that both geometric flows are related to perturbations around the $AdS_4$ Schwarzschild black hole, with the Calabi flow being more generally associated with Asymptotically locally AdS spacetimes.
- In AdS/CFT the Schwarzschild black hole is associated with a thermal state in the dual $3d$ QFT.
- Perturbations around any given AlAdS solution are associated with QFT correlators of specific operators in the state specified by the background solution:

  linearized perturbations $\longrightarrow$ 2-point functions
  2nd order perturbations $\longrightarrow$ 3-point functions
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$\Rightarrow$ Geometric flows control the behavior of certain QFT correlators at strong coupling.

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# Holography

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- We will actually ask a more general question:

  How do we set up the gravity/gauge theory duality in real-time?

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  How do we set up the gravity/gauge theory duality in real-time?

# Holography in real-time

One would like to set up a prescription as general as the Euclidean one. In particular, it should

- apply to any $n$-point function, including correlators in non-trivial states.
- apply to all QFTs with a holographic dual.
- the prescription should be fully holographic, i.e. only boundary data and regularity should suffice.
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# Motivation

Euclidean techniques suffice for many applications. However, it is clear that there are many reasons to set up the holographic prescription directly in Lorentzian signature. To mention a few:

1. holography for time-dependent backgrounds,
2. holographic description of non-equilibrium QFT,
3. computation of correlators in non-trivial states,
4. Holography vs causality,
5. Understanding the physics of black hole horizons,
6. etc. etc.

The development of a real-time formalism is also becoming urgent, as actual application, for example the modeling of the quark-gluon plasma in RHIC and LHC, require real-time techniques. Actually some of the previous work on the subject was driven by such applications [Son, Starinets], [Herzog, Son](2002)

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2. Lorentzian prescription
3. Examples
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# Basic Dictionary

Let us start by briefly reviewing the basics of holography. In the low energy approximation, where the bulk theory is approximated by supergravity the basic holographic dictionary is [GKP,W (1998)]:

1. There is 1-1 correspondence between local gauge invariant operators $\mathcal{O}$ of the boundary QFT and bulk supergravity modes $\Phi$.
2. The fields $\phi_{(0)}$ parametrizing the boundary conditions of the bulk fields $\Phi$ are identified with the sources of dual operators.

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$$Z_{SUGRA}[\phi_{(0)}] = \int_{\Phi \sim \phi_{(0)}} \mathcal{D}\Phi \exp\left(-S[\Phi]\right) = \langle \exp(-\int_{\partial M} \phi_{(0)} \mathcal{O}) \rangle_{QFT}$$

To leading order

$$S_{on-shell}[\phi_{(0)}, \ldots] = -W_{QFT}[\phi_{(0)}, \ldots]$$

on-shell SUGRA action = generating functional of QFT connected graphs

Such a relation is however formal as both sides diverge. On the QFT side these are the usual UV divergences, dealt with by standard renormalization techniques. On the gravitational side, the infinities are due to the infinite volume of the spacetime. This issue is dealt with by the formalism of holographic renormalization, which is the precise gravitational analogue of QFT renormalization. [Henningson, KS (1998)], ...

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# Holographic renormalization

To understand holographic renormalization one needs to know some facts about asymptotically (locally) AdS spacetimes.

- These spacetimes solve the Einstein equations with a negative cosmological constant and have the following asymptotic (Fefferman-Graham) form

$$ds^2 = \frac{dr^2}{r^2} + \frac{1}{r^2} g_{ij}(x,r) dx^i dx^j$$

where

$$g_{ij}(x,r) = \mathbf{g}_{(0)ij} + r^2 g_{(2)ij} + ... + r^d \left(\log r^2 h_{(d)ij} + g_{(d)ij}\right) + ...$$

This is an expansion in $r$ (the conformal boundary of the spacetime is located at $r = 0$).

- Matter fields, e.g. scalar fields, have a similar asymptotic expansion

$$\Phi(x,r) = r^{d-\Delta} \left(\phi_{(0)} + r^2 \phi_{(2)} + ... + r^{2\Delta-d} \left(\log r^2 \psi_{(2\Delta-d)} + \phi_{(2\Delta-d)}\right) + ...\right)$$

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- The asymptotic solution is determined by solving the Einstein equations perturbatively in $r$. This procedure does not depend on the spacetime signature and yields algebraic equations that can be solved to determine the asymptotic coefficients.
- The coefficients $g_{(2n)}$ with $2n < d$, $\phi_{(2k)}$ with $2k < 2\Delta - d$ and $h_{(d)}$, $\psi_{(2\Delta-d)}$ are determined locally in terms of $g_{(0)}, \phi_{(0)}$.
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# Holographic Renormalization

Renormalized correlators can now be obtained as follows:[de Haro, KS, Solodukhin (2000)]

1. Regulate the divergences by restricting the radial coordinate to have a finite range.
2. Evaluate the action on the asymptotic solution.
3. Subtract the infinite terms by adding suitable local covariant counterterms.
4. Compute the holographic 1-point functions in the **presence of sources**.

→ This leads to a precise relation between correlation functions and asymptotics

$$\langle T_{ij} \rangle = \frac{2}{\sqrt{g_{(0)}}} \frac{\delta S^{ren}_{SUGRA}}{\delta g^{ij}_{(0)}} = \frac{d}{16\pi G} [g_{(d)ij} + X^{(d)}_{ij}(g_{(0)})].$$

where $X^{(d)}_{ij}(g_{(0)})$ are local functions of $g_{(0)}$.

$$\langle O_\Delta \rangle = \frac{1}{\sqrt{g_{(0)}}} \frac{\delta S^{ren}_{SUGRA}}{\delta \phi_{(0)}} = (2\Delta - d)\phi_{(2\Delta-d)}$$

→ Correlators satisfy all expected Ward identities,

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where $X^{(d)}_{ij}(g_{(0)})$ are local functions of $g_{(0)}$.

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→ Correlators satisfy all expected Ward identities,

$$\nabla^i \langle T_{ij} \rangle = \langle O_\Delta \rangle \partial_j \phi_{(0)}, \qquad \langle T^i_i \rangle = -(d-\Delta)\phi_{(0)} \langle O_\Delta \rangle + \mathcal{A}$$

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Renormalized correlators can now be obtained as follows:[de Haro, KS, Solodukhin (2000)]

1. Regulate the divergences by restricting the radial coordinate to have a finite range.
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# Holographic Renormalization: higher point functions

5 Since the first variation of the on-shell action was performed in complete generality, one may obtain higher-point functions by differentiating the 1-point functions w.r.t. sources and then set the sources to zero

$$\langle O_\Delta(x_1) O_\Delta(x_2) \cdots O_\Delta(x_n) \rangle \sim \left. \frac{\delta^{(n-1)} \phi_{(2\Delta-d)}(x_1)}{\delta\phi_{(0)}(x_2) \cdots \delta\phi_{(0)}(x_n)} \right|_{\phi_{(0)}=0}$$

6 Thus to solve the theory we need to know $\phi_{(2\Delta-d)}, g_{(d)}$ as a function of $\phi_{(0)}, g_{(0)}$.

→ In absence of more powerful techniques we proceed perturbatively: 2-point functions are obtained by solving linearized fluctuations, 3-point functions by solving quadratic fluctuations etc.

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# Radial Hamiltonian formalism

- The method of holographic renormalization used so far is conceptually simple, but computationally inefficient as it does not exploit the underlying conformal structure.
- For most explicit computations, it is better to use the radial Hamiltonian formalism, a Hamiltonian formulation in which the radius plays the role of time.
- One relates the regularized holographic 1-point of an operator $\mathcal{O}_\Phi$ to the radial canonical momentum $\pi_\Phi$ of the corresponding bulk field $\Phi$ [de Boer,Verlinde[2]], [Papadimitriou, KS].

$$\delta S = \int dr \left( \frac{\partial L}{\partial \Phi} - \partial_r \frac{\partial L}{\partial(\partial_r \Phi)} \right) \delta\Phi + \left[ \frac{\partial L}{\partial(\partial_r \Phi)} \delta\Phi \right]_r , \qquad L \equiv \int d^d x \sqrt{G} \mathcal{L}$$

$$\Rightarrow \frac{\delta S_{on-shell}}{\delta \Phi} = \frac{\partial L}{\partial(\partial_r \Phi)} \equiv \pi_\Phi$$

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# Radial Hamiltonian formalism: renormalization [Papadimitriou, KS (2004)]

One still has to renormalize ....

- A fundamental property of asymptotically locally AdS spacetimes is that scale transformations are part of the asymptotic symmetries and therefore every covariant quantity can be decomposed into a sum of terms each having a definite scaling.
- Thus the canonical momenta of a field dual to a dimension $k$ operator are asymptotically expanded as

$$\pi^k = \pi^k_{(d-k)} + \cdots + \pi^k_{(k)} + \tilde{\pi}^k_{(k)} \log r + \cdots$$

  with each coefficient $\pi^k_{(n)}$ having weight $n$.
- Each coefficient can be expressed (non-linearly) in terms of the asymptotic expansions, but the holographic 1-point functions are more naturally expressed in terms of the coefficients of $\pi^k$,

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# Lorentzian Issues

Let us summarize the special issues that arise in the Lorentzian set up:

1. In the Lorentzian case one has to specify initial and final conditions as well $\phi_\pm$. So the on-shell action, $S_{onshell}[\phi_{(0)}, \phi_\pm]$, depends not only $\phi_{(0)}$ but also of $\phi_\pm$.
2. The variation of the on-shell supergravity action appears to pick up additional contributions from $t = \pm\infty$,

$$\delta S_{onshell} = [\pi_r \delta\Phi]_r + [\pi_t \delta\Phi]_{t=\infty} - [\pi_t \delta\Phi]_{t=-\infty}$$

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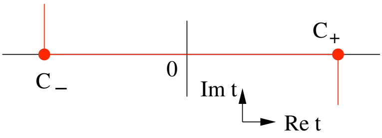
# QFT interlude

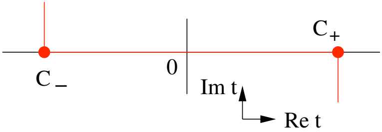
Consider the QFT path integral

$$\int_{\Psi(\vec{x},t=\pm T)=\psi_\pm(\vec{x})} [\mathcal{D}\Psi] e^{iS[\Psi]}$$

This computes the transition amplitude $\langle \psi_+(\vec{x}), T | \psi_-(\vec{x}), -T \rangle$. To compute vacuum-to-vacuum amplitudes we multiply with the wavefunctions $\langle \psi_-(\vec{x}), -T | 0 \rangle$, $\langle 0 | \psi_+(\vec{x}), T \rangle$ and integrate over $\psi_\pm$. The insertions of these wavefunctions is equivalent to extending the fields in the path integral to live along the red contour in the complex time plane:



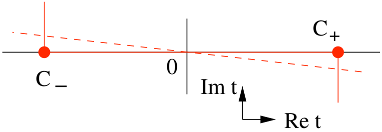
# Remarks




- The infinite vertical segments represent the wavefunctions $\langle\psi_-(\vec{x}), -T|0\rangle$, $\langle 0|\psi_+(\vec{x}), T\rangle$ as Euclidean path integrals,

$$\langle\psi_-(\vec{x}), -T|0\rangle = \lim_{\beta\to\infty}\langle\psi_-(\vec{x}), -T|e^{-\beta H}|\psi\rangle$$

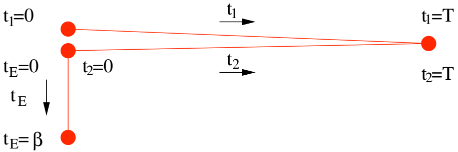
- These wavefunctions are ultimately lead to $i\epsilon$ factors in the Feynman propagator.



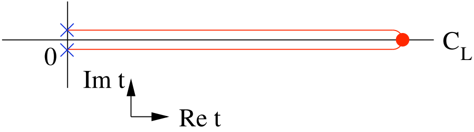
# Remarks

Correlators in non-trivial states, thermal ensembles etc. can be obtained by using different time contours. E.g.

- the real-time real contror is




- the in-in contour, used to calculate correlators $\langle in|O\cdots O|in\rangle$, is



# Lorentzian prescription

The holographic prescription is now to use "piece-wise" holography:

- Real segments are associated with Lorentzian solutions,
- Imaginary segments are associated with Euclidean solutions,
- Solutions are matched at the corners.

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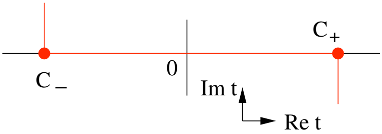
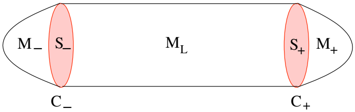
# Vacuum-to-vacuum amplitudes

To illustrate the prescription, consider vacuum-to-vacuum amplitudes for $CFT_d$. Corresponding to the time-contour

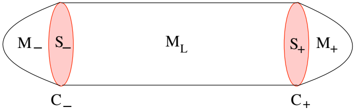



we consider the following solution:




Here $M_L$ is Lorentzian $AdS_{d+1}$ and the two caps $M_\pm$ are half of Euclidean $AdS_{d+1}$ spaces.

# Matching conditions




- Induced values of the bulk fields are continuous across $S_{\pm}$.
- The combined on-shell supergravity actions should be stationary w.r.t. variations with respect to $\phi_{\pm}$:

$$\frac{\delta}{\delta\phi_{\pm}}\Big(iI_L[\phi_{(0)},\phi_-,\phi_+] - I_E[\phi_{(0,-)},\phi_-] - I_E[\phi_{(0,+)},\phi_+]\Big) = 0$$

→ The matching conditions are equations for $\phi_{\pm}$.

→ Using the Hamilton-Jacobi relation the last condition becomes the standard Israel matching condition

$$i\pi_t|_{S_-} = \pi_\tau|_{S_-}, \qquad i\pi_t|_{S_+} = \pi_\tau|_{S_+}$$

# Fundamental bulk-boundary relation

The fundamental relation between bulk and boundary quantities reads

$$\langle 0|T\exp\left(i\int_{M_L} d^dx\sqrt{-g}\phi_{(0)}\mathcal{O}\right)|\mathbf{0}\rangle = \exp\left(iI_L[\phi_{(0)},\phi_-,\phi_+]-I_E[0,\phi_+]-I_E[\mathbf{0},\phi_-]\right)$$

- In this expesssion $\phi_\pm$ are the values determined via the matching conditions.
- We have set $\phi_{(0,-)}=\phi_{(0,+)}=0$ since we are interested in vacuum-to-vacuum correlators. One can consider non-trivial *in* and *out* states by turning on these sources.

→ This is a fully holographic prescription. Everything is determined by boundary conditions and regularity in the interior.

→ The Euclidean caps can be also be thought of as Hartle-Hawking wavefunctions.

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- We have set $\phi_{(0,-)} = \phi_{(0,+)} = 0$ since we are interested in vacuum-to-vacuum correlators. One can consider non-trivial *in* and *out* states by turning on these sources.

→ This is a fully holographic prescription. Everything is determined by boundary conditions and regularity in the interior.

→ The Euclidean caps can be also be thought of as Hartle-Hawking wavefunctions.

# Fundamental bulk-boundary relation

The fundamental relation between bulk and boundary quantities reads

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# Correlators

- Having set up the prescription one can verify that there are no additional ambiguities.
- A well known problem in the computation of 2-point functions is that the linearized field equations do not have a unique solution with Dirichlet boundary conditions.
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# Massive scalar field in $AdS_3$

As the simplest yet illustrative example we consider a free massive scalar field in $AdS_3$,

$$S = \frac{1}{2} \int d^3x \sqrt{|G|} (-\partial_\mu \Phi \partial^\mu \Phi - m^2 \Phi^2).$$

The dimension of $\mathcal{O}$ is $\Delta = 1 + \sqrt{1 + m^2} = 1 + l$ with $l \in \{0, 1, 2, \ldots\}$.
We want to solve

$$(\Box - m^2)\Phi(t, \phi, r) = 0$$

in the AdS$_3$ background,

$$ds^2 = -(r^2 + 1)dt^2 + \frac{dr^2}{r^2 + 1} + r^2 d\phi^2 \, ,$$

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# Solution of $(\Box - m^2)\Phi(t, \phi, r) = 0$

The solution to this equation is well known:

- Non-normalizable modes: $\Phi(t, \phi, r) \sim e^{-i\omega t + ik\phi} f(\omega, \pm k, r)$ with

$$f(\omega, k, r) \sim r^{l-1} + \ldots + r^{-l-1}\alpha(\omega, k, l)[\ln(r^2) + \beta(\omega, k, l)] + \ldots$$

- Normalizable modes: $\Phi(t, \phi, r) \sim e^{-i\omega^{\pm}_{nk} t + ik\phi} g(\omega_{nk}, |k|, r)$ with

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# Most general solution with prescribed boundary data

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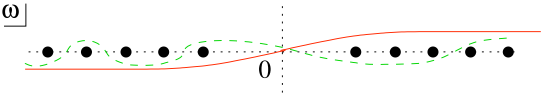
# Choice of contour




- We are free to specify any contour that avoids the poles, for example the green or the red contour. However the difference between any two contours is a sum over residues and the latter are exactly equal to normalizable modes.
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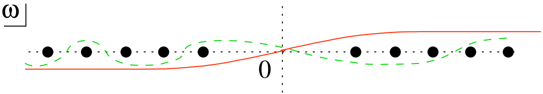


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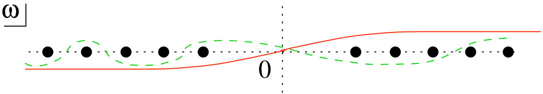


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# Euclidean solutions

We will now show that the matching conditions determine $c^{\pm}_{nk}$.

- Consider the solution on the 'initial cap', so on the space specified by the metric,

$$ds^2 = (r^2+1)d\tau^2 + \frac{dr^2}{r^2+1} + r^2 d\phi^2$$

with $-\infty < \tau \leq 0$, so that we have half of Euclidean AdS space.

- Had the bulk been the entire Euclidean AdS space, the Klein-Gordon equation would have a unique regular solution given boundary data. In particular, with zero sources the unique regular solution is identically equal to zero.
- In our case the sources are zero but we only consider half of the space, so solutions that would be excluded are now allowed because they are only singular at the other half of the space,

$$\Phi(\tau,\phi,r) = \sum_{n,k} d^{-}_{nk} e^{-\omega^{-}_{nk}\tau + ik\phi} g(\omega_{nk},|k|,r)\,,$$

with thus far arbitrary coefficients $d^{-}_{nk}$.

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# 2-point function

- Following our earlier discussion we can now extract the 2-point function from $r^{-l-1}$ term in the asymptotic expansion of full solutions. This leads to

$$\langle 0|T\mathcal{O}(t,\phi)\mathcal{O}(0,0)|0\rangle = \frac{l+1}{4\pi^2 i}\sum_k \int_{\mathbf{C}} d\omega e^{-i\omega t+ik\phi}\alpha(\omega,|k|,l)\beta(\omega,|k|,l).$$

with the contour $C$ being the same as for the bulk solution, which was completely fixed by the matching to the caps. This is the standard Feynman prescription leading to time-ordered correlators.

- Performing the $\omega$ integral leads to

$$\langle 0|T\mathcal{O}(t,\phi)\mathcal{O}(0,0)|0\rangle = \frac{C_l}{[\cos(t-i\epsilon t)-\cos(\phi)]^{\Delta}},$$

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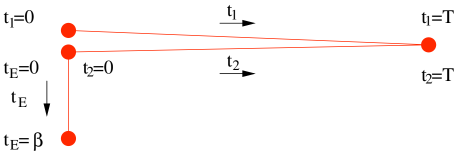
# Thermal 2-point function from Thermal AdS

A fairly straightforward extension is the computation of the thermal 2-point function using a scalar field in thermal AdS. The relevant time contour is




and this implies the following matching conditions

$$\Phi_1(0,\phi,r) = \Phi_E(\beta,\phi,r)$$
$$\partial_{t_1}\Phi_1(0,\phi,r) = i\partial_{t_E}\Phi_E(\beta,\phi,r)$$
$$\Phi_1(T,\phi,r) = \Phi_2(T,\phi,r)$$
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# Thermal 2-point function from thermal AdS

Carrying out the computation for both operators inserted in the first real segment leads to

$$\langle 0|T\mathcal{O}(t,\phi)\mathcal{O}(0,0)|0\rangle_{\beta} = \sum_{n\in\mathbb{Z}} \frac{C_l}{[\cos(t+in\beta)-\cos(\phi)]^{\Delta}} \,,$$

- This is a sum over images in imaginary time of the zero temperature result, as it should be, since thermal AdS is obtained by identification in the time direction of global AdS.
- It satisfies the Kubo-Martin-Schwinger (KMS) condition.
- Considering operators inserted on both real segments results in the $2\times 2$ matrix of Schwinger-Keldysh propagators.

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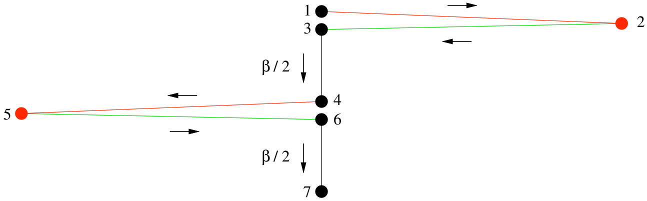
# Thermal 2-point function from the BTZ black hole

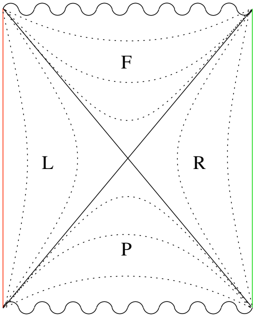
A more challenging example is the computation of the thermal propagator using a scalar field in the non-rotating massive BTZ black hole. It is more convenient to use the following thermal contour:

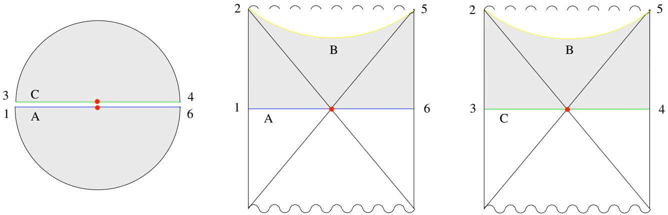



This a more convenient choice because it is easier to solve the various matching conditions. To fill in this contour we need two copies of half of the Lorentzian eternal BTZ and two copies of half of the Euclidean BTZ.

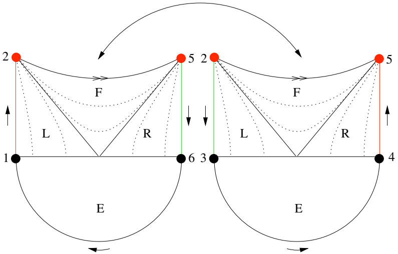
# Bulk solution corresponding to time contour

The Penrose diagram for the eternal BTZ black is






Left figure: $\phi = 0$ slice: $\tau = 0$ at points 1,3 and $\tau = \beta/2$ at points 4,6.



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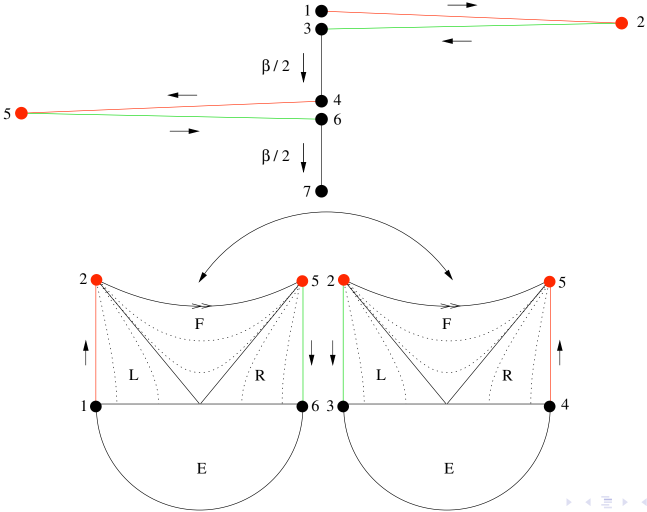

# 2-point function

- The 2-point function is

$$\langle T\mathcal{O}(t,\phi)\mathcal{O}(0,0)\rangle_{\beta} \sim \sum_{m\in\mathbb{Z}} \frac{1}{[\cosh(t)-\cosh(\phi+2\pi\sqrt{M}m)]^{l+1}}$$

  where $M$ is the mass of the BTZ black hole.

- This is also a sum over images reflecting the fact that the BTZ is a quotient of $AdS_3$.
- The result agrees with results in the literature (obtained using the fact that BTZ is the quotient of $AdS_3$) and obeys the KMS condition.
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# Concluding remarks

- We have outlined a connection of geometric flows with $3d$ QFT using holography.
- We have present a general prescription for holographic computation in real time.
  - The prescription amounts to "filling-in" the complex time contour with bulk solution: real segments with Lorentzian solutions and imaginary segments with Euclidean solutions.
  - This prescription fulfils all requirements described earlier: it allows for computation of $n$-point functions in any holographic QFT and in non-trivial states. It is fully holographic and all information is encoded in classical bulk dynamics.

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# On-going and future work

Using these techniques one would like to return to the $AdS_4$ case and

- compute the 2-point function of the stress energy using the linearized perturbations around the Schwarzschild solution.
- understand the implications of the connection with the geometric flows.
- in the case of Calabi flow/Robinson-Trautman spacetimes, compute higher point functions.

These and related computations are in progress ....