Gravitational instantons, Ricci flows and integrable structures

Marios Petropoulos

CPHT – Ecole Polytechnique – CNRS

Workshop on
Field Theory and Geometric Flows
LMU, Munich, November 2008

inspired from work with I. Bakas, F. Bourliot & D. Orlando

Highlights

Framework

Gravitational instantons have been investigated in the past with motivations similar to gauge instantons

- Potentially describe non-perturbative transitions in quantum gravity
- Provide real-time gravitational backgrounds by analytic continuation

Solving Einstein equations in general is a hard task – substantially simplified assuming self-duality

- Equations become *first order* – like self-dual Yang–Mills
- Often related to remarkable integrable systems

Homogeneity of spatial sections is another important assumption

- Borrowed from cosmology
- Allows for more specific and solvable ansätze

Homogeneous geometries are also at the heart of the Ricci-flow developments

- Naturally arise as building blocks of Hamilton's programme
- Lead – at least in 3D – to integrable Ricci-flow equations similar to the self-dual equations of Einstein's gravity

Ricci flows also describe the renormalization-group evolution of 2D sigma models – can mimic time evolution in string theory

Here

4D Euclidean gravity with Bianchi IX "spatial" sections

- Involves homogeneous spaces with *3-sphere* geometry
- Accounts for some celebrated geometries like *Eguchi–Hanson* or *Taub-NUT*
- Is related to closed FRW universes – upon real-time rotation

More specifically

- Review the self-duality analysis in the Bianchi IX class
- Recast the equations in terms of integrable systems (Lagrange or Darboux–Halphen) with particular emphasis to the issue of *fully anisotropic* solutions
- Make contact with Ricci-flow equations of homogeneous 3-sphere spaces

Highlights

Solving Einstein equations in 4D

Metric and torsionless connection one-form $\omega^a{}_b$ *and curvature two-form* $\mathcal{R}^a{}_b$ *in an orthonormal frame:* $\mathrm{d}s^2 = \delta_{ab}\theta^a\theta^b$

- Cartan structure equations: $\mathrm{d}\theta^a + \omega^a{}_b \wedge \theta^b = 0$, $\omega^a{}_b = -\omega^b{}_a$
- Riemann tensor: $\mathcal{R}^a{}_b = \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = \frac{1}{2} R^a{}_{bcd}\theta^c \wedge \theta^d$
- Bianchi identity: $\mathrm{d}\mathcal{R}^a{}_b + \omega^a{}_c \wedge \mathcal{R}^c{}_b - \mathcal{R}^a{}_c \wedge \omega^c{}_b = 0$
- Cyclic identity: $\mathcal{R}^a{}_b \wedge \theta^b = 0 \Leftrightarrow R^a{}_{bcd}\epsilon^{bcde} = 0$
- Dual Riemann: $\tilde{\mathcal{R}}^a{}_b = \frac{1}{2}\epsilon^a{}_{bc}{}^d\mathcal{R}^c{}_d = \frac{1}{2}\tilde{R}^a{}_{bcd}\theta^c \wedge \theta^d \Leftrightarrow$ $R_{abcd} = \frac{1}{2}\epsilon_{abef}\tilde{R}^{ef}{}_{cd}$
- Ricci tensor: $R^a{}_b = R^{ac}{}_{bc} = \frac{1}{2}\epsilon^{acef}\tilde{R}_{bcef}$

Gravitational instantons: non-singular Euclidean-signature solutions with finite action and preferably localised (asymptotically locally flat) – here in vacuum

- Vacuum equations: $R_{ab} = 0 \Leftrightarrow \tilde{\mathcal{R}}^c{}_d \wedge \theta^d = 0$
- Yang–Mills paradigm: (anti)self-dual curvature solutions
  - $\mathcal{R}^a{}_b = \pm\tilde{\mathcal{R}}^a{}_b$ & Cyclic id. $\Rightarrow$ vacuum eqs.
  - $\mathcal{R}^a{}_b = \pm\tilde{\mathcal{R}}^a{}_b \Leftrightarrow \omega^a{}_b = \pm\omega^a{}_b$ up to local $O(4)$ frame rotations $(\omega^a{}_b = \frac{1}{2}\epsilon^a{}_{bc}{}^d\omega^c{}_d)$

  Imposing self-duality to the connection leads to 1st order equations – exhaustive if *all* gauge transformations are considered
- Note: self-duality of the Weyl tensor accounts for non-vacuum equations (e.g. Fubini–Study)

The Bianchi IX metrics [Bianchi 1897, Taub, 1951]

The 4D Bianchi IX metrics admit $SU(2)$-homogeneous 3D "spatial" sections

- Metric: $\mathrm{d}s^2 = (\gamma_1\gamma_2\gamma_3)^2\,\mathrm{d}T^2 + (\gamma_i\sigma^i)^2$
  - $\{\sigma^i, i = 1, 2, 3\}$: left-invariant Maurer–Cartan forms of $SU(2)$

$$\begin{cases} \sigma^1 = \sin\vartheta\sin\psi\mathrm{d}\varphi + \cos\psi\mathrm{d}\vartheta \\ \sigma^2 = \sin\vartheta\cos\psi\mathrm{d}\varphi - \sin\psi\mathrm{d}\vartheta \\ \sigma^3 = \cos\vartheta\mathrm{d}\varphi + \mathrm{d}\psi \end{cases}$$

  with $0 \le \vartheta \le \pi, 0 \le \varphi \le 2\pi, 0 \le \psi \le 4\pi$ (Euler angles)
  - $\mathrm{d}\sigma^i + \frac{1}{2}\epsilon^i{}_{jk}\sigma^j \wedge \sigma^k = 0$
- Convenient parameterization: $\Omega^i = \gamma_j\gamma_k$

$$\mathrm{d}s^2 = \Omega_1\Omega_2\Omega_3\mathrm{d}T^2 + \frac{\Omega^2\Omega^3}{\Omega^1}\left(\sigma^1\right)^2 + \frac{\Omega^3\Omega^1}{\Omega^2}\left(\sigma^2\right)^2 + \frac{\Omega^1\Omega^2}{\Omega^3}\left(\sigma^3\right)^2$$

The functions $\Omega^i(T)$ define the solution

- Must be *positive*
- Zeros of a single Ω are *genuine* curvature singularities with

$$ds^2 \approx d\tau^2 + \frac{\Xi}{\tau^{2/3}}\left(\sigma^1\right)^2 + \Upsilon\tau^{2/3}\left(\left(\sigma^2\right)^2 + \left(\sigma^3\right)^2\right)$$

  (here for linearly vanishing Ω^1 with Ξ, Υ constants)
- Zeros of 2 Ωs or poles of some Ω can be of different nature, *potentially* removable coordinate singularities
- If *two* Ωs are equal the isometry group is promoted to $SU(2) \times U(1)$ (axial symmetry)
- Full isotropy requires Ωs be *all* equal

The general self-duality equations

Imposing self-duality on the connection within the present metric ansatz leads to the Lagrange system (special version of the Euler top)

$$\dot{\Omega}^1 = -\Omega^2\Omega^3, \quad \dot{\Omega}^2 = -\Omega^3\Omega^1, \quad \dot{\Omega}^3 = -\Omega^1\Omega^2$$

Imposing self-duality on the curvature leads to one more possibility: the Darboux–Halphen system

$$\begin{cases} \dot{\Omega}^1 = \Omega^2\Omega^3 - \Omega^1\left(\Omega^2 + \Omega^3\right) \\ \dot{\Omega}^2 = \Omega^3\Omega^1 - \Omega^2\left(\Omega^3 + \Omega^1\right) \\ \dot{\Omega}^3 = \Omega^1\Omega^2 - \Omega^3\left(\Omega^1 + \Omega^2\right) \end{cases}$$

The corresponding connection is self-dual up to an appropriate $O(4)$ gauge transformation

These equations exhaust all possibilities (similarly obtained by requiring vacuum-Kähler structure) [Gibbons, Pope, 1979]

Highlights

19th century integrable systems

Search for integral lines of a vector field: $\frac{\mathrm{d}\Omega^i}{\mathrm{d}T} = V^i(\Omega)$ *with* $V^i = -D^i\Phi$ *(*$D^i = K^{ij}\partial_{\Omega^j}$*,* $\{\Omega^j\} \equiv$ *coordinates)*

- Lagrange:
  - Extension of rigid-body motion
  - Solvable *à la* Jacobi with elliptic functions
- Darboux–Halphen:
  - Darboux's work on "triply orthogonal surfaces" [Darboux, 1878]
  - Solved by Halphen using modular forms [Halphen, 1881]

2Oth century physics

Typical systems appearing in general self-dual Yang-Mills reductions since the late '70s – all integrable systems were even thought to be SDYM reductions [Ward, 1985]

- Darboux–Halphen system studied a lot over the recent years [Takhtajan, 1992; Maciejewski, Strelcyn, 1995; Chakravarty, Halburd, Ablowitz, 2003]
- The relation of Darboux–Halphen with Bianchi IX is recent (90s) with no use of its power in constructing Bianchi IX solutions or applying them e.g. in the study of scattering of $SU(2)$ BPS monopoles [Gibbons, Pope, 1979; Manton, 1981; Atiyah, Hitchin, 1985; Gibbons, Manton, 1986]
- Darboux–Halphen system is also the Ricci flow describing the evolution of $SU(2)$-homogeneous 3D geometries [Sfetsos, unpubl.; Bakas, Orlando, Petropoulos, 2006]

Highlights

The Ricci flow describes the evolution of a geometry governed by the equation

$$\frac{\mathrm{d}g_{ij}}{\mathrm{d}t} = -R_{ij}$$

- Major role in 3D: Poincaré, Thurston, Hamilton, ...
- Homogeneous 3D manifolds: all 9 classes studied using e.g. asymptotic methods (smooth convergence towards an Einstein geometry or singular behaviour) [Isenberg, Jackson, 1992; Chow, Knopf, 2004]
- The case of the 3-spheres: $\mathrm{d}s^2 = \sum_i \gamma_i(t) \left(\sigma^i\right)^2$
  - define $\mathrm{d}t = \gamma_1\gamma_2\gamma_3 \mathrm{d}T, \Omega^i = \gamma_j\gamma_k$ [Cvetič, Gibbons, Lü, Pope, 2001; Bakas, Orlando, Petropoulos, 2006]

$$\text{Ricci Flow} \equiv \text{Self}-\text{dual Bianchi IX}$$

  - asymptotic behaviour: $\gamma^i \sim 1/T \equiv$ round sphere (Einstein) of vanishing radius (IR fixed point in the $SU(2)$ sigma model)

Highlights

Lagrange and Darboux–Halphen systems differ drastically

- Lagrange: algebraically integrable – possesses polynomial first integrals
- Darboux–Halphen:
  - not algebraically integrable in the anisotropic case – potential difficulties in the initial-value problem of the Ricci flow
  - requires the use of modular forms – solution-generating pattern based on $PSL(2, \mathbb{C})$

Self-dual Bianchi IX: hard to generalize Taub-NUT – our aim – (Darboux–Halphen) – easy to generalize Eguchi–Hanson (Lagrange)

[Eguchi, Hanson, April 1978; Belinskii, Gibbons, Page, Pope, June, 1978]

Lagrange system

Jacobi resolution → expressions in terms of elliptic functions and two parameters: $l_1 \equiv \left(\Omega^1\right)^2 - \left(\Omega^2\right)^2$, $l_2 \equiv \left(\Omega^1\right)^2 - \left(\Omega^3\right)^2$ *– here we use a better radial coordinate* *[Belinskii, Gibbons, Page, Pope, June, 1978]*

$$
\mathrm{d}s^2 = \frac{\mathrm{d}\rho^2}{\sqrt{(1-16\xi_1/\rho^4)(1-16\xi_2/\rho^4)(1-16\xi_3/\rho^4)}} + \frac{\rho^2}{4}\sqrt{\frac{(1-16\xi_2/\rho^4)(1-16\xi_3/\rho^4)}{(1-16\xi_1/\rho^4)}}\left(\sigma^1\right)^2
+ \frac{\rho^2}{4}\sqrt{\frac{(1-16\xi_3/\rho^4)(1-16\xi_1/\rho^4)}{(1-16\xi_2/\rho^4)}}\left(\sigma^2\right)^2 + \frac{\rho^2}{4}\sqrt{\frac{(1-16\xi_1/\rho^4)(1-16\xi_2/\rho^4)}{(1-16\xi_3/\rho^4)}}\left(\sigma^3\right)^2
$$

Without loss of generality: $\xi_3 = 0$

- If $\xi_1 \neq \xi_2 \neq 0 \to SU(2)$ isometry $(\Omega_1 \neq \Omega_2 \neq \Omega_3)$
- If $\xi_2 = \xi_3 = 0$ or $\xi_1 = \xi_2 \neq 0 \to SU(2) \times U(1)$ isometry: Eguchi–Hanson I & II (2 Ωs equal)
- If $\xi_1 = \xi_2 = \xi_3 \to [SU(2)]^2$ isometry: flat space (Ωs equal)

Generic case: $\xi_1 \neq \xi_2 \neq 0$

- Space asymptotically Euclidean
- Genuine non-removable curvature singularities at $\rho^4 = 0, 16\xi_1, 16\xi_2$ (vanishing of one Ω)

Eguchi–Hanson I: $\xi_1 = \xi_2 = 0$

Same singular behaviour at $\rho^4 = 16\xi_3$

Eguchi–Hanson II: $\xi_1 = \xi_2 = \xi \neq 0$

- Regular curvature at $\rho^4 = 16\xi$, singular at $\rho = 0$
- The metric exhibits a singularity at $\rho^4 = 16\xi$

$$
\begin{aligned}
\mathrm{d}s^2 &\approx \mathrm{d}\tau^2 + \sqrt{\xi}\left(\left(\sigma^1\right)^2 + \left(\sigma^2\right)^2\right) + \tau^2\left(\sigma^3\right)^2 \\
&= \mathrm{d}\tau^2 + \tau^2\left(\mathrm{d}\psi + \cos\vartheta\mathrm{d}\varphi\right)^2 + \sqrt{\xi}\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2\right)
\end{aligned}
$$

$\tau = 0$: conical singularity $\rightarrow$ harmless, removable:
$0 \leq \psi \leq 4\pi \rightarrow 0 \leq \psi \leq 2\pi$

- locally the manifold is $\mathbb{R}^2 \times S^2$ with $\mathbb{R}^2 \rightarrow$ point on S^2 as $\tau \rightarrow 0$: bolt
- space asymptotically *locally* Euclidean: S^3 at infinity $\rightarrow S^3/\mathbb{Z}_2$

Lagrange's Ωs have zeros: potential curvature singularities

- Regularity (up to conical removable defects) requires the confluence of the zeros
- Achieved thanks to symmetry enhancement ($SU(2) \rightarrow SU(2) \times U(1)$)

Darboux–Halphen system

The resolution and the nature of the solutions strongly depends on whether Ωs are all different or not – much harder when all different → "Darboux–Halphen" Bianchi IX case was not studied in general

[e.g. in Gibbons, Pope, 1979]

- Cyclic invariance and first order → Ωs never cross
- $0 < \Omega^1_0 < \Omega^2_0 < \Omega^3_0 \Rightarrow 0 < \Omega^1 < \Omega^2 < \Omega^3 \; \forall T > T_0$
  proof: assume $T_* > T_0$ with $\Omega^1(T_*) \equiv \Omega^1_* = 0 < \Omega^2_* < \Omega^3_*$
  equations → $\dot{\Omega}^2_* = \dot{\Omega}^3_* = -\Omega^2_*\Omega^3_* < 0$ and $\dot{\Omega}^1_* = \Omega^2_*\Omega^3_* > 0$
  so Ω_1 vanishes while being increasing from negative values

Ricci flow: positive initial conditions guarantee later positivity
Bianchi IX: zeros are not excluded

Consider the system in the complex plane: $\omega^i(z)$, $z \in \mathbb{C}$ *[see e.g. Takhtajan, 1992]*

- ωs are regular, univalued, holomorphic in a region with movable boundary (dense set of essential singularities) – the location of this boundary accurately determines the solution
- If ω^i is a solution

$$\tilde{\omega}^i(z) = \frac{1}{(cz+d)^2}\omega^i\left(\frac{az+b}{cz+d}\right) + \frac{c}{cz+d}$$

$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in PSL(2, \mathbb{C})$ is another solution with singularity boundary moved according to $z \longrightarrow {}^{az+b}/_{cz+d}$ [Halphen, 1881]

Even more in the fully anisotropic case – our main motivation here

- No algebraic first integrals (no arbitrary accuracy in the relation solution/initial conditions)
- Solution reads: $\omega^i(z) = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}z}\log\mathcal{E}^i(z)$
  - $\mathcal{E}^i(z)$ triplet of *weight-2 modular forms* of $\Gamma(2) \subset PSL(2,\mathbb{C})$
  - if λ is a solution of Schwartz's equation

$$\mathcal{E}^1 = \frac{\mathrm{d}\lambda/\mathrm{d}z}{\lambda}, \quad \mathcal{E}^2 = \frac{\mathrm{d}\lambda/\mathrm{d}z}{\lambda-1}, \quad \mathcal{E}^3 = \frac{\mathrm{d}\lambda/\mathrm{d}z}{\lambda(\lambda-1)}$$

- Real solutions: $\Omega^\ell(T) = i\omega^\ell(iT) = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}T}\log\mathcal{E}^\ell(iT)$

Modular properties allow to relate large-T and small-T

- $\Omega^{1,2,3}(T) = -\frac{1}{T^2}\Omega^{2,1,3}\left(\frac{1}{T}\right) + \frac{1}{T}$
- For *finite* $\Omega_0^i \equiv \Omega^i(0)$

$$\Omega^i = \frac{1}{T} + \text{subleading } at\ T \text{ large}$$

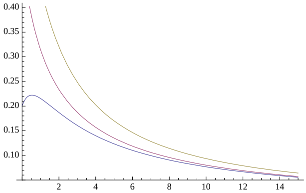

Figure: Generic behaviour for $0 < \Omega_0^1 < \Omega_0^2 < \Omega_0^3$

Consequences for the Ricci flow: universal convergence towards the round sphere

$$\mathrm{d}s^2 \approx \frac{1}{\sqrt{T}} \left(\left(\sigma^1\right)^2 + \left(\sigma^2\right)^2 + \left(\sigma^3\right)^2 \right)$$

Consequences for the Bianchi IX: existence of a "nut" at large T

$$\begin{aligned} \mathrm{d}s^2 &\approx \tfrac{\mathrm{d}T^2}{T^3} + \tfrac{1}{T} \left(\left(\sigma^1\right)^2 + \left(\sigma^2\right)^2 + \left(\sigma^3\right)^2 \right) \\ &= \mathrm{d}\tau^2 + \tfrac{\tau^2}{4} \left(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\varphi^2 + \left(\mathrm{d}\psi + \cos\vartheta \mathrm{d}\varphi\right)^2 \right) \end{aligned}$$

Nut singularity: harmless coordinate singularity $\longrightarrow$ space is $\mathbb{R}^4$ around $\tau = 0$

The original Halphen solution corresponds in this language to $\lambda = \vartheta_2^4/\vartheta_3^4$

$$\begin{cases} \omega_{\mathrm{H}}^1 = \frac{\pi}{6i}\left(E_2 - \vartheta_2^4 - \vartheta_3^4\right) \\ \omega_{\mathrm{H}}^2 = \frac{\pi}{6i}\left(E_2 + \vartheta_3^4 + \vartheta_4^4\right) \\ \omega_{\mathrm{H}}^3 = \frac{\pi}{6i}\left(E_2 + \vartheta_2^4 - \vartheta_4^4\right) \end{cases}$$

$E_2 = \frac{12}{i\pi}\frac{\mathrm{d}}{\mathrm{d}z}\log\eta$ is the pseudo-modular form of weight 2

Solution found by Atiyah and Hitchin in 1985 as the Bianchi IX solution relevant for describing the configuration space of two slowly moving BPS $SU(2)$ *Yang–Mills–Higgs monopoles*

Peculiar but non-generic solution: defined for $T > 0$ with $\Omega^1_{\mathrm{H}} < 0 < \Omega^3_{\mathrm{H}} < \Omega^2_{\mathrm{H}}$ – bad for the Ricci flow

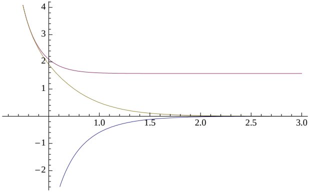

Figure: Halphen original solution (also Atiyah–Hitchin)

- At $T \to \infty$: $SU(2) \to SU(2) \times U(1)$ exponentially fast $\Omega^{1,3}_{\mathrm{H}} \approx \mp 4\pi \exp -\pi T, \Omega^2_{\mathrm{H}} \approx {}^{\pi}/_{2} + 4\pi \exp -2\pi T$
- Pole at $T = 0$: $\Omega^1_{\mathrm{H}} \approx -{}^{\pi}/_{2T^2}, \Omega^{2,3}_{\mathrm{H}} \approx {}^{1}/_{T}$

Led to the lore that anisotropic Bianchi IX solutions always have some negative Ωs – not true according to the previous analysis

$PSL(2,\mathbb{C})$ transformations can move the pole away from zero and restore positivity – at least in a range of T

$$\Omega^i(T) = \frac{1}{(CT+D)^2}\Omega^i_{\mathrm{H}}\left(\tfrac{AT+B}{CT+D}\right) + \frac{C}{CT+D}$$

defined for $\frac{AT+B}{CT+D} > 0$ with $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in SL(2,\mathbb{R})$

- Assume $\lim_{T\to\infty} {}^{AT+B}/_{CT+D} = {}^{A}/_{C} > 0 \Rightarrow \Omega^i(T) \approx {}^{1}/_{T}$: the geometry has a nut
- Two poles: $T_\infty = -{}^{D}/_{C}$ and $T_0 = -{}^{B}/_{A}$ with $T_\infty < T_0$
  - $\Omega^1 \approx -\frac{\pi}{2A^2}\frac{1}{(T-T_0)^2} < 0 \Rightarrow$ metric flips sign
  - $\Omega^{2,3} \approx \frac{1}{T-T_0}$
  - $T = T_0$ is a Taubian infinity: S^1 fiber over $\mathbb{R}^3$
    $\mathrm{d}s^2 \approx -\mathrm{d}\tau^2 - \tau^2\left(\left(\sigma^2\right)^2 + \left(\sigma^3\right)^2\right) - \frac{2A^2}{\pi}\left(\sigma^1\right)^2$

$\Omega^i(T)$ are defined for $T < T_\infty$ or $T_0 < T$ with reflected behaviour

There is always a T_* s.t. $T_0 < T_* < \infty$ with $\Omega^1_* = 0 < \Omega^3_* < \Omega^2_*$

- Good for the Ricci flow as long as $T > T_*$
- Bad for the Bianchi IX: curvature singularity at $T = T_*$

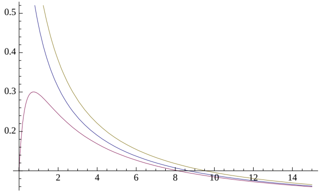

Figure: Generic solution for $T_0 < T$ and $0 < \Omega^1 < \Omega^3 < \Omega^2$

This exhausts the analysis for the anisotropic Bianchi IX solutions: they all possess a nut but the would-be Taubian infinity ($T = T_0$) is behind a genuine singularity ($T = T_*$)

Extra symmetry: Darboux–Halphen system with $\Omega^1 = \Omega^2 \neq \Omega^3$

- algebraic solutions

$$\Omega^{1,2} = \frac{1}{T - T_0} \quad \Omega^3 = \frac{T - T_*}{(T - T_0)^2}$$

- T_0: simple pole for $\Omega^{1,2}$, double for Ω^3
- T_*: zero for Ω^3
- Choose $T_* < T_0$: the zero is pushed behind the pole

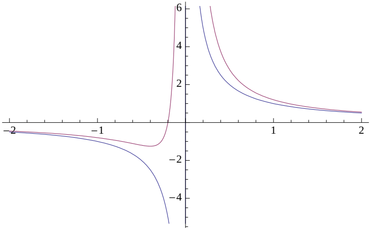

Figure: Generic solution $\Omega^1 = \Omega^2 < \Omega^3$

The axisymmetric Darboux–Halphen solutions correspond to the celebrated self-dual Taub-NUT Bianchi IX geometries

$$
\begin{aligned}
\mathrm{d}s^2 &= \frac{T - T_*}{(T - T_0)^4}\mathrm{d}T^2 \\
&\quad + \frac{T - T_*}{(T - T_0)^2}\left(\left(\sigma^1\right)^2 + \left(\sigma^2\right)^2\right) + \frac{1}{T - T_*}\left(\sigma^3\right)^2 \\
&= \frac{r+m}{r-m}\frac{\mathrm{d}r^2}{4} + m^2\frac{r-m}{r+m}\left(\sigma^3\right)^2 + \frac{1}{4}\left(r^2 - m^2\right)\left(\sigma^2\right)^2
\end{aligned}
$$

- $T - T_0 = {}^2/_{m(r-m)}$ and $T_0 - T_* = {}^1/_{m^2}$
- Non-singular self-dual Taub-NUT: $m^2 > 0$
  - nut at $r = m \Leftrightarrow T \to \infty$
  - Taubian infinity at $r \to \infty \Leftrightarrow T \to T_0$ (pole of the Ωs)
    $\mathrm{d}s^2 \approx \mathrm{d}\tau^2 + \tau^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2\right) + m^2\left(\mathrm{d}\psi + \cos\vartheta\mathrm{d}\varphi\right)^2$

Highlights

Conclusions

Relation between general Bianchi IX self-dual geometries and recreational 19th century integrable structures

- Lagrange system: "Eguchi–Hanson family"
- Darboux–Halphen system: "Taub-NUT family"

Our aim: study general anisotropic gravitational instantons of the Taub-NUT family – non-algebraic solutions

- Involve modular forms: powerful tool for unravelling explicitly the general structure of the solutions – not appreciated sofar
- Modular structure: the solutions *are* positive and well-behaved at large T (nut) but always have a zero which translates into an unavoidable curvature singularity before the would-be Taubian infinity

Some relevant questions

- The original Halphen solution is non-generic – is the corresponding Bianchi IX stable under perturbations?
- What is the geometric origin/interpretation of the $SL(2, \mathbb{C})$ acting as a solution-generator?

Confirmation of the expectation: good gravitational instantons exist only when symmetry is enhanced to $SU(2) \times U(1)$

- Lagrange system: Eguchi–Hanson II
- Darboux–Halphen system: self-dual Taub-NUT with $m^2 > 0$

The non-algebraic positive solutions are however good solutions of the anisotropic Ricci-flow equations for $SU(2)$-homogeneous geometries

Self-dual Bianchi IX equations match $SU(2)$ Ricci-flow equations (for Darboux–Halphen) – is this a peculiarity of $SU(2)$?

Many relevant issues in string theory: Ricci flow $\equiv$ RG flow

- Stabilisation to finite-radius round S^3 requires torsion (WZW)
- Squashed S^3 requires torsion *and* magnetic flux [Kiritsis, Kounnas, 1994; Israël, Kounnas, Orlando, Petropoulos, 2005]
- Full 4D Eguchi–Hanson requires more sophisticated tools [Carlevaro, Israël, Petropoulos, 2008]
- More generally: relation time evolution/Liouville field/RG flow?