Logarithmic Sobolev inequality on gradient solitons and Perelman's reduced volume

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Munich

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  For any $\sigma > 0$ and any function $u \geq 0$ with $u^{1/2} \in W^{1,2}(\mathbb{R}^n)$, write $u = \frac{e^{-f}}{(4\pi\sigma)^{n/2}}$ with $\int_M u = 1$, then

$$\int_{\mathbb{R}^n} \left(\sigma|\nabla f|^2 + f - n\right) u \, dx \geq 0$$

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- There exists a proof via the heat equation and the entropy formula.

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(N, 2004, JGA) Then

$$\frac{d\mathcal{W}}{dt} = -2t \int_M \left(|\nabla_i \nabla_j \varphi - \frac{1}{2t} g_{ij}|^2 + R_{ij}\varphi_i\varphi_j \right) v \, d\mu.$$

In particular, if M has nonnegative Ricci curvature, $\mathcal{W}(v,t)$ is monotone decreasing along the heat equation.

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- The above can be made rigorous into a proof. To make sense of the convergence, one has to do a time dependent scaling.
- On the other hand, the proof can be not be made to work on general manifolds.
- The result fails to hold on a nonflat complete Riemannian manifold with nonnegative Ricci curvature. (Due to Bakry-Concordet-Ledoux).

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- 2) Any Einstein metric.
- 3) $(\mathbb{R}^n, g)$ with $f = \frac{1}{4}|x|^2$.
- Non trivial examples are constructed by Koiso, Cao, Feldman-Ilamenen-Knopf, B. Yang, Dancer-Wang etc. Besides that it is a self-similar solution to Ricci flow, the gradient soliton arises in the singularity analysis of Ricci flow.

Basic facts and identities

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- Associated to the metric g and the *potential function* f, there exists a family of metrics $g(\eta)$, a solution to Ricci flow $\frac{\partial}{\partial \eta} g(\eta) = -2\,\mathrm{Ric}(g(\eta))$, with the property that $g(0) = g$, the original metric, and a family of diffeomorphisms $\phi(\eta)$, which is generated by the vector field $X = \frac{1}{\tau} \nabla f$, such that $\phi(0) = \mathrm{id}$ and $g(\eta) = \tau(\eta)\phi^*(\eta) g$ with $\tau(\eta) = 1 - \eta$, as well as $f(x, \eta) = \phi^*(\eta) f(x)$. Namely it gives a self-similar (shrinking) family of metrics which is a solution to the Ricci flow.

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and

$$S + |\nabla f|^2 - \frac{f}{\tau} = \frac{\mu_s(\tau)}{\tau}.$$

Estimates on f and volume

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- Lemma
  Assume that the scalar curvature $S \geq -A$ for some $A > 0$. Let $r(x)$ be the distance function to a fixed point $o \in M$ with respect to $g(\eta)$ metric. Then there exists $\delta_0 = \delta_0(M, f, \tau)$ and positive constants C_2, C_3 depending on M, f, τ such that for any $\delta \leq \delta_0$,

$$f(x) \geq \delta r^2(x) \tag{0.1}$$

  for $r(x) \geq C_2$ and

$$f(x) \leq C_3 r^2(x), \qquad |\nabla f|(x) \leq C_3(r(x) + 1) \tag{0.2}$$

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  - The case $|\,\mathrm{Ric}\,|$ is bounded is trivial. The case $\mathrm{Ric} \geq 0$ without the upper bound requires some effort. The more general case $S \geq -A$ was due to Fang-Man-Zhang.

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- Proposition

  Let (M, g) *be a nonflat gradient shrinking soliton with* $\mathrm{Ric} \geq 0$*.*
  Then $\mathcal{V}(M) = 0$*, where* $\mathcal{V}(M) := \lim_{r\to\infty} \frac{V(o,r)}{r^n}$.

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  Proposition

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$$V(o, R+1) \leq V(o, R_0+1) e^{\frac{C(n,\delta)}{R_0}} \left(\frac{R-a}{R_0-a}\right)^{n-\delta}. \tag{0.3}$$

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Theorem

Assume that (M, g, f) is a gradient shrinking soliton so that the scalar curvature $S \geq -A$ for some $A > 0$. Then there exists a geometric invariant (under the isometry) μ_s which depends only on the value of f and S at the minimum point of f under the normalization $\frac{1}{(4\pi\tau)^{n/2}} \int_M e^{-f} d\Gamma_\tau = 1$, and is independent of τ, such that for $\tau > 0$ and any compact supported smooth function $\rho = \frac{e^{-\psi}}{(4\pi\tau)^{n/2}}$ with $\int_M \rho \, d\Gamma_\tau = 1$, where $d\Gamma_\tau$ is the volume element of g^τ, we have that

$$\int_M \left(\tau(|\nabla\psi|^2_\tau + S(\cdot, \tau)) + \psi - n\right) \rho \, d\Gamma_\tau \geq -\mu_s.$$

Moreover, for this geometric constant μ_s the above inequality is sharp. In the case that $|R_{ijkl}|$ is bounded, $\mu_s \geq 0$.

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- Let $v = \log \rho + f + \frac{n}{2} \log(4\pi\tau)$, called the pointwise relative entropy.

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$$\begin{aligned} \frac{d}{dt} N_{\rho,f}(t) &= \int_M v_t \rho + v \rho_t \, d\mu \\ &= \int_M (\Delta v)\rho + \langle \nabla v, \nabla \rho \rangle + v \operatorname{div}(\rho \nabla v) \\ &= -\int_M |\nabla v|^2 \rho \, d\mu. \end{aligned} \tag{0.4}$$

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- There is a Bochner type formula

$$\left(\frac{\partial}{\partial t} - \Delta\right)|\nabla v|^2 = -2v_{ij}^2 + 2\langle\nabla(\langle\nabla v, \nabla \log \rho\rangle), \nabla v\rangle - 2R_{ij}v_iv_j. \tag{0.5}$$

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$$\begin{aligned} \frac{d}{dt} F_{\rho,f}(t) &= \int_M \left(-2v_{ij}^2 - 2R_{ij} v_i v_j\right) \rho \, d\mu \\ &\quad + \int_M \langle \nabla |\nabla v|^2, \nabla f \rangle \rho - 2\langle \nabla \langle \nabla f, \nabla v \rangle, \nabla v \rangle \rho \, d\mu \\ &= \int_M \left(-2v_{ij}^2 - 2(R_{ij} + f_{ij}) v_i v_j\right) \rho \, d\mu. \end{aligned} \tag{0.6}$$

The dynamics

The dynamics

- Use the soliton equation (not essentially since one just needs an inequality here)

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- Also

$$\frac{d}{dt}N_{\rho,f}(t) \geq \tau \frac{d}{dt}F_{\rho,f}(t).$$

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- The Fokker-Planck equation/dynamics was studied by many people including Arnold-Markowich-Toscani-Unterreiter, Carrillo-Toscani, Del Pino-Dolbeault, Otto-Villani, Sturm etc.

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- Recall Perelman's entropy functional

$$\mathcal{W}(g^\tau, u, \tau) \doteqdot \int_M \left(\tau(|\nabla\psi|^2 + S) + \psi - n\right) u \, d\Gamma_\tau$$

is defined for $u = \frac{e^{-\psi}}{(4\pi\tau)^{n/2}}$ with $\int_M u \, d\Gamma_\tau = 1$.

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- Theorem implies that for (M, g^τ), $W(g^\tau, u, \tau) \geq -\mu_s$. Namely Perelman's μ-invariant

$$\mu(g^\tau, \tau) \doteqdot \inf_{\int_M u = 1} \mathcal{W}(g^\tau, u, \tau)$$

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- From soliton equation and its consequence it is easy to see that

$$\tau(2\Delta f - |\nabla f|^2 + S) + f - n = -\mu_s.$$

  Hence $u = \frac{e^{-f}}{(4\pi\tau)^{n/2}}$ is the minimizer for Perelman's $\mu(g, \tau)$

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  - In RG-flow, there is a result asserting (with some conditions) that there exists a invariant $c(t)$ which is non-increasing along the flow and non-negative.
  - Motivated by this we show the following result we have the following result.

  Theorem

  Let (M, g) be a gradient shrinking soliton with bounded curvature. Let f be the normalized potential function as above. Then $\mu_s \geq 0$.

The proof

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- The first is the Li-Yau-Hamilton type estimate for the fundamental solution to the conjugate heat equation of Perelman.
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being the $\delta_x(y)$ at $t = t_0$. By a result of Perelman, see also Chau-Tam-Yu and a paper by myself (CAG, 2006), we know that

$$v_H(y, t) \doteq (t_0 - t)\left(2\Delta\varphi - |\nabla\varphi|^2 + S\right) + \varphi - n \leq 0$$

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Proposition

Assume that on a complete Riemannian manifold (M, g), $\mu(g, 1) > \infty$ *and* $\mathrm{Ric} \geq -A$ *and* $S \leq B$ *for some positive numbers* A *and* B*. Then for any* $0 < \sigma < 1$,

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- $$\mathcal{W}_0(t) \doteqdot \int_M \left(\tau|\nabla\varphi|^2 + \varphi - n\right) u$$
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- It has been confirmed by Yokota recently. The key is a connection between LSI inequality/entropy and the reduced volume of Perelman.

The reduced distance and volume

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$$\ell_{x_0}(x, \bar{\tau}) := \inf_{\gamma} \frac{1}{2\sqrt{\bar{\tau}}} \int_0^{\bar{\tau}} \sqrt{\tau} \left(|\gamma'(\tau)|^2 + R \right) \, d\tau \tag{0.10}$$

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- And the reduced volume:

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- Most importantly $\tilde{V}_{x_0, T_0}(\tau)$ is monotone nonincreasing in τ.

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- A gap theorem: there exists a $\epsilon(n)$ such that if $\mathcal{V}_{T_0} \geq 1-\epsilon(n)$, M is Euclidean space.
- The gap theorem corresponds to a local regularity theorem I proved before (Asian Journal, 2007), after the derivation of the local monotonicity formulae joint with Ecker, Knopf and Topping (Crelle 2008).