RG flows in conformal field theory

Matthias Gaberdiel
ETH Zurich

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based on work with
S. Fredenhagen, C. Keller, A. Konechny and C. Schmidt-Colinet.

Conformal perturbation theory

Perturbations of conformal field theories play an important role in different contexts, for example

integrable perturbations in statistical mechanics.

marginal perturbations (moduli) in string theory.

Conformal perturbation theory

Most work has been done in the past on

- bulk perturbations of bulk theories

  Zamolodchikov, Cardy & Ludwig, ...

- boundary perturbations of boundary theories

  Affleck & Ludwig, Shatashvili, Recknagel & Schomerus, Friedan & Konechny, ...

Mixed bulk boundary case

In the following study combined bulk-boundary problem.

Main question: how do boundary conditions (D-branes) behave under bulk deformations?

Concentrate on the case when bulk deformation is (exactly) marginal in the bulk (closed string modulus).

[Ignore backreaction --- higher order in string perturbation theory.]

[Fischler,Susskind]
[Keller]

Leading order RG equations

Consider the perturbation

$$S = S^* + \sum_i \tilde{\lambda}_i \int \phi_i(z)\, d^2 z + \sum_j \tilde{\mu}_j \int \psi_j(x)\, dx$$

bulk perturbation

boundary perturbation

Can make coupling constants dimensionless by introducing a length scale

$$\tilde{\lambda}_i = \lambda_i\, l^{h_{\phi_i}-2}\,, \qquad \tilde{\mu}_j = \mu_j\, l^{h_{\psi_j}-1}\,,$$

Leading order analysis

In Wilsonian approach take as UV cut-off

$$|z^i_k - z^{i'}_{k'}| > l \ , \qquad |x^j_k - x^{j'}_{k'}| > l \ , \qquad \mathrm{Im} z > \frac{l}{2} \ .$$

Then rescale the length scale

$$l \mapsto (1+\delta t) l$$

and adjust the coupling constants so that the free energy remains unchanged,

$$\frac{\delta}{\delta t} e^S = 0 \ .$$

RG equations

Explicit dependence:

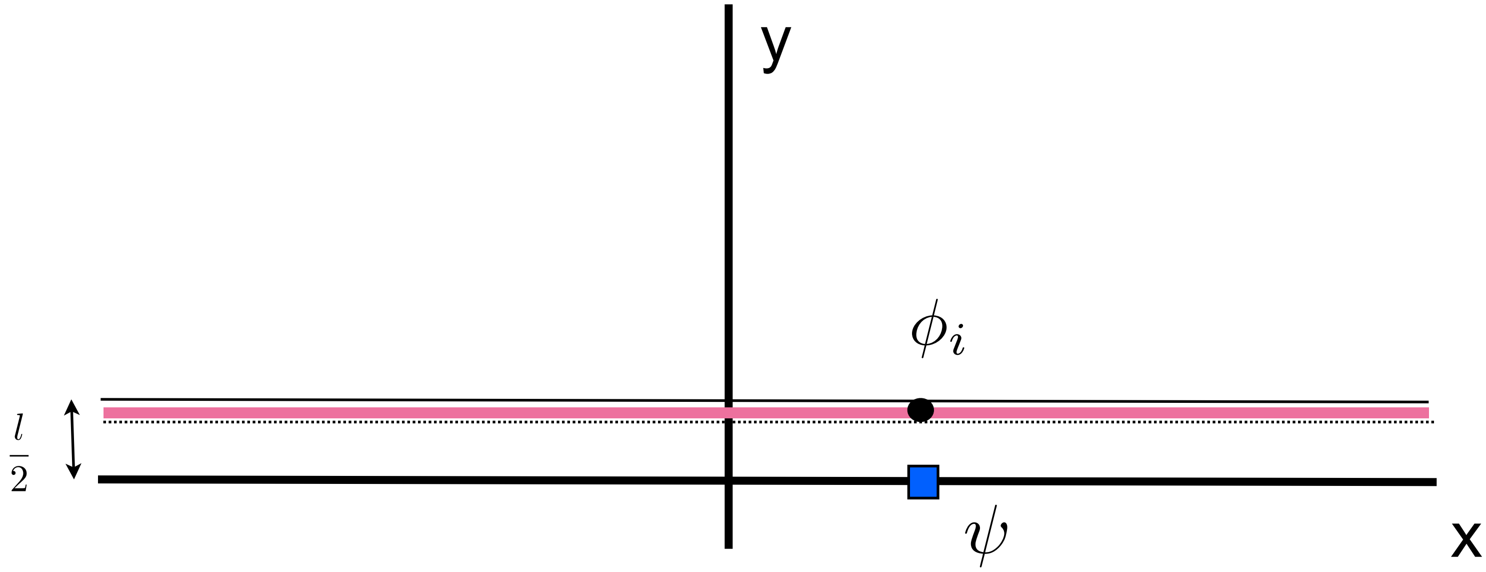
$$\lambda_i \to (1+(2-h_{\phi_i})\delta t)\lambda_i \ , \qquad \mu_j \to (1+(1-h_{\psi_j})\delta t)\mu_j \ .$$

Implicit dependence:

$|z_k^i - z_{k'}^{i'}| > l \ \Rightarrow \ \delta\lambda_k = \pi C_{ijk}\lambda_i\lambda_j\delta t$ $\quad C_{ijk}$: bulk OPE coefficient

$|x_k^j - x_{k'}^{j'}| > l \ \Rightarrow \ \delta\mu_k = D_{ijk}\mu_i\mu_j\delta t$ $\quad D_{ijk}$: boundary OPE coefficient

Bulk boundary term



$$\text{Im}z > \frac{l}{2} \;\Rightarrow\; \delta\mu_k = \frac{\delta t}{2} B_{ik}\lambda_i$$

B_{ik}: bulk-boundary OPE coefficient

RG equations

Altogether we thus find the RG equations:

$$\begin{aligned} \dot{\lambda}_k &= (2 - h_{\phi_k})\lambda_k + \pi C_{ijk}\,\lambda_i\lambda_j + \mathcal{O}(\lambda^3) \\ \dot{\mu}_k &= (1 - h_{\psi_k})\mu_k + \frac{1}{2} B_{ik}\,\lambda_i + D_{ijk}\,\mu_i\mu_j + \mathcal{O}(\mu\lambda, \mu^3, \lambda^2) \end{aligned}$$

couples bulk and boundary

[Fredenhagen, MRG, Keller]

Universality

This contribution is universal if the corresponding boundary field is marginal. Otherwise can remove it by coupling constant redefinition

$$\mu_k \mapsto \tilde{\mu}_k = \mu_k + \frac{B_{ik}}{(1 - h_{\psi_k})} \lambda_i \ .$$

This corresponds to modifying the bulk field as

$$\tilde{\phi}_i(x,y) = \phi_i(x,y) - \sum_k \frac{B_{ik}}{(1 - h_{\psi_k})} \psi_k(x) \delta(y) \ .$$

[MRG, Konechny, Schmidt-Colinet]

Exact marginality on disc

Thus we have shown that an exactly marginal bulk perturbation ϕ_i is only exactly marginal in the presence of a boundary if the bulk-boundary OPE coefficients vanish

$$B_{ik} = 0$$

for all marginal boundary fields ψ_k .

Example

As an example consider the theory of a free compactified boson on a circle of radius R, for which all conformal boundary conditions are known.

For all values of R we have the usual Dirichlet & Neumann branes.

But the remainder of the moduli space of conformal D-branes depends in a very sensitive manner on the value of R:

The D-brane moduli space

- if $R = \frac{M}{N} R_{sd}$ then the additional part of the moduli space of conformal D-branes is

$$SU(2)/\mathbb{Z}_M \times \mathbb{Z}_N \ .$$

[Friedan]
[MRG, Recknagel]

- if R is an irrational multiple of the self-dual radius, then the additional part of the moduli space is just the interval

$$(-1, 1) \ .$$

[Friedan], [Janik]

Bulk modulus

Thus the moduli space of D-branes depends strongly on the radius, which is described by an exactly marginal bulk field.

In particular, a brane associated to a generic group element in SU(2) is not any longer conformal if the radius is perturbed --- expect an RG flow!

The WZW case

For simplicity we consider in the following the theory at the self-dual radius (M=N=1), where it is equivalent to the SU(2) WZW model at level k=1.

The moduli space of conformal branes is then simply SU(2), where we write an arbitrary group element as

$$g = \begin{pmatrix} a & b^* \\ -b & a^* \end{pmatrix}$$

$$[|a|^2 + |b|^2 = 1]$$

b=0: Dirichlet brane

a=0: Neumann brane

Conformal branes

From the point of view of the SU(2) description, the exactly marginal bulk operator that corresponds to changing the radius is then the operator of conformal dimension (1,1)

$$\Phi = J^3 \bar{J}^3 \ .$$

Exact marginality

Exact marginality requires, in particular, that the perturbing field continues to have conformal dimension (1,1), even after the perturbation.

For closed string correlators this implies (to first order in perturbation theory) that the 3-point self-coupling vanishes:

$$\Phi(x,\bar{x})\Phi(y,\bar{y}) = \cdots + \frac{0}{|x-y|^2}\Phi(y,\bar{y}) + \cdots \quad .$$

Obviously, this is the case in the above example.

Bulk RG equation

This implies, in particular, that the bulk RG equation vanishes

$$\dot{\lambda} = 0 \ .$$

What about conformal invariance in the presence of the boundary?

Exact marginality on disc

To check for exact marginality on the disc, we calculate the perturbed 1-point function on the upper half plane, i.e.

$$\langle \Phi(w) \rangle_\lambda = \langle \Phi(w) \rangle + \lambda \int_{\mathbb{H}^+} d^2 z \, \langle \Phi(z) \, \Phi(w) \rangle + \cdots$$

A necessary condition for exact marginality is then that

$$\langle \Phi(w) \rangle_\lambda = \frac{\mathrm{C}}{|\mathrm{Im} w|^2} \, .$$

SU(2) level 1

For the case of the D-brane described by the group element g, the first order perturbation equals (here ϵ is a UV cutoff)

$$\lambda \, \mathrm{Tr}\left([t^3, g t^3 g^{-1}]^2\right) \frac{\pi}{4|\mathrm{Im} w|^2} \left(\log \epsilon - \frac{1}{2} \log |\mathrm{Im} w|^2 + \mathcal{O}(\epsilon)\right)$$

if prefactor is non-zero

modifies functional dependence!

Exact marginality

The prefactor equals

$$\mathrm{Tr}([t^3, g\, t^3\, g^{-1}]^2) = -8|a|^2|b|^2 \ .$$

Thus the radius perturbation is only exactly marginal if a=0 or b=0, i.e. if the brane is a standard Neumann or Dirichlet brane!

This ties in nicely with the fact that only the standard Neumann and Dirichlet branes exist for all radii!

WZW example

In fact, the exactly marginal bulk perturbation by $\Phi = J^3\,\bar{J}^3$ has a non-vanishing bulk-boundary OPE coefficient

$$B_{\Phi c} = -2\sqrt{2}\,|a|\,|b|$$

with the marginal boundary current corresponding to

$$t^c = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -e^{i\varphi} \\ e^{-i\varphi} & 0 \end{pmatrix}\,, \quad \text{where} \quad a\,b^* = |ab|e^{i\varphi}\,.$$

Resulting RG equation

The resulting RG equation therefore has the form

$$\dot{\mu}_c = \frac{1}{2} B_{\Phi c}\,\lambda + \cdots \ .$$

This boundary current modifies the boundary condition g by

$$\delta g = i\, t^c\, g = \frac{1}{\sqrt{2}} \begin{pmatrix} -a\frac{|b|}{|a|} & b^*\frac{|a|}{|b|} \\ -b\frac{|a|}{|b|} & -a^*\frac{|b|}{|a|} \end{pmatrix} \ .$$

Boundary flow

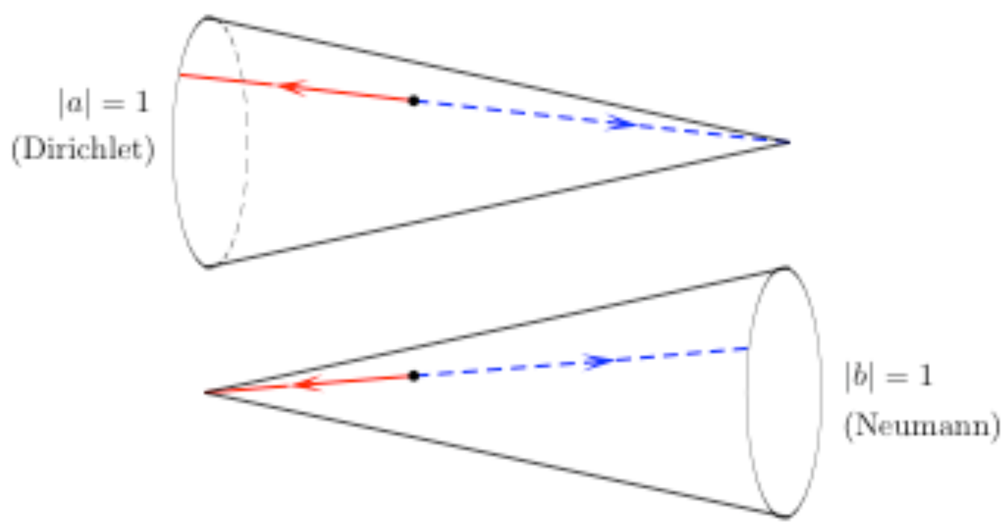
This leaves the phases of a and b unmodified, but decreases the modulus of a, while increasing that of b.

Increase radius $(\lambda > 0)$: brane flows to boundary condition
with b=0 --- Dirichlet brane.

Decrease radius $(\lambda < 0)$: brane flows to boundary condition
with a=0 --- Neumann brane.

The flow on SU(2)

In fact, one can integrate the RG equations exactly in the boundary coupling (at first order in the bulk perturbation), and one finds that the RG flow is along a geodesic on SU(2).



[Fredenhagen, MRG, Keller]

increase radius

decrease radius

Generalisations

It is not difficult to generalise this analysis in a number of ways.

For the case of a rational radius, $R = \frac{M}{N} R_{sd}$, we find essentially the same result:

Increase radius $(\lambda > 0)$: brane flows to boundary condition with b=0 --- M Dirichlet branes.

Decrease radius $(\lambda < 0)$: brane flows to boundary condition with a=0 --- N Neumann branes.

SU(2) at higher level

The analysis also carries over directly to the current-current deformations of the SU(2) WZW model at higher level.

At large level these deformations can be interpreted as changing the metric, the B-field and the dilaton

$$e^{-2\phi(\psi)} = \frac{1-(1-R^2)\cos^2\psi}{R} \, .$$

[Hassan & Sen, Giveon & Kiritsis, Förste]

$$g = \begin{pmatrix} \sin\psi\, e^{i\theta} & \cos\psi\, e^{-i\varphi} \\ -\cos\psi\, e^{i\varphi} & \sin\psi\, e^{-i\theta} \end{pmatrix}$$

Geometrical interpretation

At large level these deformations can be interpreted as changing the metric, the B-field and the dilaton

$$e^{-2\phi(\psi)} = \frac{1-(1-R^2)\cos^2\psi}{R} \ .$$

Our analysis then shows that the brane flows to the position on the group manifold where the brane tension is minimal (dilaton maximal).

General groups

The analysis is also not specific to SU(2). For a general group G we have the flow equation

$$\dot{g} = \frac{\lambda}{2} \left[t^\alpha, g\, t^\alpha g^{-1} \right] g \ ,$$

$$\text{bulk field:} \quad \Phi = J^\alpha \bar{J}^\alpha$$

$$\text{boundary condition:} \quad g$$

Gradient flow

This flow is a gradient flow:

$$\dot{g} = -\nabla V(g) \ , \qquad V(g) = -\frac{\lambda}{2}\,\mathrm{Tr}\Big(t^\alpha\, g\, t^\alpha\, g^{-1}\Big) \ .$$

This is probably even more generally true!

cf. [Friedan, Konechny]

N=1 superconformal case

The analysis is finally very similar for N=1 superconformal branes on the circle theory.

[MRG, Schlotterer]

In particular, can use that the circle theory is the orbifold of the superaffine theory which, at R=1, has a description as a SU(2) WZW model at level k=2.

[Dixon, Ginsparg,Harvey]
[MRG, Israel, Rabinovici]

No RG-flow

Even if there is no non-trivial RG flow, the RG analysis is useful in understanding the behaviour of the branes under the bulk deformation.

However, this then requires an analysis to higher order in perturbation theory.

RG generalities

Recall from above that the general form of the combined RG equations is (assume for simplicity that there is only one bulk field)

$$
\begin{aligned}
\dot{\lambda} &= (2-\Delta)\lambda + \mathcal{C}_{\phi\phi\phi}\lambda\lambda + \mathcal{O}(\lambda^3) \\
\dot{\mu}_i &= (1-h_i)\mu_i + B_{\phi i}\lambda + \sum_{jk}\mathcal{D}_{ijk}\mu_j\mu_k + \sum_j \mathcal{E}_{\phi j}\lambda\mu_j + \cdots \; .
\end{aligned}
$$

We are interested in the situation where no marginal boundary field is switched on by the bulk deformation.

Higher order analysis

Then we can redefine boundary coupling constants

$$\mu_k \mapsto \mu_k + \frac{B_{\phi k}}{(1-h_k)}\lambda \ .$$

so as to bring the boundary RG equations into the form

$$\dot{\mu}_i = D_{ij}(\lambda)\mu_j + \mathcal{O}(\mu^2) \ ,$$

where

$$D_{ij}(\lambda) = (1-h_i)\delta_{ij} + \lambda\tilde{\mathcal{E}}_{ij} + \mathcal{O}(\lambda^2)$$

$$\tilde{\mathcal{E}}_{ij} = \mathcal{E}_{ij} - 2\sum_r \mathcal{D}_{ijr}\frac{B_{\phi r}}{(1-h_r)} \ .$$

Dimension shift

Diagonalise matrix $D_{ij}(\lambda)$ --- to leading order in λ eigenvalues describe shift in conformal dimension:

$$
\begin{aligned}
\dot{\mu} &= (1-h)\mu + \mathcal{E}\mu\lambda + \mathcal{O}(\lambda^2, \mu^2) \\
&= (1-h+\mathcal{E}\lambda)\mu + \mathcal{O}(\lambda^2, \mu^2)
\end{aligned}
$$

describes shift in conformal dimension

Dimension shift

In fact these eigenvalues are explicitly given by

$$\tilde{\mathcal{E}}_{ii} = \mathcal{E}_{ii} - 2 \sum_r \mathcal{D}_{iir} \frac{B_{\phi r}}{(1 - h_r)} \, .$$

One easily checks that this expression is indeed universal!

[MRG, Konechny, Schmidt-Colinet]

Dimension shift

Furthermore, we can give an explicit formula for it in terms of an integrated correlation function:

$$\mathcal{E} = \lim_{\delta \to 0} \left[\int_{\delta}^{\pi-\delta} d\vartheta \, \langle \phi(e^{i\vartheta}) \psi(0) \psi(\infty) \rangle - \sum_{h_r < 1} D_{\psi\psi r} \frac{B_{\phi r}}{(1-h_r)} \left(\frac{1}{2\delta} \right)^{1-h_r} \right] .$$

[This was calculated in a Wilsonian scheme, but since the result is universal, it is independent of the scheme that is used.]

[MRG, Konechny, Schmidt-Colinet]

Neumann example

The simplest example is that of a single Neumann brane on the circle of radius R.

As we have seen above, for a Neumann brane no RG flow is induced as we change the radius of the circle.

However, the conformal dimension of the momentum operators depends on the radius:

$$\psi = e^{ikX} \ , \qquad h_k = k^2 = \frac{n^2}{R^2} \ , \quad n \in \mathbb{Z} \ .$$

Change in conformal dimension

The change in conformal dimension can be determined from

$$\mathcal{E} = \lim_{\delta \to 0} \left[2 \int_{\delta}^{\pi - \delta} d\vartheta \langle e^{-ikX}(\infty)\, \partial X(e^{i\vartheta})\, \bar{\partial} X(e^{-i\vartheta})\, e^{ikX}(0) \,\rangle - \sum_{h_r < 1} D_{r\psi}{}^{\psi} \frac{B_{\phi}{}^{r}}{1 - h_r} \left(\frac{1}{2\delta} \right)^{1 - h_r} \right] .$$

$$2 \langle\, e^{-ikX}(\infty)\, \partial X(e^{i\vartheta})\, \bar{\partial} X(e^{-i\vartheta})\, e^{ikX}(0) \,\rangle = -2\, k^2 + \frac{1}{4 \sin^2 \vartheta} \, .$$

cancels singularity
in integral

only identity is
switched on

Change in conformal dimension

Indeed, the integral of the correlation function is

$$2\int_{\delta}^{\pi-\delta} d\vartheta\,\langle\, e^{-ikX}(\infty)\;\partial X(e^{i\vartheta})\,\bar{\partial}X(e^{-i\vartheta})\,e^{ikX}(0)\,\rangle = -2\,k^2\pi + \frac{1}{2\delta}\ .$$

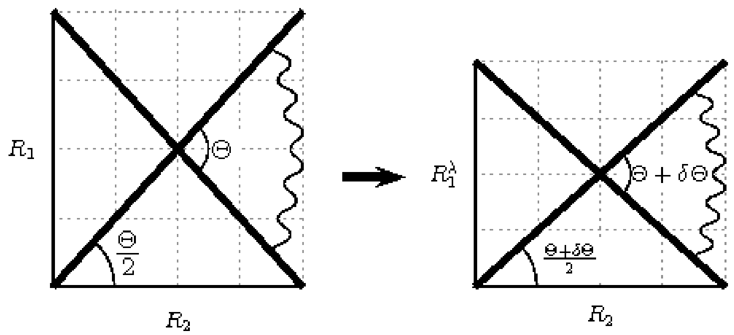
$$\mathcal{E} = -2\,k^2\pi$$

Change in conformal dimension is then, as expected

$$\delta h = -\mathcal{E}\lambda = 2k^2\,\pi\lambda = -2k^2\,\frac{\delta R}{R}\ .$$

More interesting example

As a more interesting example consider the following problem:



The bulk boundary OPE coefficient always vanishes for boundary changing operators, and hence there is no term linear in λ for the boundary changing RG equation.

Dimension shift

The above formula for the shift in conformal dimension of the boundary changing field gives

$$\mathcal{E} = \lim_{\delta\to 0}\left[\int_{\delta}^{\pi-\delta} d\vartheta \,\langle \phi(e^{i\vartheta})\psi(0)\psi(\infty)\rangle - \sum_{h_r<1} D_{\psi\psi r}\frac{B_{\phi r}}{(1-h_r)}\left(\frac{1}{2\delta}\right)^{1-h_r}\right] .$$

$$\frac{1}{2}e^{-i\Theta}\frac{z^{\nu}}{\bar{z}^{\nu}}\frac{z(1-\nu)+\bar{z}\nu}{z(z-\bar{z})^2} + c.c.$$

$$[\Theta = \pi\nu\,,\ \ z = e^{i\vartheta}]$$

again important contribution from boundary preserving fields (identity).

Dimension shift

After integration one then finds

$$\mathcal{E} = -\frac{1}{2}\,(2\nu - 1)\,\sin\Theta$$

leading to correct formula for shift of conformal dimension

$$\delta h = -\mathcal{E}\lambda = \frac{\lambda}{2}\,(2\nu - 1)\,\sin\Theta\ .$$

$$\left[h = \frac{1}{2}\nu(1-\nu)\ ,\ \delta\nu = -\sin\Theta\,\lambda\right]\ .$$

[MRG, Konechny, Schmidt-Colinet]

Conclusions

Described the effect of bulk perturbations on boundary conditions. In particular, it can

- break conformal symmetry if marginal field is switched on --- induces RG flow
- change conformal dimensions of boundary fields --- calculable at higher order

Conclusions

Both effects are of interest in string theory:

- break conformal symmetry if marginal field is switched on --- induces RG flow

  captures dependence of brane moduli space on bulk moduli space --- can be calculated in explicit examples (e.g. D2-branes on quintic)

[Baumgartl, Brunner, MRG]

Conclusions

Both effects are of interest in string theory:

- break conformal symmetry if marginal field is switched on --- induces RG flow

- change conformal dimensions of boundary fields --- calculable at higher order

important for stability analysis of D-branes.