Gradient property of the boundary rg flow for supersymmetric 1+1d quantum field theories

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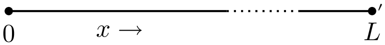
1-d quantum systems

quantum mechanics

- a Hilbert space $\mathcal{H}$
- a self-adjoint hamiltonian operator $H \geq 0$ on $\mathcal{H}$
  (H generates translation in time, $t \mapsto e^{itH}$)

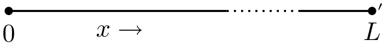
in a one dimensional space (e.g., a quantum wire)



i.e., the algebra of operators (observables) is generated by operators (operator-valued distributions) $\mathcal{O}_\alpha(x)$, localized in a one dimensional space:

$$[\mathcal{O}_\alpha(x), \mathcal{O}_{\alpha'}(x')] = 0 \qquad x \neq x'$$

But a physical wire is not a one-dimensional continuum:



$$x \in \{0, 1\epsilon, 2\epsilon, 3\epsilon, \ldots, N\epsilon\} \qquad N = L/\epsilon \gg 1$$

We are interested in systems where quantum phenomena are correlated over large distances compared to the microscopic scale ϵ.

This happens when the system goes critical at very low temperature.

We are interested in doing things with the system at some typical distance of order L, with $\epsilon \ll L$.

We want to get close to the limit $\epsilon \to 0$.

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I argued that circuits made of bulk-critical quantum wire, joined at boundaries and junctions, would be ideal for asymptotically large-scale quantum computing. The $c = 24$ monster-symmetric wire would be especially ideal. [cond-mat/0505084,0505085]

Abstract examples: the nonlinear models

Let M be a manifold. Put a copy of M at each point x.

$$\mathcal{H} = \bigotimes_x L_2(M) = L_2\left(\prod_x M\right)$$

The states in $\mathcal{H}$ are the L_2 functions of the classical field ϕ

$$\phi \in \prod_x M \ : x \mapsto \phi(x) \in M$$

The hamiltonian is parametrized by the metric g

$$H = \sum_x \ \epsilon^{-1} \left[\Delta_{\phi(x)} + \ \mathrm{dist}^2\left(\phi(x), \phi(x+\epsilon)\right)\right]$$

generalizing the Heisenberg model, where $(M, g) =$ round S^3.

Formally, in the limit $\epsilon \to 0$,

$$H = \int dx \ \left[g^{-1}(\pi(x), \pi(x)) + g(d\phi(x), d\phi(x))\right]$$

where

$$[\pi(x),\, \phi(x')] = \delta(x - x')$$

We say that g is naively *dimensionless*. No inconvenient powers of ϵ appear when we take the formal limit.

But this formal limit is valid only when M is asymptotically large, when $g \to \infty$.

More generally, as we send $\epsilon \to 0$, we have to make the metric g depend on ϵ in a certain way

$$\epsilon \frac{\partial}{\partial \epsilon} g = \beta(g)$$

so that our measurements at distances of order L are independent of ϵ in the limit.

We can calculate $\beta(g)$ as an expansion in powers of g^{-1},

$$\beta(g) = -\mathrm{Ricci}(g) + R^2(g) + \cdots$$

This is the renormalization group flow. The continuum limit $\epsilon \to 0$ requires stability in the far past.

Bulk-critical systems

I will be considering systems that are exactly critical (in the bulk): fixed points of the RG, $\beta = 0$. The geometric analogy would be Ricci-flat.

These systems are scale-invariant, because scaling x is equivalent to scaling ϵ, and nothing depends on ϵ.

A fair number of general theorems can be proved about such 1-d quantum systems.

It turns out that these systems are conformally invariant, not just scale invariant. In consequence, the Virasoro algebra acts on $\mathcal{H}$. The Virasoro algebra is the central extension of $\mathrm{diff}(S^1)$,

$$[L_m,\, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n}$$

In one spatial dimension, the rigid structure of quantum field theory combined with the constraint of conformal invariance seems to be one of those optimal mathematical objects which have a rich collection of realizations, that are almost classifiable.

A theorem about conformally invariant 1-d systems

Unitary representation theory of the Virasoro algebra

$$c \geq 1 \quad \text{or} \quad c \in \{1 - 6/m(m+1) : \; m = 2, 3, 4, \ldots\}$$

For the nonlinear models, $c = \dim(M) \geq 1$.

The thermal partition function of the 1-d system

$$Z = \mathrm{tr}\; e^{-\beta H}$$

It can be shown that, for $L/\beta \gg 1$,

$$\ln Z = \frac{\pi c}{6}\frac{L}{\beta} + o\left(\frac{L}{\beta}\right)$$

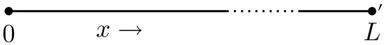
where c is the central charge of the Virasoro algebra (appropriately normalized).

So the unitary representation theory of the Virasoro algebra yields a theorem on the possible values of this physical measurement.

Boundary conditions

It might seem, after we have restricted ourselves to systems with $\beta = 0$, that there is nothing more to be said about the RG flow on such systems.

But I left something out.



We need to impose some boundary condition at $x = 0$, and also at $x = L$.

The boundary condition is additional data parametrizing the system.

For example, in the nonlinear model, we might require $\phi(0) \in N$, for N a sub-manifold of M, and $\phi(L) \in N'$, for another sub-manifold N'.

For the general bulk-critical system, we consider the space of all possible boundary conditions, parametrized by coordinates λ^a.

The boundary partition function $z(\lambda, \beta)$

$$\ln Z = \ln z(\lambda, \beta) + \frac{\pi c}{6}\frac{L}{\beta} + \ln z(\lambda', \beta) + o(1)$$

λ being the boundary condition at $x = 0$ and λ' the boundary condition at $x = L$.

The boundary renormalization group flow

$$\left(\epsilon\frac{\partial}{\partial\epsilon} + \beta^a(\lambda)\frac{\partial}{\partial\lambda^a}\right)\ln z = 0$$

Entropy and boundary entropy

$$S = \left(1 - \beta\frac{\partial}{\partial\beta}\right)\ln Z = s + \frac{\pi c}{3}\frac{L}{\beta} + s'$$

Gradient formula for the boundary entropy [DF & A. Konechny, 2004]

$$\frac{\partial s}{\partial\lambda^a} = -g_{ab}\beta^b(\lambda)$$

The gradient formula implies that s decreases along the RG flow

$$-\epsilon \frac{\partial s}{\partial \epsilon} = \beta^a \frac{\partial s}{\partial \lambda^a} = -\beta^a g_{ab}(\lambda) \beta^b \leq 0$$

with equality iff $\beta = 0$.

The gradient formula implies 2nd law of boundary thermodynamics

$$\beta \frac{\partial s}{\partial \beta} = -\epsilon \frac{\partial s}{\partial \epsilon} = -\beta^a g_{ab}(\lambda) \beta^b \leq 0$$

The boundary behaves thermodynamically like an isolated system.

No general gradient formula is known for the bulk

The c-theorem says that c extends to the non-scale-invariant bulk systems so that

$$\beta^i \frac{\partial c}{\partial \lambda^i} = -\beta^i g_{ij}^{bulk} \beta^j \leq 0$$

but this has not been derived from a general bulk gradient formula.

Supersymmetric 1-d systems, critical in the bulk

a conserved fermionic super-charge

$$H = \hat{Q}^2$$

A 2nd gradient formula for susy boundaries [DF & A. Konechny, 2008]

$$\frac{\partial \ln z}{\partial \lambda^a} = -g^S_{ab}(\lambda)\beta^b(\lambda)$$

The λ^a now restricted to the susy coupling constants.

implying

$$-\epsilon\frac{\partial \ln z}{\partial \epsilon} = \beta\frac{\partial \ln z}{\partial \beta} = \beta^a\frac{\partial \ln z}{\partial \lambda^a} = -\beta^a g^S_{ab}\beta^b \leq 0$$

so $\ln z$ decreases under the RG flow. and the boundary energy $-\partial \ln z/\partial\beta$ is nonnegative.

The boundary behaves thermodynamically like an isolated supersymmetric system:

$$-\frac{\partial \ln Z}{\partial \beta} = Z^{-1}\,\mathrm{tr}\left(e^{-\beta H}H\right) = Z^{-1}\,\mathrm{tr}\left(e^{-\beta H}\hat{Q}^2\right) \geq 0$$

Action principal for open string theory

$\beta^a = 0$ is the classical string equation of motion, so the gradient formula gives an action principle: the equation of motion is the stationarity condition on an action function. The super gradient formula was originally conjectured in string theory in the form

$$\frac{\partial z}{\partial \lambda^a} = -G^S_{ab}\beta^b \qquad G^S_{ab} = z\,g^S_{ab}$$

z is the string action, not the physical $\ln z$. The bosonic boundary gradient formula had been indirectly conjectured in string theory: the equivalence to the physical formula was not as obvious.

In the remainder of the talk, I will sketch a direct proof that

$$\beta \frac{\partial \ln z}{\partial \beta} \leq 0$$

i.e., that the susy boundary energy is non-negative, that $\ln z$ decreases under the RG flow.

The steps are exactly the same as used in the proof of the gradient formula. At the end I will flash a few slides of the proof of the gradient formula, then pose some questions, then, if time permits, explain a crucial technical lemma.

Assumption: a locally conserved supercharge density

energy density and super-charge density

$$H = \int_0^L dx \, \mathcal{H}(t,x)$$

$$\hat{Q} = \int_0^L dx \, \hat{\rho}(t,x)$$

$$[\hat{Q}, \, \hat{\rho}(t,x)]_+ = 2\mathcal{H}(t,x)$$

local conservation of super-charge

$$\partial_t \hat{\rho}(t,x) + \partial_x \hat{\jmath}(t,x) = 0$$

Boundary charge operators

boundary hamiltonian

$$h(t) = \lim_{\epsilon \longrightarrow 0} \int_0^\epsilon dx\, \mathcal{H}(t, x)$$

boundary super-charge

$$\hat{q}(t) = \lim_{\epsilon \longrightarrow 0} \int_0^\epsilon dx\, \hat{\rho}(t, x)$$

super-partners

$$[\hat{Q},\, \hat{q}(t)]_+ = 2h(t)$$

thermodynamic boundary energy

$$-\frac{\partial \ln z}{\partial \beta} = \langle h \rangle$$

Separate $\hat{Q}$ into boundary and bulk parts at $x = \epsilon$

$$\hat{q}_\epsilon(t) = \int_0^\epsilon dx\ \hat{\rho}(t,x) \qquad \hat{Q}_{bulk}(t) = \int_\epsilon^L dx\ \hat{\rho}(t,x)$$

$$\hat{Q} = \hat{q}_\epsilon(t) + \hat{Q}_{bulk}(t)$$

locality implies

$$[\hat{Q}_{bulk}(0),\ \hat{q}(0)]_+ = 0$$

so

$$\langle 2h \rangle = \langle [Q,\ \hat{q}(0)]_+ \rangle = \langle [\hat{q}_\epsilon(0),\ \hat{q}(0)]_+ \rangle$$

but this equation is useless at $\epsilon = 0$, where it would trivially give the positivity result.

The problem is that $\langle [\hat{q}(0),\ \hat{q}(0)]_+ \rangle$ is divergent.

The boundary cannot be separated from the bulk, in general.

Use time dependence

Fourier transform and define response functions

$$g_\epsilon(\omega) = \int_{-\infty}^{\infty} dt\; e^{i\omega t} \left\langle [\hat{q}_\epsilon(t),\, \hat{q}(0)]_+ \right\rangle$$

$$G_\epsilon^{\pm}(\omega) = \pm \int_0^{\pm\infty} dt\; e^{i\omega t} \left\langle [\hat{Q}_{bulk}(t),\, \hat{q}(0)]_+ \right\rangle$$

so

$$2\pi\delta(\omega)\big\langle 2h \big\rangle = g_\epsilon(\omega) + G_\epsilon^{+}(\omega) + G_\epsilon^{-}(\omega)$$

bulk super-conformal invariance implies

$$G_\epsilon^{+}(i\pi/\beta) = 0 = G_\epsilon^{-}(-i\pi/\beta)$$

so

$$\int \frac{d\omega}{2\pi}\, \frac{\pi^2/\beta^2}{\omega^2 + \pi^2/\beta^2}\, G_\epsilon^{\pm}(\omega) = 0$$

so

$$\langle 2h \rangle = \int \frac{d\omega}{2\pi} \, \frac{\pi^2/\beta^2}{\omega^2 + \pi^2/\beta^2} \, g_\epsilon(\omega)$$

Now take $\epsilon \to 0$:

$$g(\omega) = \int_{-\infty}^{\infty} dt \; e^{i\omega t} \left\langle [\hat{q}(t), \, \hat{q}(0)]_+ \right\rangle$$

$$\beta \frac{\partial \ln z}{\partial \beta} = -\beta \langle h \rangle = -\frac{\beta}{2} \int \frac{d\omega}{2\pi} \, \frac{\pi^2/\beta^2}{\omega^2 + \pi^2/\beta^2} \, g(\omega)$$

which is uv-finite as long as $\dim[g(\omega)] < 1$, i.e., $\dim[\hat{q}] < 1$.

We have $g(\omega) \geq 0$ and $g(\omega) = 0$ iff $\hat{q} = 0$, so

$$\beta \frac{\partial \ln z}{\partial \beta} \leq 0$$

with equality iff the boundary is critical (superconformal).

The gradient formula

boundary operators

$$[\hat{Q},\, \hat{\phi}_a(t)]_+ = \phi_a(t)$$

$$\frac{\partial \ln z}{\partial \lambda^a} = \beta \langle \phi_a \rangle$$

boundary beta-functions

$$\hat{q} = -2\beta^a \hat{\phi}_a$$

$$h = \frac{1}{2}[\hat{Q},\, \hat{q}]_+ = -\beta^a \phi_a$$

$$\Lambda \frac{\partial \ln z}{\partial \Lambda} = \beta \frac{\partial \ln z}{\partial \beta} = -\beta \langle h \rangle = \beta \langle \beta^a \phi_a \rangle = \beta^a \frac{\partial \ln z}{\partial \lambda^a}$$

$$\langle \phi_a \rangle = \langle [\hat{Q}, \hat{\phi}_a(0)]_+ \rangle = \langle [\hat{q}_\epsilon(t) + \hat{Q}_{bulk}(t), \hat{\phi}_a(0)]_+ \rangle$$

$$\begin{aligned} g_a(\omega) &= \int_{-\infty}^{\infty} dt\ e^{i\omega t} \langle [\hat{q}(t), \hat{\phi}_a(0)]_+ \rangle \\ &= \int_{-\infty}^{\infty} dt\ e^{i\omega t} \langle [-2\beta^b \hat{\phi}_b(t), \hat{\phi}_a(0)]_+ \rangle \\ &= -2\beta^b g_{ab}(\omega) \end{aligned}$$

$$\begin{aligned} \langle \phi_a \rangle &= \int \frac{d\omega}{2\pi} \frac{\pi^2/\beta^2}{\omega^2 + \pi^2/\beta^2}\, g_a(\omega) \\ &= -2\beta^b \int \frac{d\omega}{2\pi} \frac{\pi^2/\beta^2}{\omega^2 + \pi^2/\beta^2}\, g_{ab}(\omega) \end{aligned}$$

$$\frac{\partial \ln z}{\partial \lambda^a} = -g^S_{ab}\beta^b$$

$$g_{ab}(\omega) = \int_{-\infty}^{\infty} dt\ e^{i\omega t} \left\langle [\hat{\phi}_b(t),\ \hat{\phi}_a(0)]_+ \right\rangle$$

$$\begin{aligned}
g^S_{ab} &= 2\beta \int \frac{d\omega}{2\pi}\ \frac{\pi^2/\beta^2}{\omega^2 + \pi^2/\beta^2}\ g_{ab}(\omega) \\
&= \pi \int dt\ e^{-\pi|t|/\beta} \left\langle [\hat{\phi}_b(t),\ \hat{\phi}_a(0)]_+ \right\rangle \\
&= 2\pi \int_0^{\beta} d\tau\ \sin\left(\frac{\pi\tau}{\beta}\right) \left\langle \hat{\phi}_b(-i\tau),\ \hat{\phi}_a(0) \right\rangle
\end{aligned}$$

Some questions

1. Why do we need bulk conformal invariance?
2. Why do we need canonical uv behavior in the boundary?
   - no negative dimension boundary operators
   - no strongly irrelevant boundary operators
3. Does the result apply to composite boundaries/junctions?
4. Can $\ln z$ (and/or s) be bounded below?

Bulk conformal invariance and zeros of response functions

$$\partial_t \hat{Q}_{bulk}(t) = \int_{\epsilon}^{L} dx\,[-\partial_x \hat{\jmath}(t,x)] = \hat{\jmath}(t,\epsilon)$$

Define response functions

$$R_a^{\pm}(\omega) = \pm \int_0^{\pm\infty} dt\; e^{i\omega t - \delta|t|} \langle [i\hat{\jmath}(t,\epsilon),\, \hat{\phi}_a(0)]_+ \rangle$$

$R_a^+(\omega)$ is analytic in the upper half-plane, $R_a^-(\omega)$ in the lower.

Use the conservation equation

$$G_{a,\epsilon}^{\pm}(\omega) = \pm \int_0^{\pm\infty} dt\; e^{i\omega t - \delta|t|} \langle [\hat{Q}_{bulk}(t),\, \hat{\phi}_a(0)]_+ \rangle = \frac{R_a^{\pm}(\omega)}{\omega \pm i\delta}$$

$\tau = it$, $0 < \tau < \beta$

$$\left\langle \hat{\jmath}(-i\tau,\epsilon)\,\hat{\phi}_a(0)\right\rangle = \int \frac{d\omega}{2\pi i}\,\frac{e^{-\omega\tau}}{1+e^{-\omega\beta}}\left[R^+(\omega)+R^-(\omega)\right]$$

poles at

$$\omega_n = \frac{2\pi i n}{\beta} \qquad n \in \frac{1}{2}+\mathbb{Z}$$

so

$$\left\langle \hat{\jmath}(-i\tau,\epsilon)\,\hat{\phi}_a(0)\right\rangle = \beta^{-1}\sum_n e^{-\omega_n\tau}\left[\theta(n)R^+(\omega_n)-\theta(-n)R^-(\omega_n)\right]$$

but

$$j(-i\tau,x) = AG(x+i\tau)+\bar{A}G(x-i\tau)$$

so

$$R_a^+(i\pi/\beta) = 0 = R_a^-(-i\pi/\beta)$$