# *Green functions for the Ricci flow and their applications*

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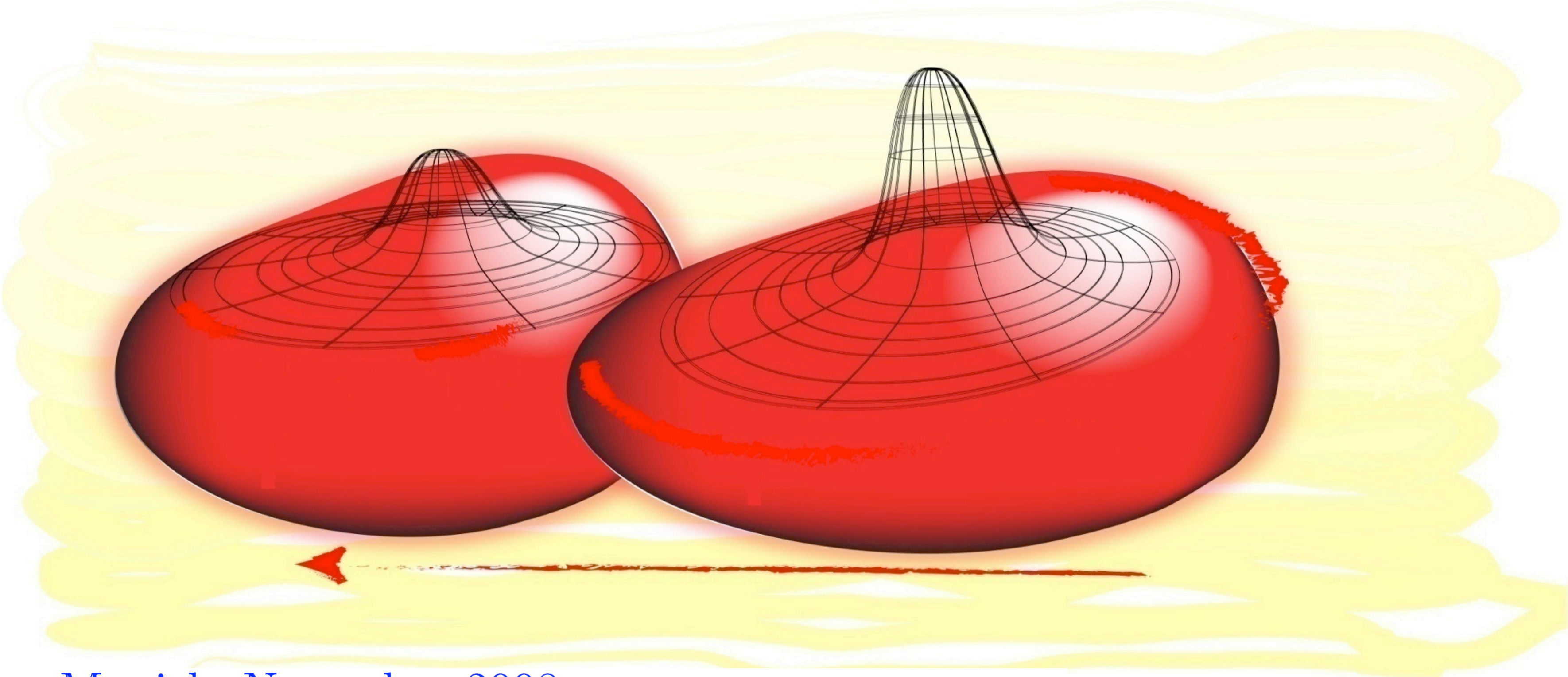

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Workshop on Field Theory and Geometric Flows

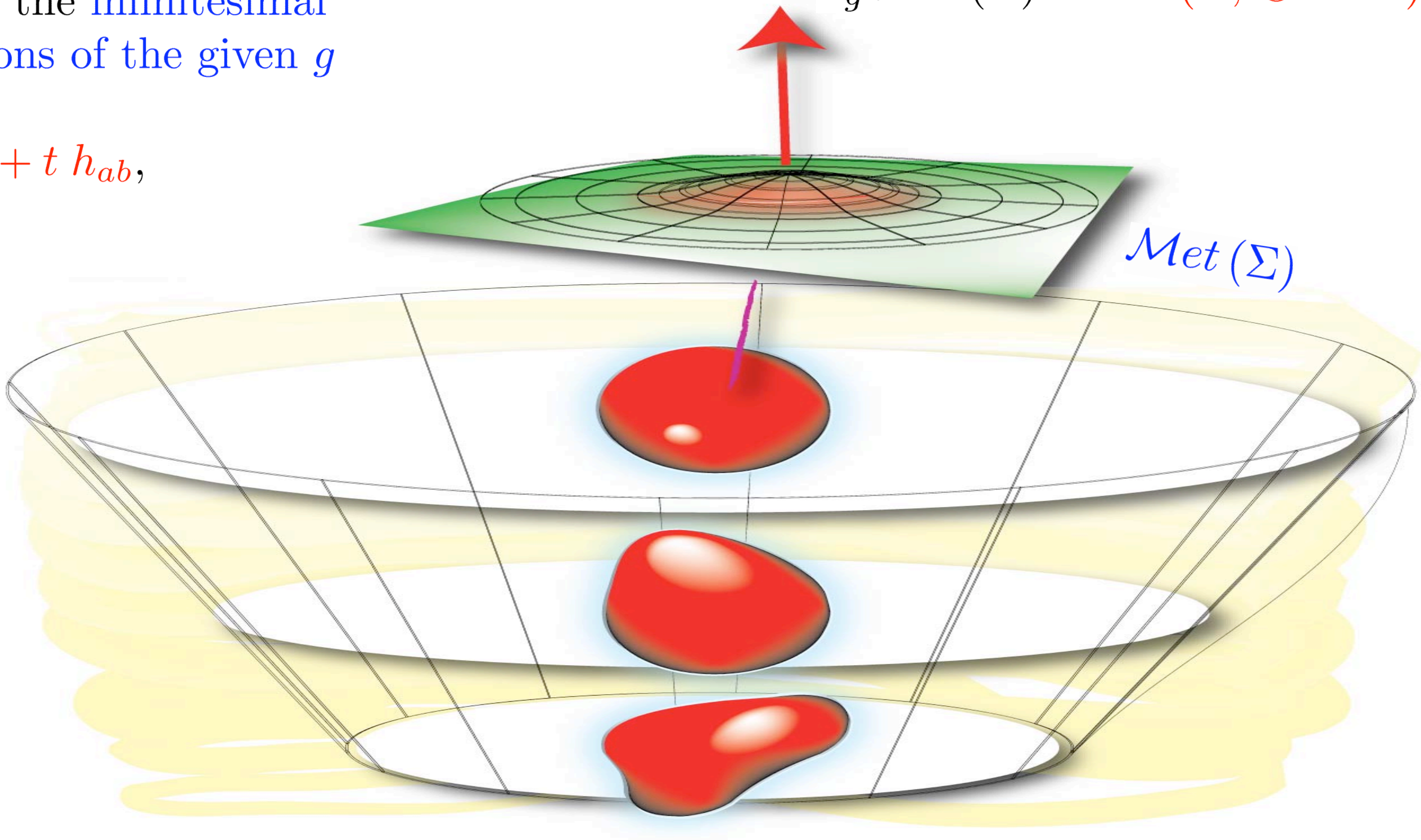
$(\Sigma, g)$ three-dimensional compact Riemannian Manifold without boundary thought of as a point in the Space of Riemannian structures $\frac{\mathcal{M}et\,(\Sigma)}{\mathcal{D}iff\,(\Sigma)}$

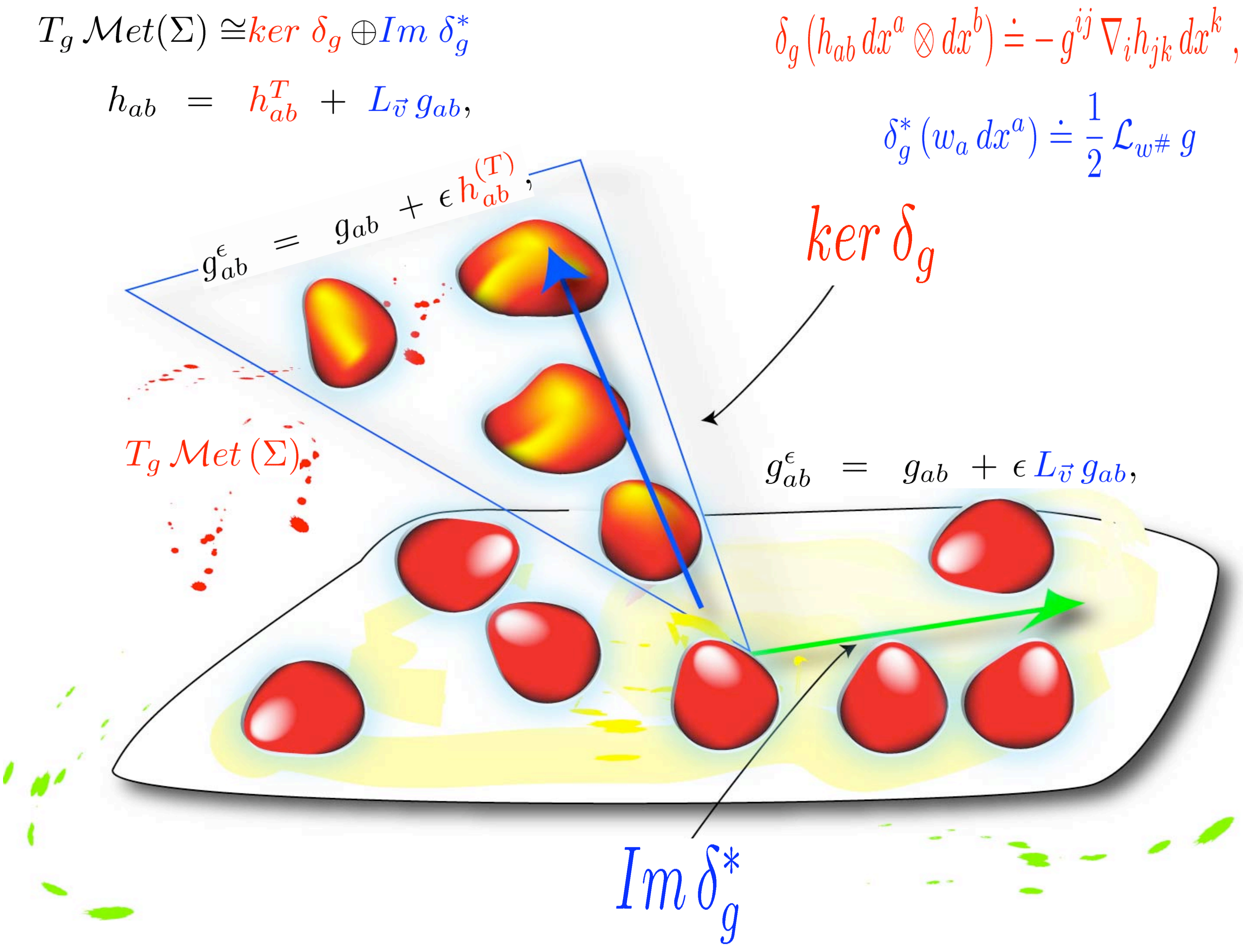
# Tangent space $T_g\,\mathcal{M}\,(\Sigma)$

represents the infinitesimal deformations of the given $g$

$$g^{(t)}_{ab} = g_{ab} + t\,h_{ab},$$

$T_g\,\mathcal{M}et(\Sigma) \cong C^\infty(\Sigma, \otimes^2 T^*\Sigma)$




$T_g\, \mathcal{M}et(\Sigma) \cong ker\ \delta_g \oplus Im\ \delta_g^*$
$h_{ab} = h^T_{ab} + L_{\vec{v}}\, g_{ab},$
$\delta_g\,(h_{ab}\, dx^a \otimes dx^b) \doteq -g^{ij}\, \nabla_i h_{jk}\, dx^k\,,$
$\delta_g^*\,(w_a\, dx^a) \doteq \frac{1}{2}\, \mathcal{L}_{w\#}\, g$
$g^{\epsilon}_{ab} = g_{ab} + \epsilon\, h^{(T)}_{ab}\,,$
$ker\,\delta_g$
$T_g\, \mathcal{M}et\,(\Sigma)$
$g^{\epsilon}_{ab} = g_{ab} + \epsilon\, L_{\vec{v}}\, g_{ab},$
$Im\,\delta_g^*$

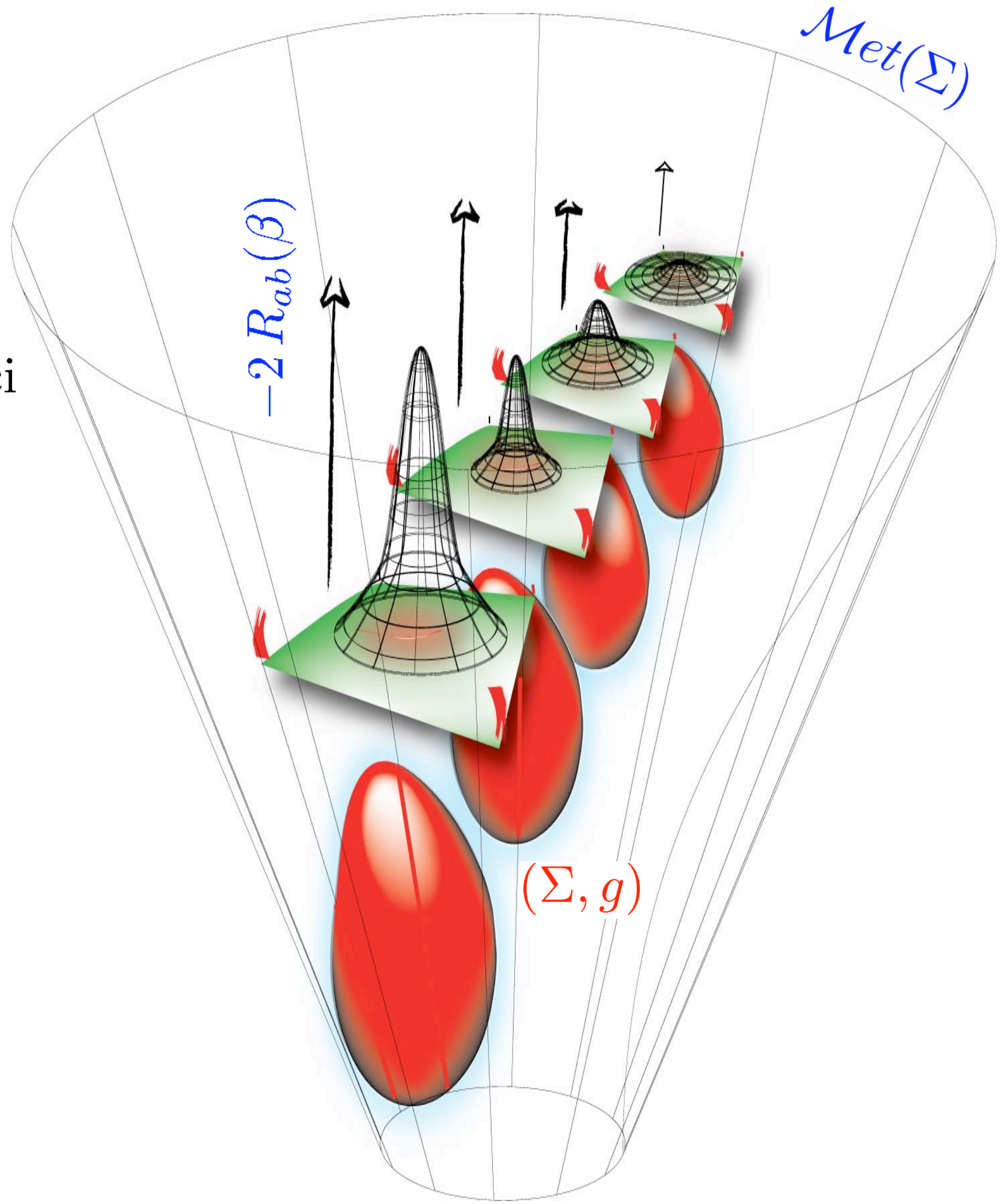
Ricci Flow as a (weakly–parabolic) dynamical system on:

$$\mathcal{M}et(\Sigma) \longrightarrow \mathcal{M}et(\Sigma)$$
$$(\Sigma, g) \mapsto (\Sigma, g(\beta)) \,,$$

defined by deforming the metric $(\Sigma, g)$ in the direction of (-) its Ricci tensor thought of as a (non–trivial) vector in $T_g\,\mathcal{M}et(\Sigma)$, *i.e.*,

$$\begin{cases} \frac{\partial}{\partial\beta} g_{ab}(\beta) = -2R_{ab}(\beta), \\ g_{ab}(\beta = 0) = g_{ab}, \end{cases}$$

$$0 \leq \beta \leq \beta^* < T_0$$



Together with the Ricci Flow,
$\beta \mapsto g_{ab}(\beta)$, $0 \leq \beta \leq \beta^*$,
we shall consider also the
backward Ricci flow

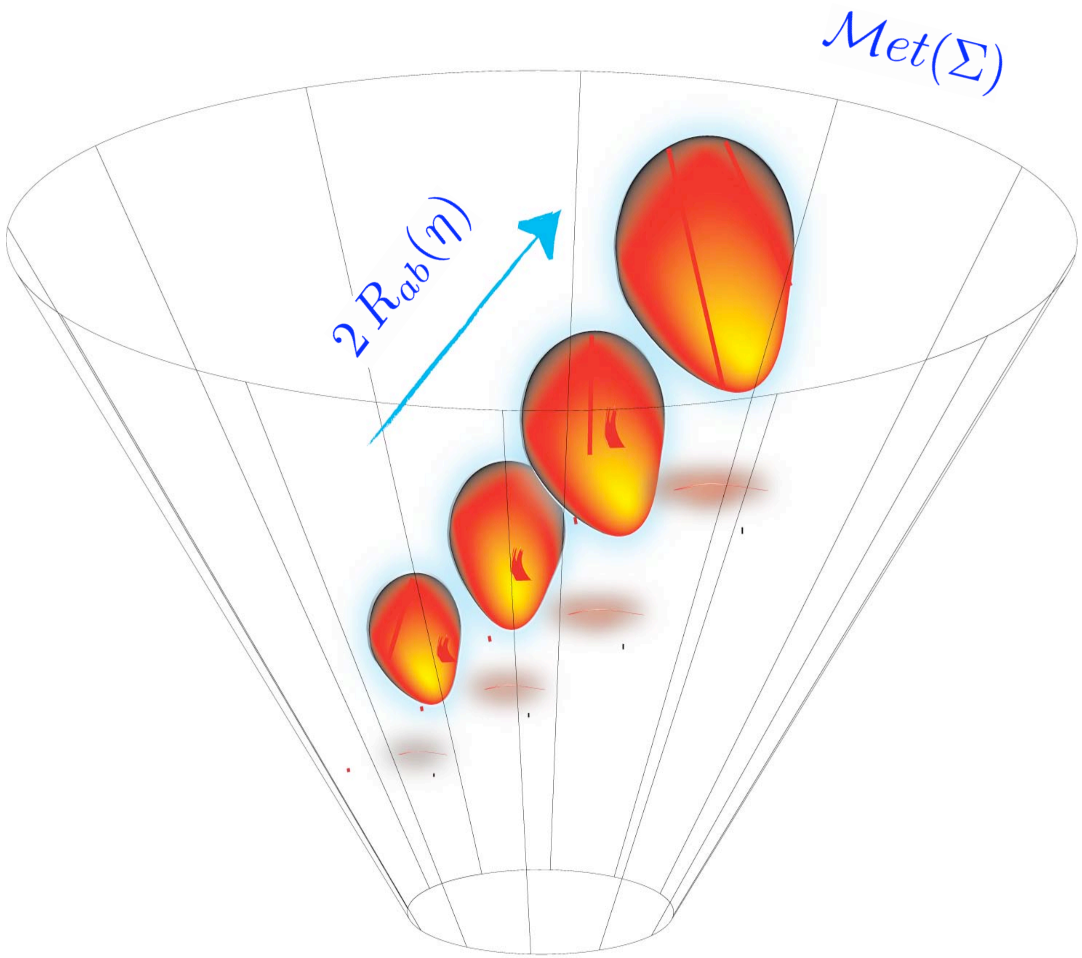
$$\begin{array}{ccc} \mathcal{M}et(\Sigma) & \longrightarrow & \mathcal{M}et(\Sigma) \\ (\Sigma, g) & \mapsto & (\Sigma, g(\eta)) \, , \end{array}$$

obtained upon time–reversal:

$$\eta \mapsto g_{ab}(\eta \doteq \beta^* - \beta).$$

$$\begin{cases} \frac{\partial}{\partial \eta} g_{ab}(\eta) = 2R_{ab}(\eta), \\ \\ g_{ab}(\eta = 0) = g_{ab}(\beta^*), \end{cases}$$

$$0 \leq \eta \leq \beta^*$$



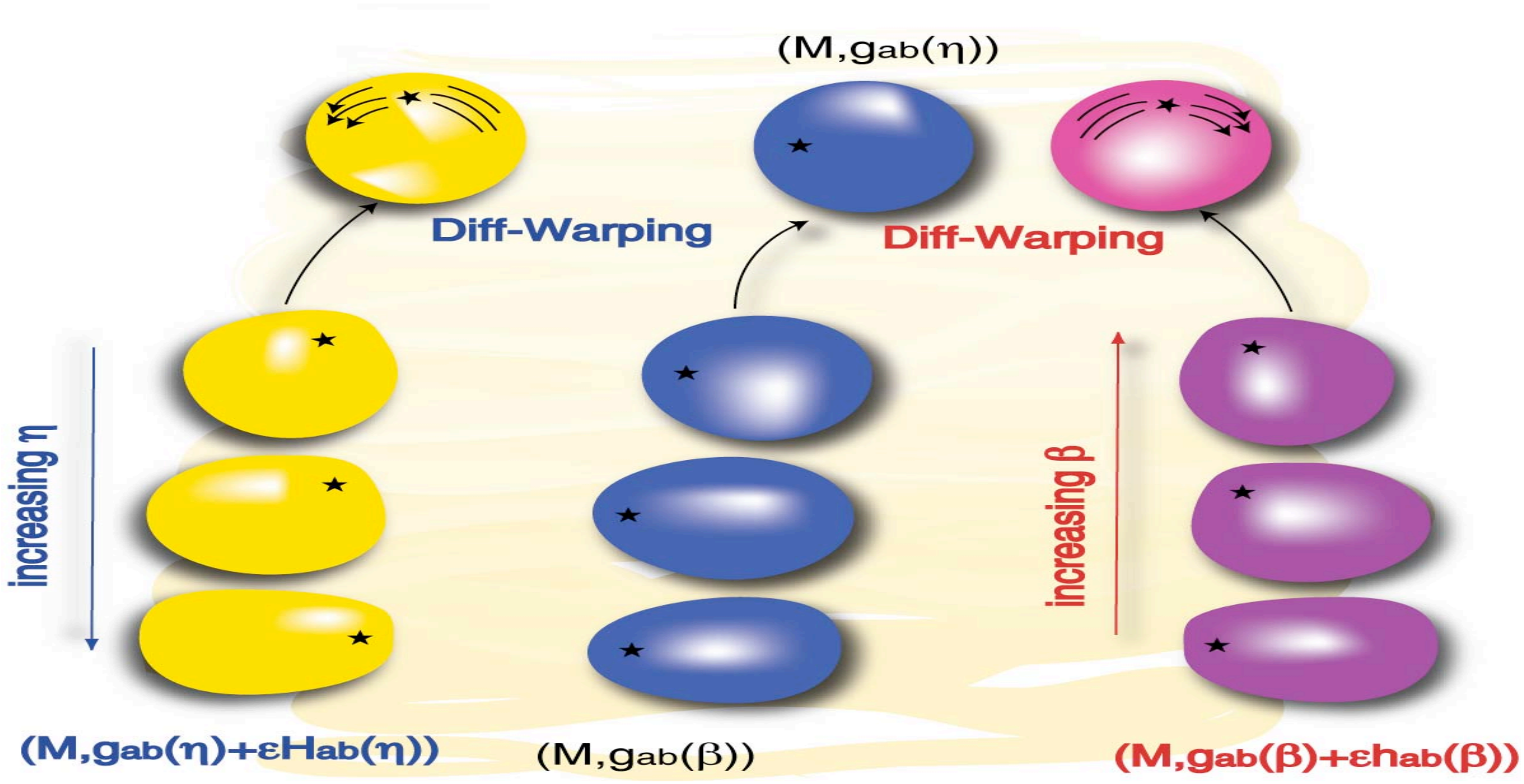
Along with the Ricci and the backward Ricci flow it is natural to consider the: Linearized Ricci flow and the Conjugate Linearized Ricci flow

representing the evolution of infinitesimal deformations

$$g^{(t)}_{ab}(\beta) = g_{ab}(\beta) + t\, h_{ab}(\beta),$$

$$g^{ab}_{(t)}(\eta) = g^{ab}(\eta) + t\, H^{ab}(\eta),$$

$$\frac{d}{d\eta}\int_{\Sigma} H^{ab}(\eta)\, h_{ab}(\eta)\, d\mu_{g(\eta)} = 0\,.$$



The flows $\beta \mapsto h_{ab}(\beta)$ and $\eta \mapsto H^{ab}(\eta)$ are associated with the weakly-parabolic initial value problems

$$\frac{\partial}{\partial\beta} h_{ab} = \Delta_L h_{ab} + 2\,[\delta^*\,\delta\, G(h)]_{ab}\ ,$$

$$h_{ab}(\beta = 0) = h_{ab}\ .$$

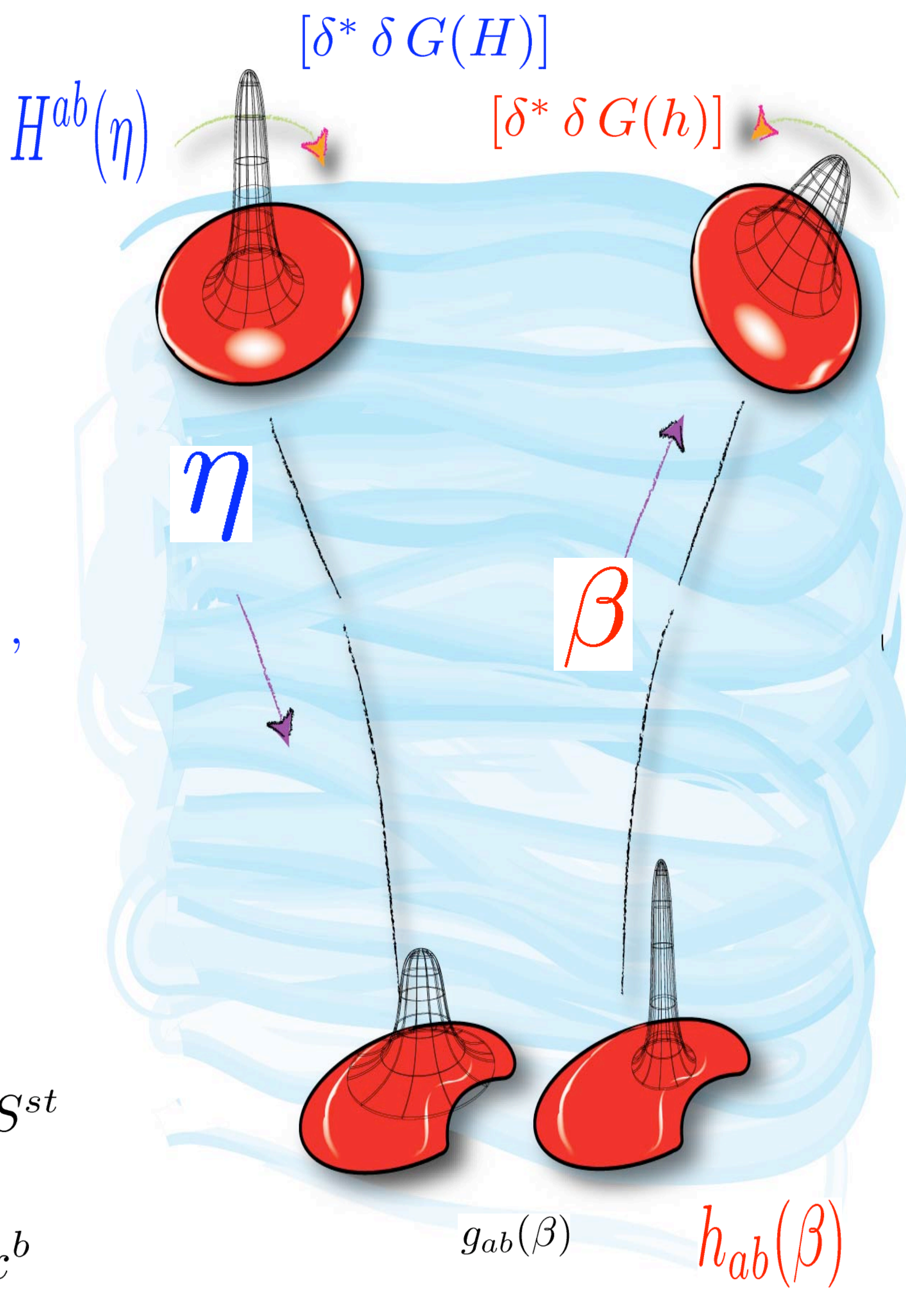
$$\frac{\partial}{\partial\eta} H^{ab} = \Delta_L H^{ab} + 2\,[\delta^*\,\delta\, G(H)]^{ab} - R\, H^{ab}\ ,$$

$$H^{ab}(\eta = 0) = H^{ab}\ .$$

$$\frac{d}{d\eta}\int_{\Sigma} H^{ab}(\eta)\, h_{ab}(\eta)\, d\mu_{g(\eta)} = 0\ .$$

$$\Delta_L S_{ab} \doteq \triangle S_{ab} - R_{as} S^s_b - R_{bs} S^s_a + 2 R_{asbt} S^{st}$$

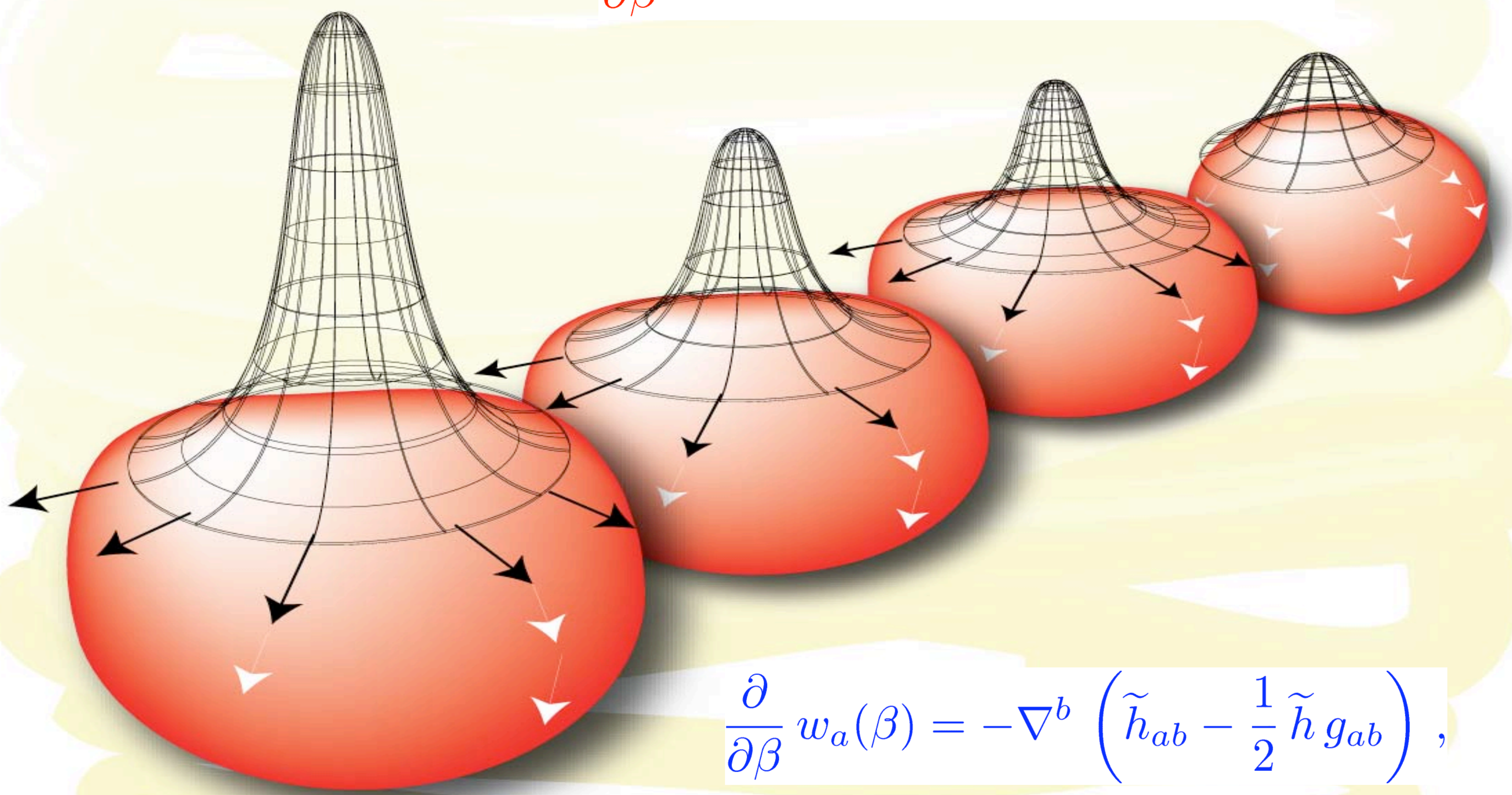
$$G(g, S) \doteq \left(S_{ab} - \frac{1}{2} g^{ik} S_{ik}\, g_{ab}\right) dx^a \otimes dx^b$$



These flows can be made manifestly parabolic:

*e.g.* for the forward flow:

$$\frac{\partial}{\partial\beta}\,\widetilde{h}_{ab} = \Delta_L \widetilde{h}_{ab}\,, \quad \widetilde{h}_{ab}(\beta = 0) = h_{ab}$$



$$\frac{\partial}{\partial\beta}\,w_a(\beta) = -\nabla^b \left(\widetilde{h}_{ab} - \frac{1}{2}\,\widetilde{h}\,g_{ab}\right)\,,$$

$$h_{ab}(\beta) \doteq \widetilde{h}_{ab}(\beta) + \mathcal{L}_{w(\beta)}\,g_{ab}(\beta),$$

Henceforth, when we speak of the Linearized Ricci flow we shall always mean the flow solution, for $0 \leq \beta < T_0$, of the parabolic i.v.p. defined on $T\mathcal{M}et(\Sigma)$ by

$$\frac{\partial}{\partial\beta} g_{ab}(\beta) = -2R_{ab}(\beta), \quad g_{ab}(\beta = 0) = g_{ab},$$

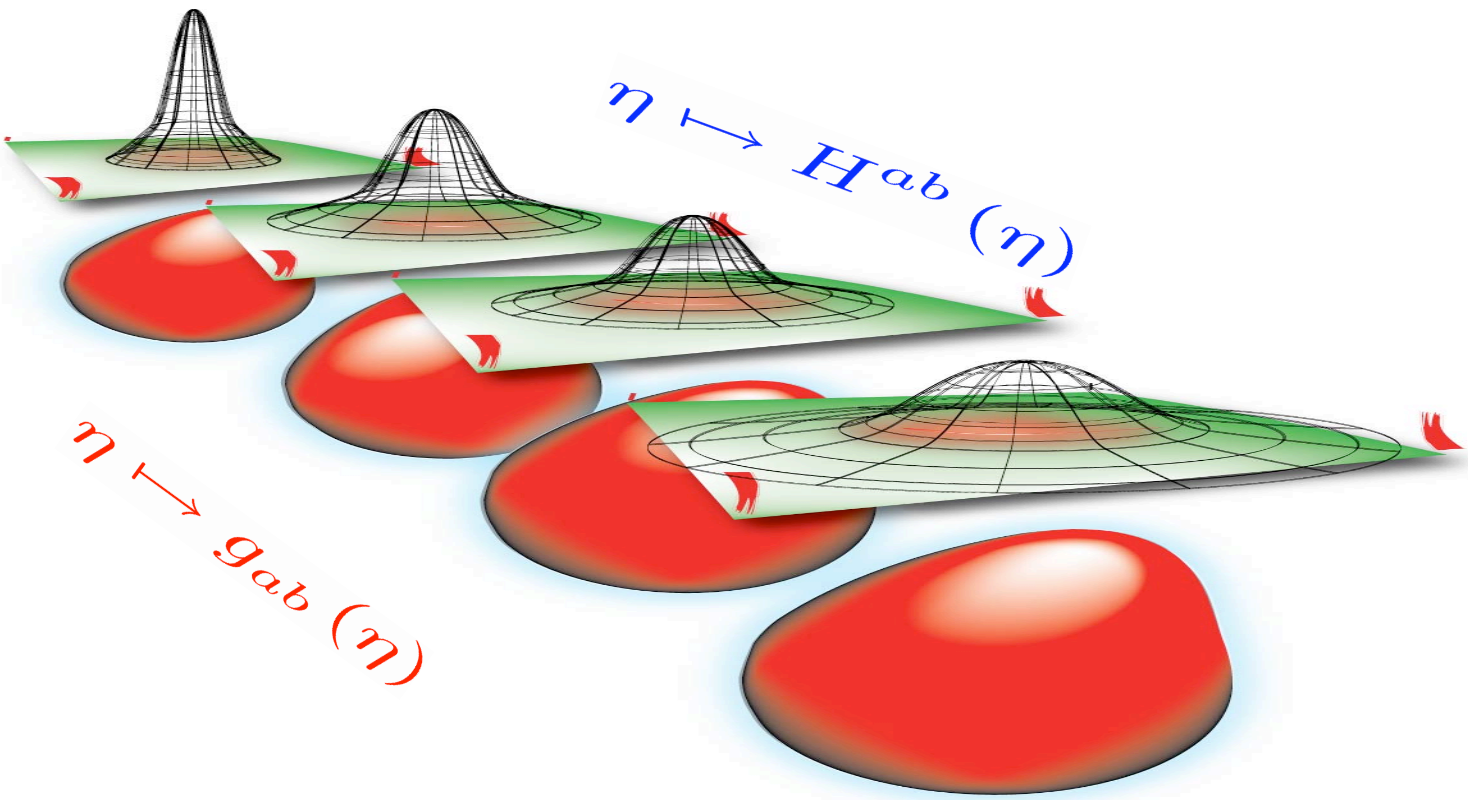
$$\frac{\partial}{\partial\beta} h_{ab} = \Delta_L h_{ab}\,, \quad h_{ab}(\beta = 0) = h_{ab}$$

[redacted]

And the Conjugate Linearized Ricci flow can be represented by the flow solution, for $0 \leq \eta \leq \beta^*$, of the parabolic i.v.p. defined on $T\,\mathcal{M}et(\Sigma)$ by

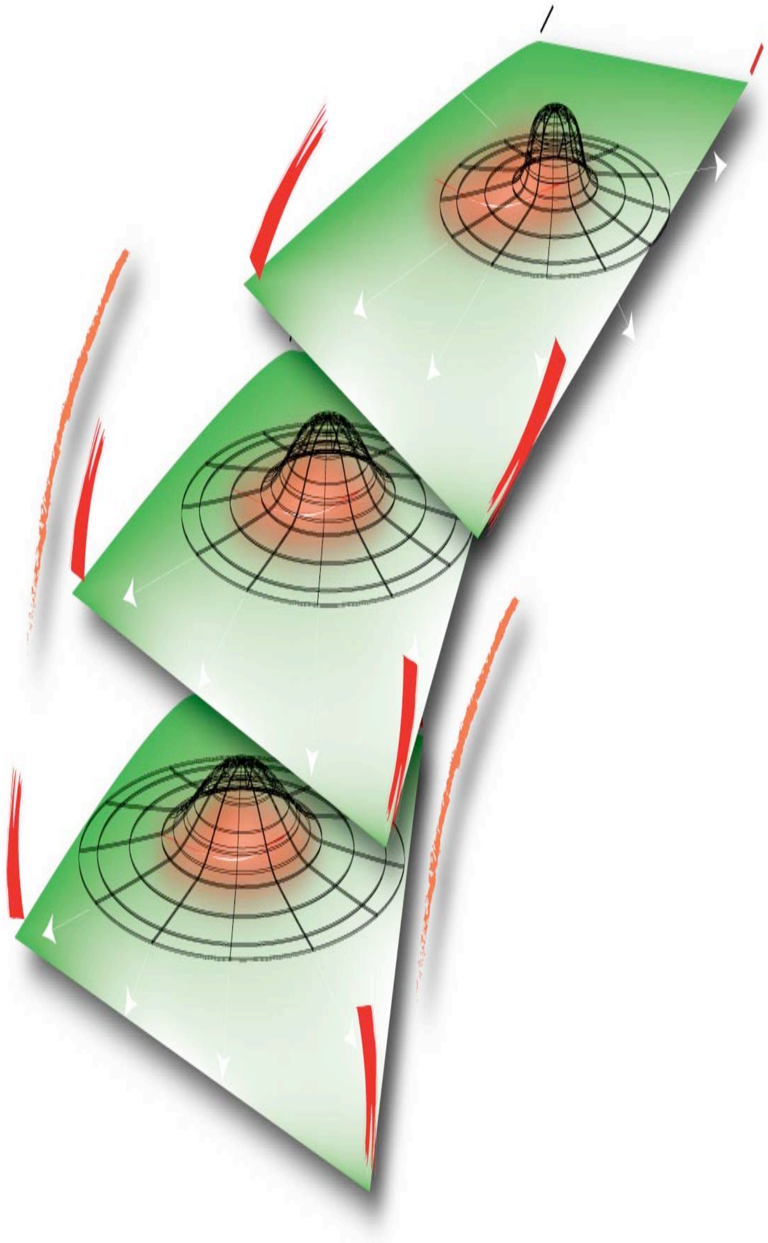
$$\frac{\partial}{\partial \eta} g_{ab}(\eta) = 2R_{ab}(\eta), \;\; g_{ab}(\eta = 0) = g_{ab}(\beta^*),$$

$$\frac{\partial}{\partial \eta} H^{ab} = \Delta_L H^{ab} - R\, H^{ab}\,, \quad H^{ab}(\eta = 0) = H^{ab}$$



The Linearized Ricci flow preserves the subspace $Im\,\delta^*_{g(\beta)}$ of $T_{g(\beta)}\,\mathcal{M}et\,(\Sigma)$:

$$\frac{\partial}{\partial\beta}\mathcal{L}_{\vec{v}}\,g_{ab} = \Delta_L\mathcal{L}_{\vec{v}}\,g_{ab}\;,$$



where, along $\beta \mapsto g_{ab}(\beta)$, $0 \leq \beta < T_0$,
$\beta \mapsto v_a(\beta)$ denotes the flow solution of the parabolic initial value problem

$$\frac{\partial}{\partial\beta}v_a(\beta) = \triangle_d\, v_a(\beta),$$

$$v_a(\beta = 0) = v_a,$$

where $v \in C^\infty(\Sigma, T^*\Sigma)$ is a given covector field and

$$\triangle_d\, v_a(\beta) \doteq \triangle\, v_a(\beta) - \mathcal{R}^b_a\, v_b(\beta),$$

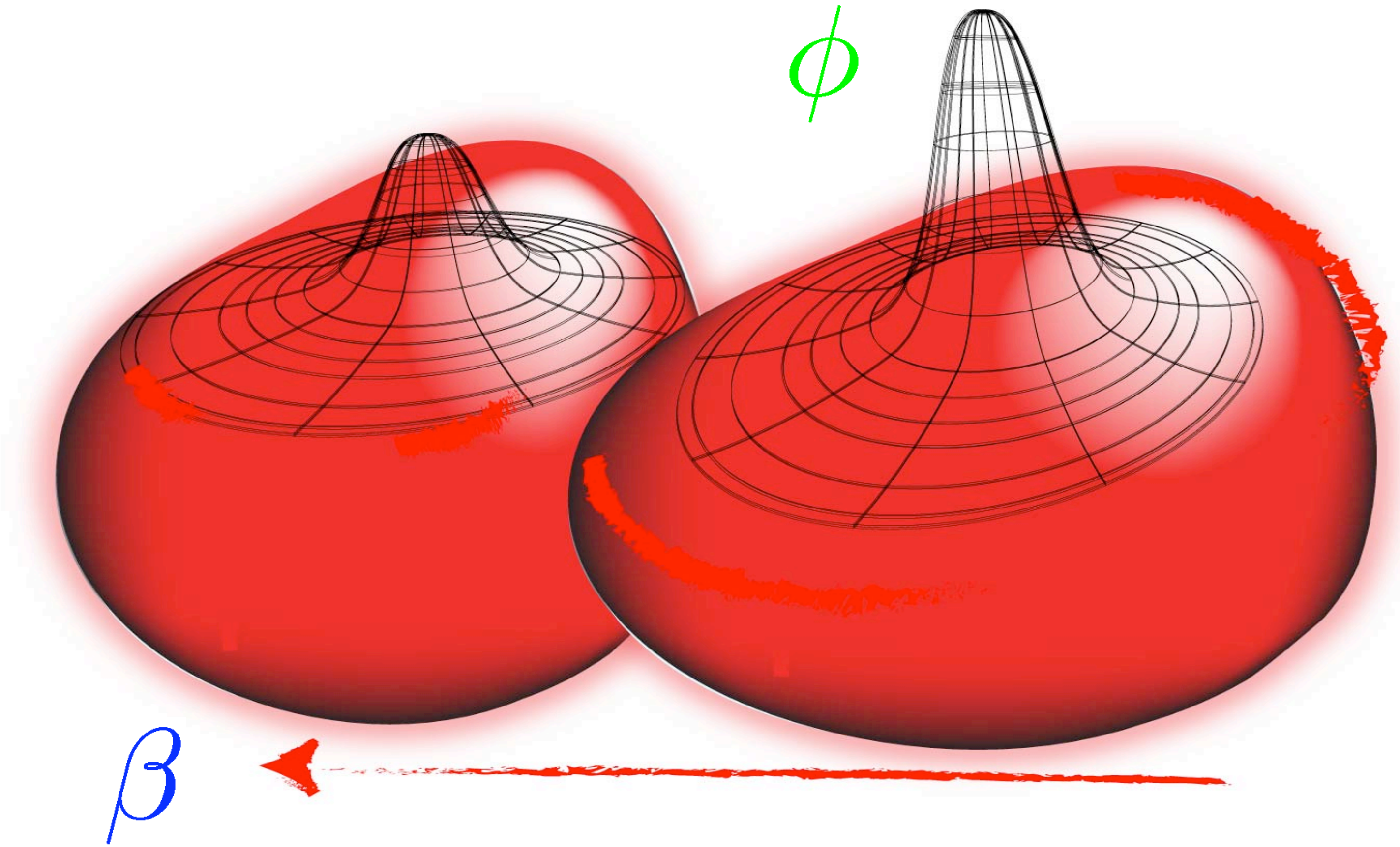
In Particular, if along a given Ricci flow $\beta \mapsto g_{ab}(\beta)$, $0 \leq \beta < T_0$, $\beta \mapsto \phi(\beta)$ denote the heat flow

$$\frac{\partial}{\partial\beta}\phi(\beta) = \triangle\, \phi(\beta),$$

$$\phi(\beta = 0) = \phi_{(0)},$$

Then

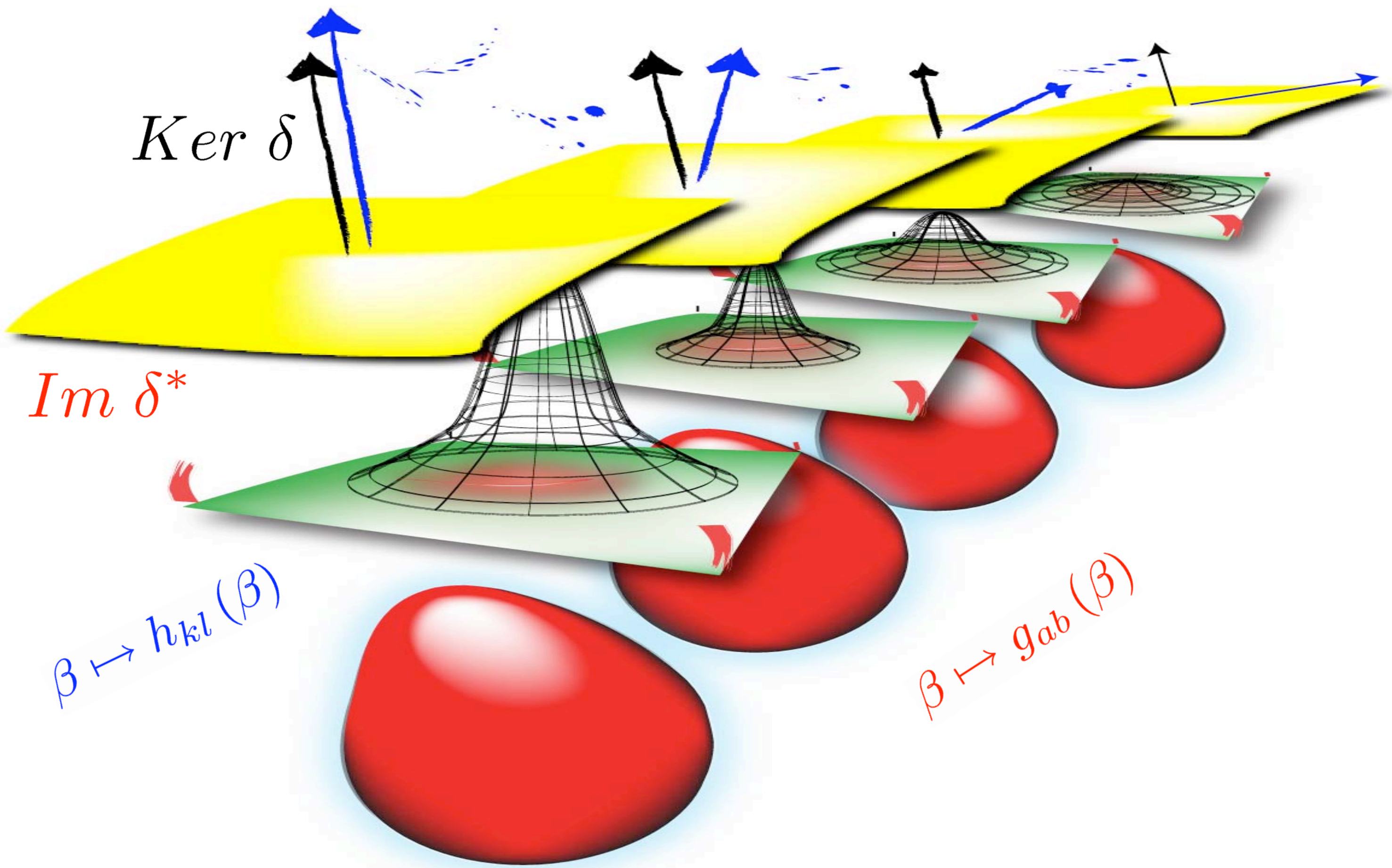
$$\frac{\partial}{\partial\beta}\, Hess\, \phi_{ab} = \Delta_L\, Hess\, \phi_{ab}\ ,$$



The Linearized Ricci flow does not preserve the subspace $Ker\,\delta_{g(\beta)}$ of $T_{g(\beta)}\,\mathcal{M}et\,(\Sigma)$: the evolution $\beta \mapsto \delta_{g(\beta)}\,h(\beta)$

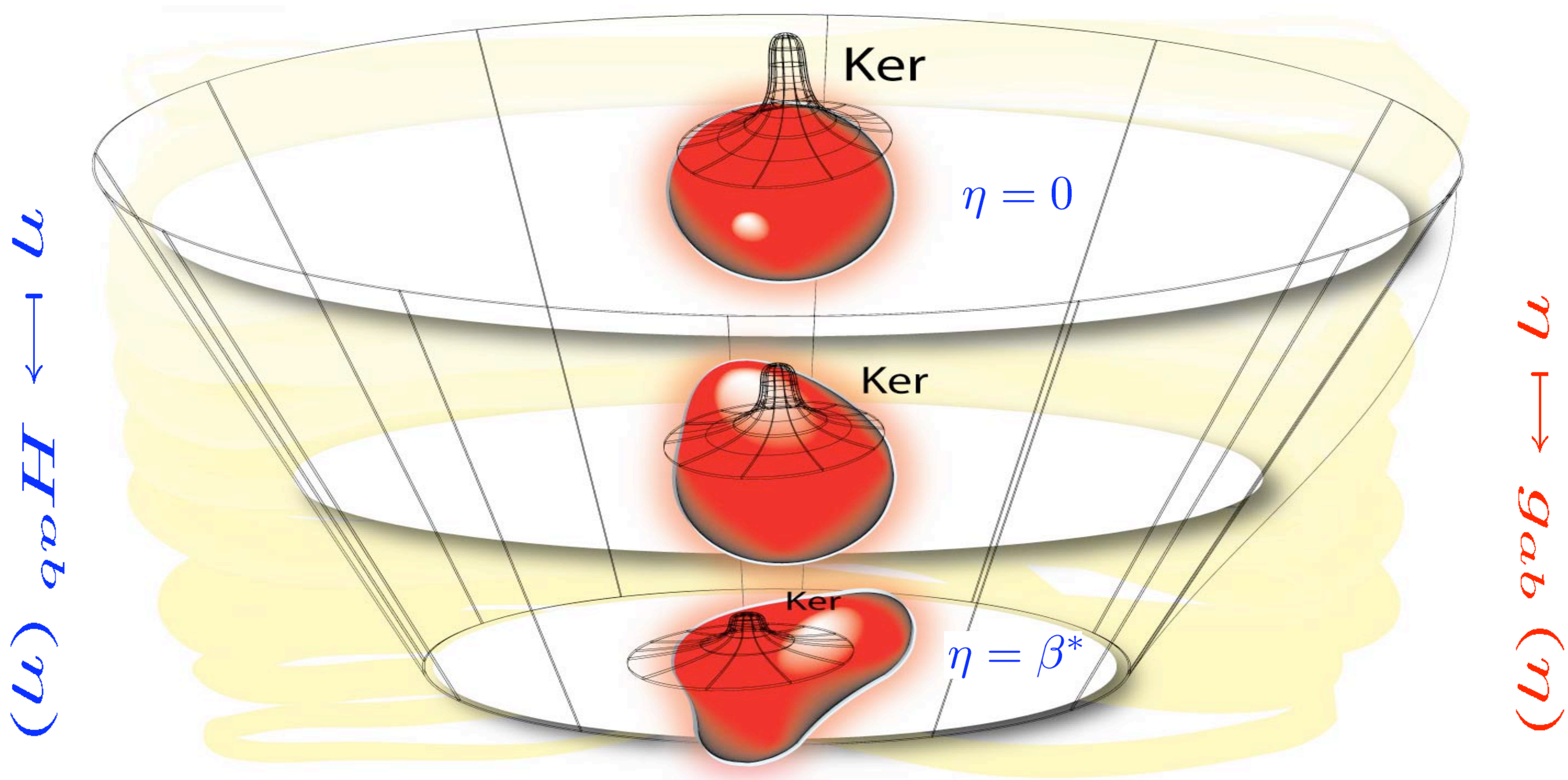
does not admit, in general, the solution $\beta \mapsto \nabla^a\,h_{ab}(\beta) = 0$, $0 \leq \beta < T_0$, if $\nabla^a\,h_{ab}(\beta = 0) = 0$

$$\frac{\partial}{\partial\beta}\,\nabla^k\,h_{kl} = \triangle\,\nabla^k\,h_{kl} - R^a_l\,\nabla^k\,h_{ka} + h_{ab}\,\nabla_l\,R^{ab} + 2R^{ik}\,\nabla_i\,h_{kl} - 2h_{ik}\,\nabla^i\,R^k_l\ ,$$



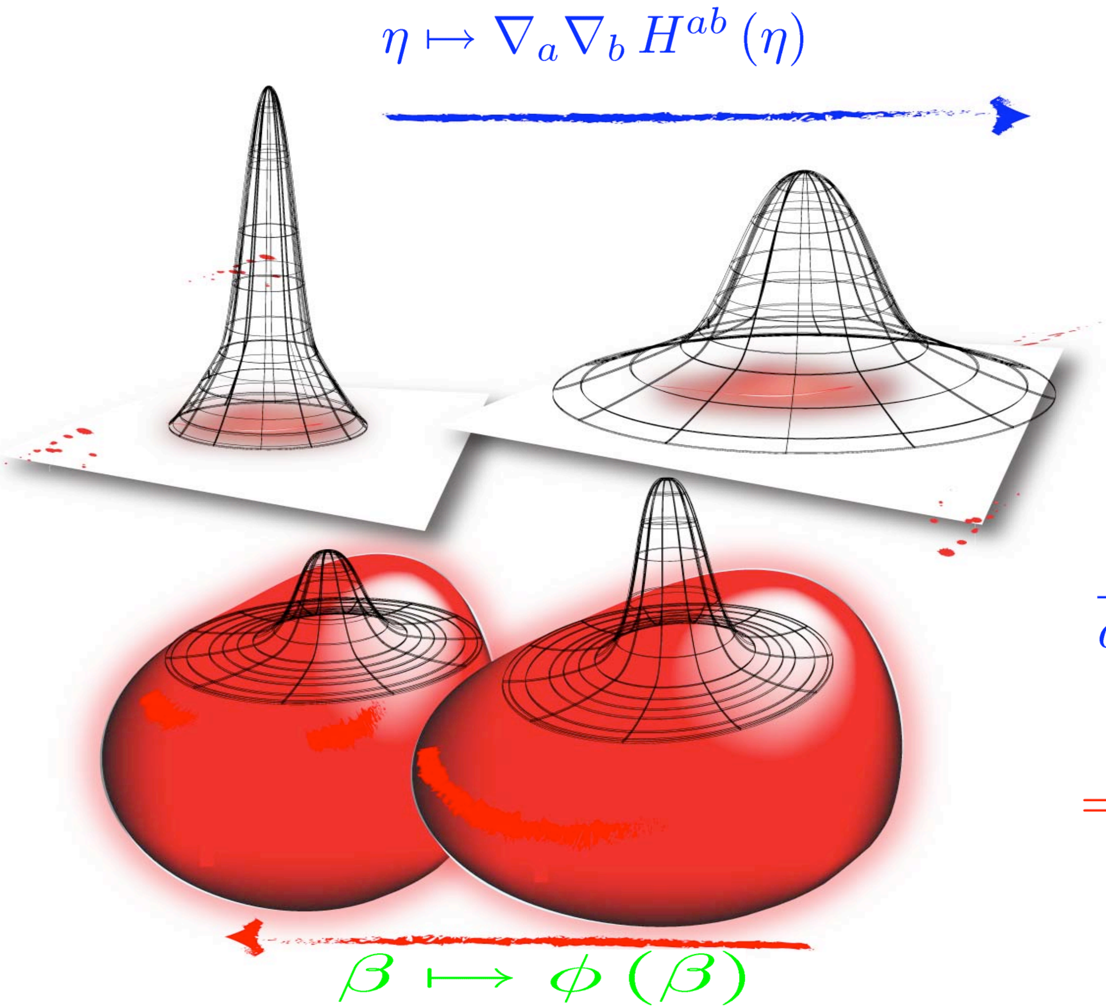
The subspace $Ker\,\delta_{g(\beta)}$ of $T_{g(\beta)}\,\mathcal{M}et\,(\Sigma)$ is preserved by the Conjugate Linearized Ricci flow since $\eta \mapsto \delta_{g(\eta)}\,H(\eta)$ evolves according to

$$\frac{\partial}{\partial\eta}\nabla_a\,H^{ab} = \triangle\,\nabla_a\,H^{ab} - R^b_a\,\nabla_j\,H^{aj} - R\,\nabla_a\,H^{ab}\;,$$



Note that $\eta \mapsto \nabla_a \nabla_b H^{ab}(\eta)$ evolves according to the Conjugate scalar heat equation

$$\frac{\partial}{\partial \eta} \nabla_a \nabla_b H^{ab} = \triangle \nabla_a \nabla_b H^{ab} - R \nabla_a \nabla_b H^{ab} \ ,$$




For a solution $\beta \mapsto \phi(\beta)$ of the forward heat equation, we have the conjugation

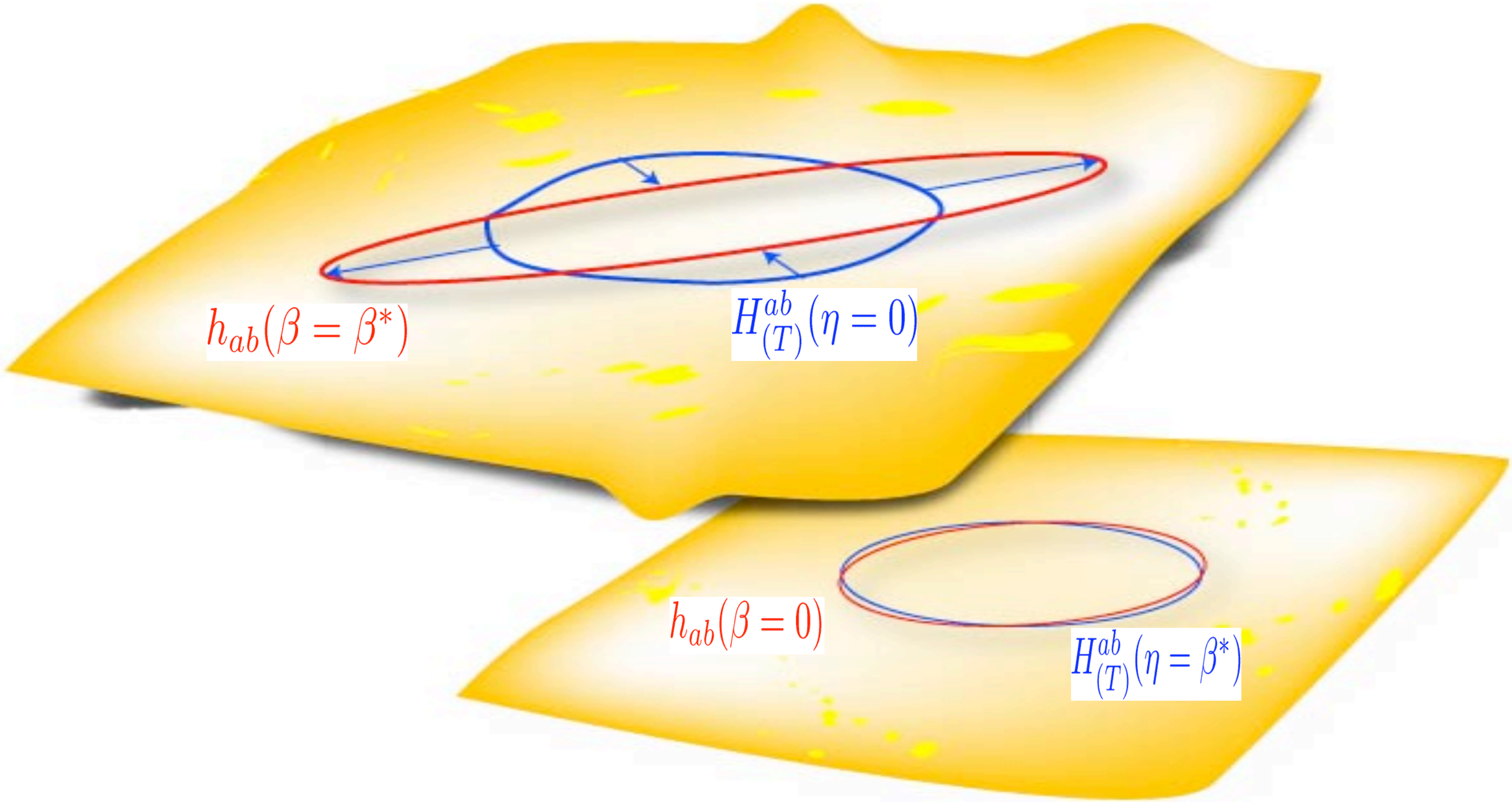
$$\frac{d}{d\eta} \int_\Sigma \phi \nabla_a \nabla_b H^{ab}(\eta)\, d\mu_{g(\eta)} = 0 \ .$$

$$= \frac{d}{d\eta} \int_\Sigma Hess_{ab}\phi \ H^{ab}(\eta)\, d\mu_{g(\eta)}$$

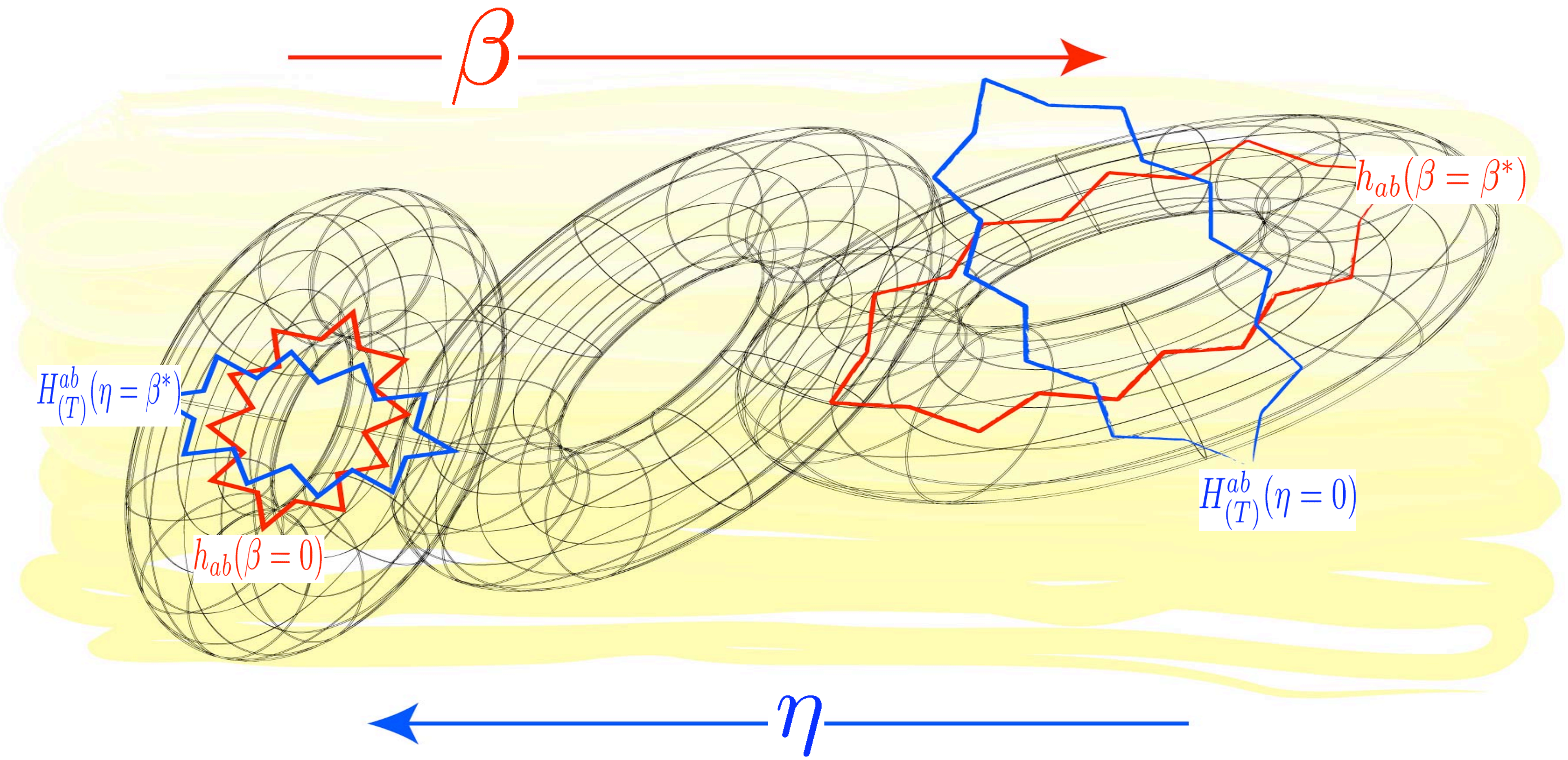
As a further consequence of $L^2$–conjugation, we have:

$$\frac{d}{d\eta}\int_\Sigma H^{ab}_{(T)}(\eta)\, h^{(T)}_{ab}(\eta)\, d\mu_{g(\eta)} = 0 \ ,$$

where $\nabla_a H^{ab}_{(T)}(\eta) = 0$ and $\nabla^a h^{(T)}_{ab}(\beta) = 0$.



**Prop.** If for the given Ricci flow $Ker\,\delta_{g(\beta)}|_{\eta=0}\neq\emptyset$, and $H^{ab}_{(T)}\in Ker\,\delta_{g(\beta)}|_{\eta=0}$, with $tr\,H_{(T)}|_{\eta=0}=0$, $\int_\Sigma H^{ab}_{(T)}(\eta)\,R_{ab}(\eta)\,d\mu_{g(\eta)}|_{\eta=0}=0$, then any solution $\beta\mapsto h_{ab}(\beta)$ of the linearized Ricci flow with $\int_\Sigma H^{ab}_{(T)}(\beta)\,h_{ab}(\beta)\,d\mu_{g(\beta)}|_{\beta=0}\neq 0$, provides a non–trivial perturbation of the given Ricci flow $\beta\mapsto g_{ab}(\beta)$.



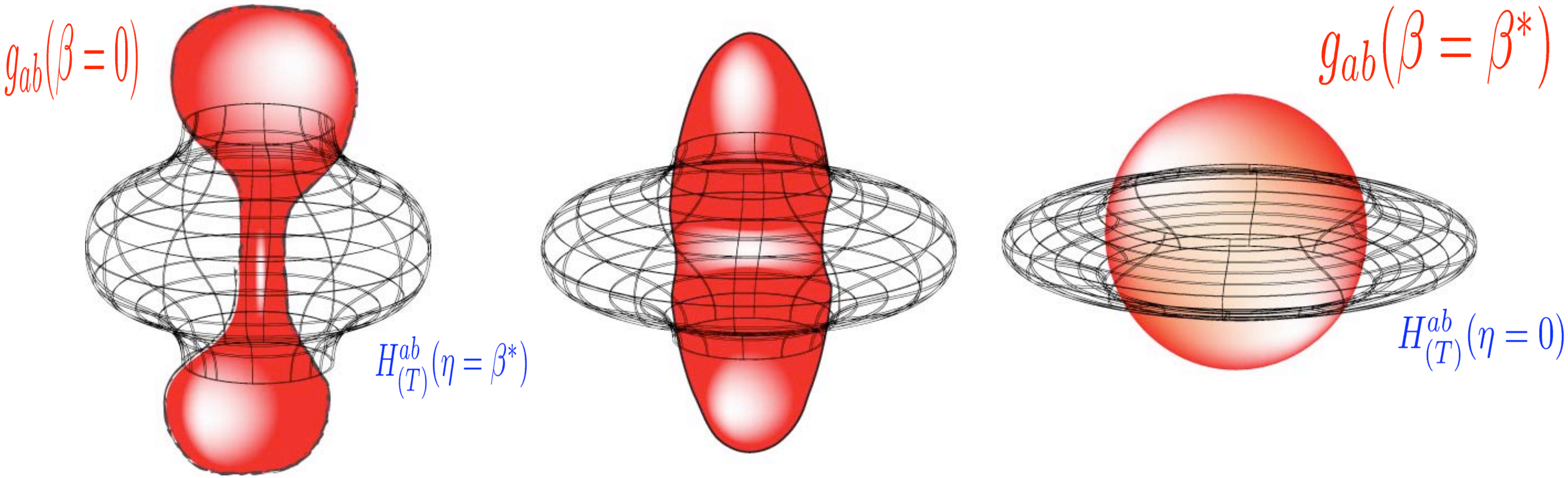
Averaging Properties of the Conjugate Linearized Ricci Flow:

**Prop.** Let $\eta \mapsto (\Sigma, g(\eta))$, $\eta \in [0, \beta^*]$ be a given backward Ricci flow, and let $\eta \mapsto H^{ab}(\eta)$, $\eta \in [0, \beta^*]$, $H^{ab}(\eta = 0) = H^{ab}_{\beta^*}$ denote the corresponding solution of the Conjugate Linearized Ricci Flow. Then,

$$\frac{d}{d\eta}\int_{\Sigma} H^{ab}(\eta)\,\mathcal{R}_{ab}(\eta)\,d\mu_{g(\eta)} = 0\ ,$$

and

$$\frac{d}{d\eta}\int_{\Sigma} H^{ab}(\eta)\,\left[g_{ab}(\eta) - 2\eta\,\mathcal{R}_{ab}(\eta)\right]\,d\mu_{g(\eta)} = 0\ .$$



**Prop.** If for the given Ricci flow $Ker\,\delta_{g(\beta)}|_{\eta=0}\neq\emptyset$, $H^{ab}_{(T)}\in Ker\,\delta_{g(\beta)}|_{\eta=0}$, then if $g_{ab}(\beta)$ evolves towards a non–trivial Ricci soliton structure

$$R_{ab}+L_{\vec{v}}\,g_{ab}-\frac{1}{2\eta}\,g_{ab}=0\,,$$

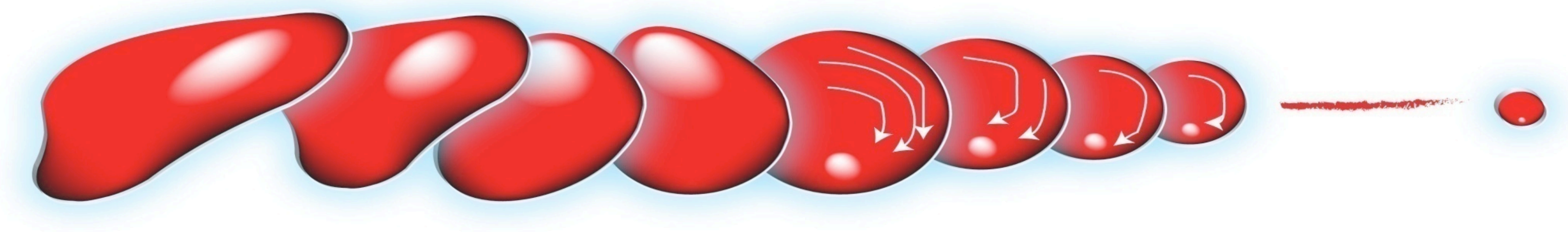
then necessarily

$$\int_{\Sigma}H^{ab}(\beta^*)\;[g_{ab}-2\beta^*\,\mathcal{R}_{ab}]\;d\mu_g\Big|_{\beta=0}=0\;.$$

Conversely, if for every $\Psi^{ab}\in Ker\,\delta_{g(\beta)}\big|_{\beta=0}$

$$\int_{\Sigma}\Psi^{ab}\;[g_{ab}-2\beta^*\,\mathcal{R}_{ab}]\;d\mu_g\Big|_{\beta=0}\neq 0\;.$$

then $\beta\mapsto g_{ab}(\beta)$, $0\leq\beta\leq\beta^*$ cannot evolve into a non–trivial (shrinking) Ricci soliton.



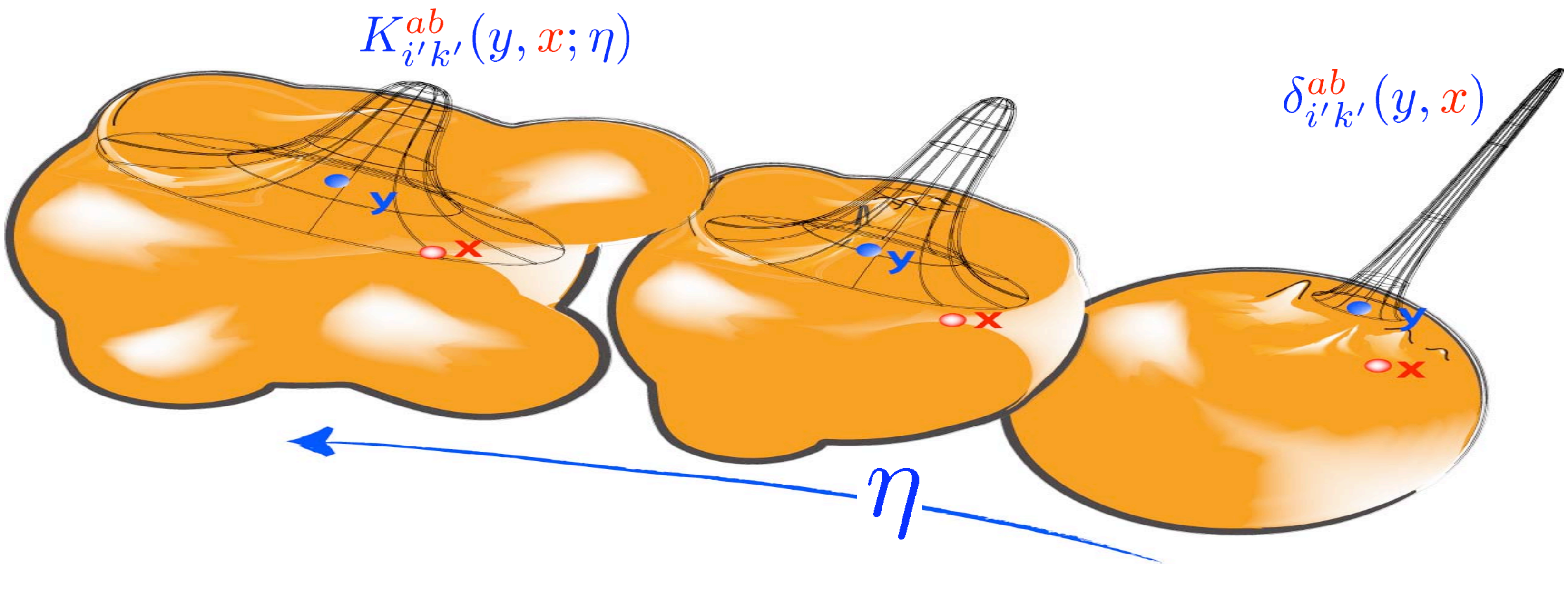
## Heat Kernel for the conjugate linearized Ricci Flow

If $(\Sigma, g_{ab}(\eta))$ is a smooth solution to the backward Ricci flow on $\Sigma_\eta \times [0, \beta^*]$ with bounded curvature, then we can consider the $g(\eta)$–dependent fundamental solution $K^{ab}_{i'k'}(y, x; \eta)$ of the conjugate linearized Ricci flow, *i.e.*,

$$\frac{\partial}{\partial \eta} K^{ab}_{i'k'}(y, x; \eta) = \Delta^{(\eta, x)}_{L} K^{ab}_{i'k'}(y, x; \eta) - \mathcal{R}(\eta, x) K^{ab}_{i'k'}(y, x; \eta) ,$$

$$\lim_{\eta \searrow 0^+} K^{ab}_{i'k'}(y, x; \eta) = \delta^{ab}_{i'k'}(y, x) ,$$

where $(y, x; \eta) \in (\Sigma \times \Sigma \backslash Diag(\Sigma \times \Sigma)) \times [0, \beta^*]$, $\eta \doteq \beta^* - \beta$, $\Delta^{(\eta,x)}_{L}$ denotes the $g(\eta)$–dependent Lichnerowicz–DeRham laplacian with respect to the variable $x$.

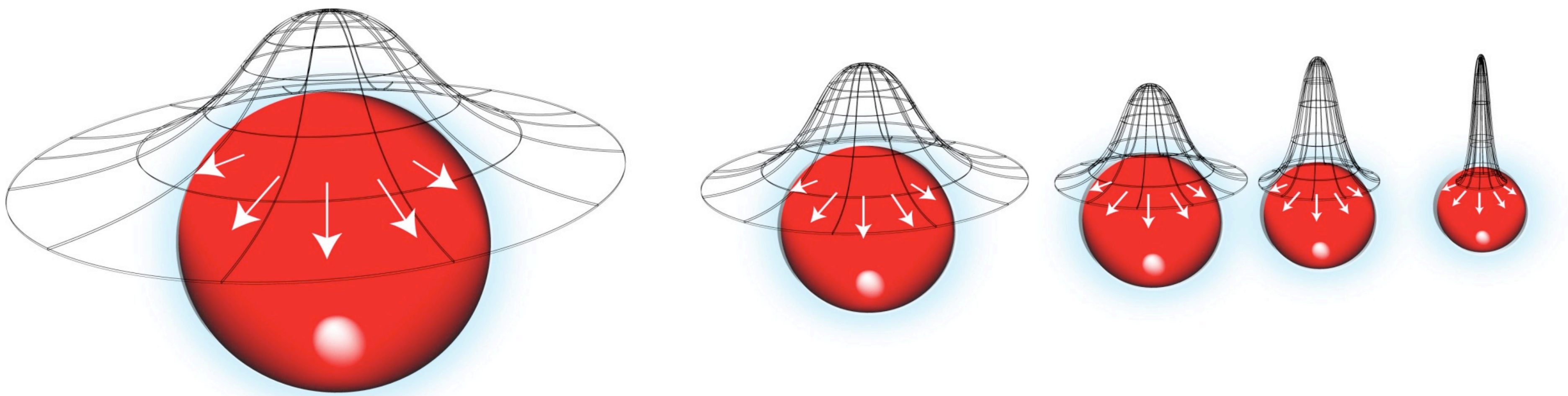


The heat kernel $K^{ab}_{i'k'}(y,x;\eta)$ can be naturally normalized along the expanding soliton on $S^3$ : Let $\bar{g}_{ab}$ the round metric on the unit 3–sphere $S^3$, and, for $\eta \in [0.\beta^*]$, let $\eta \mapsto 4\,(T_0 - \beta^* + \eta)\,\bar{g}_{ab}$ be the expanding Ricci soliton on $S^3$ with initial radius $r(\eta = 0) = 2\sqrt{T_0 - \beta^*}$ and final radius $r(\eta = \beta^*) = 2\sqrt{T_0}$. Then

$$\frac{r(\eta)^3}{3}\int_\Sigma \bar{g}^{i'k'}(y)\,K^{ab}_{i'k'}(y,x;\eta)\,\bar{g}_{ab}(x)\,d\bar{\mu}_{g(x)} = 1\;,$$

where $d\bar{\mu}_{g(x,\eta)}$ is the volume element on $(S^3, \bar{g}_{ab})$.

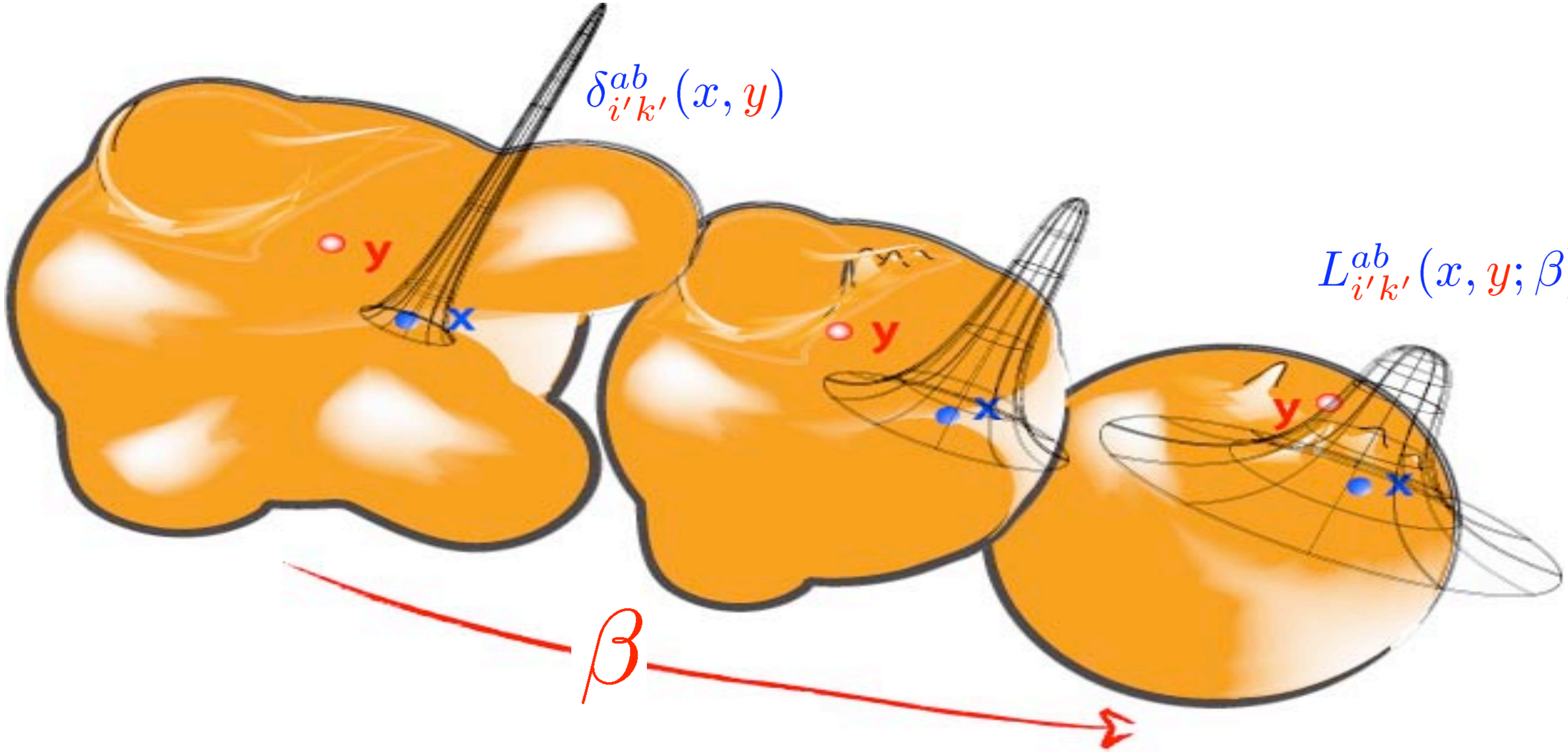
Moreover, if $(\Sigma, g_{ab}(\eta))$ is a smooth solution to a backward Ricci flow of bounded geometry on $\Sigma_\eta \times [0, \beta^*]$ with non–negative curvature operator, then $K^{ab}_{i'k'}(y,x;\eta)$, $0 \leq \eta \leq \beta^*$, is a positive integral kernel.



Similarly, along the Ricci flow on $\Sigma_\beta \times [0, \beta^*]$ we can consider the $g(\beta)$–dependent fundamental solution $L^{ab}_{i'k'}(x, y; \beta)$ of the linearized Ricci flow, *i.e.*,

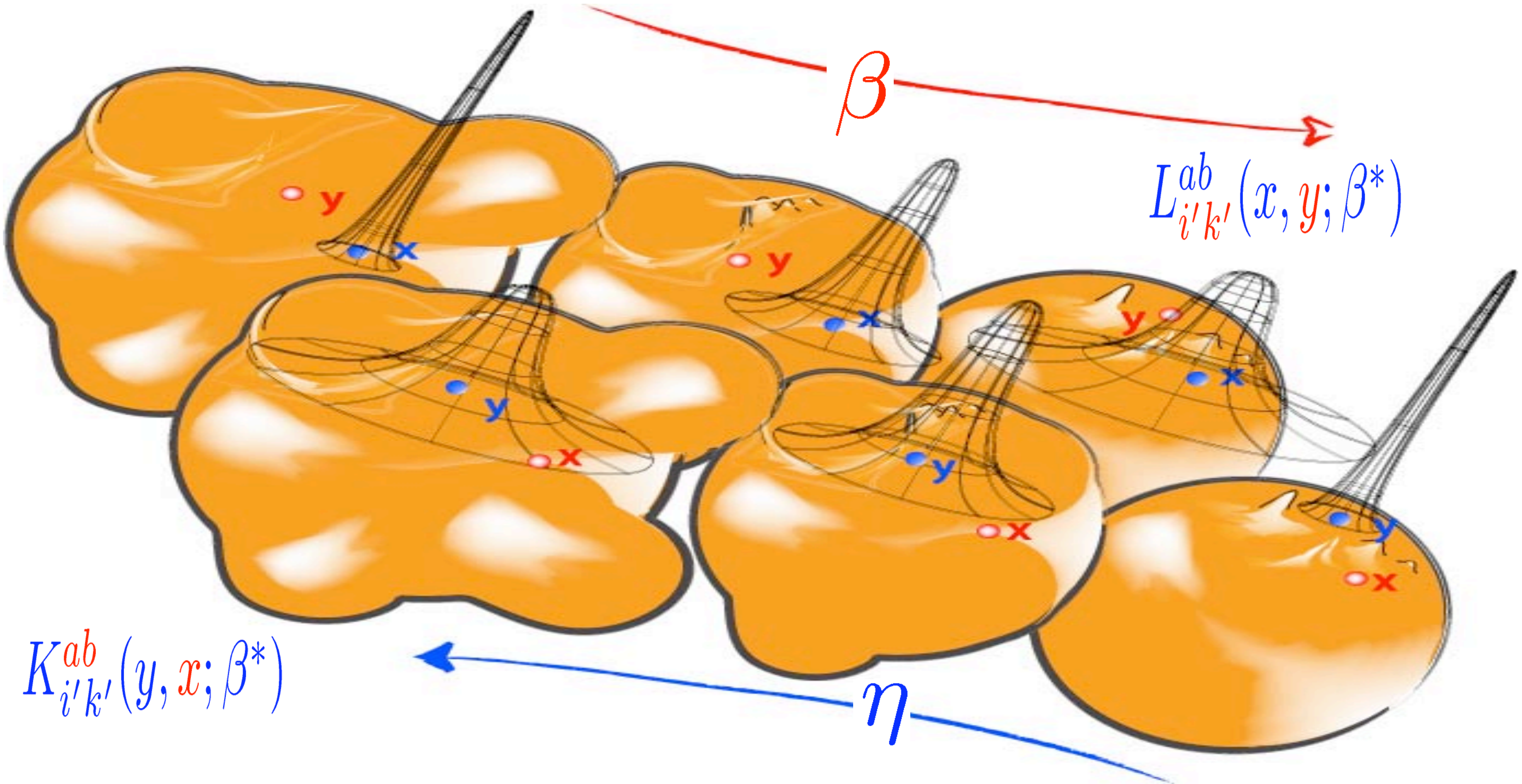
$$\frac{\partial}{\partial \beta} L^{ab}_{i'k'}(x, y; \beta) = \Delta^{(\beta, y)}_{L} L^{ab}_{i'k'}(x, y; \beta) ,$$

$$\lim_{\beta \searrow 0^+} L^{ab}_{i'k'}(x, y; \beta) = \delta^{ab}_{i'k'}(x, y) ,$$



Along the given Ricci Flow,
$\beta \mapsto g_{ab}(\beta)$, $0 \leq \beta \leq \beta^* < T_0$,
the $L^2$–duality between the
linearized and the conjugate
linearized flow implies

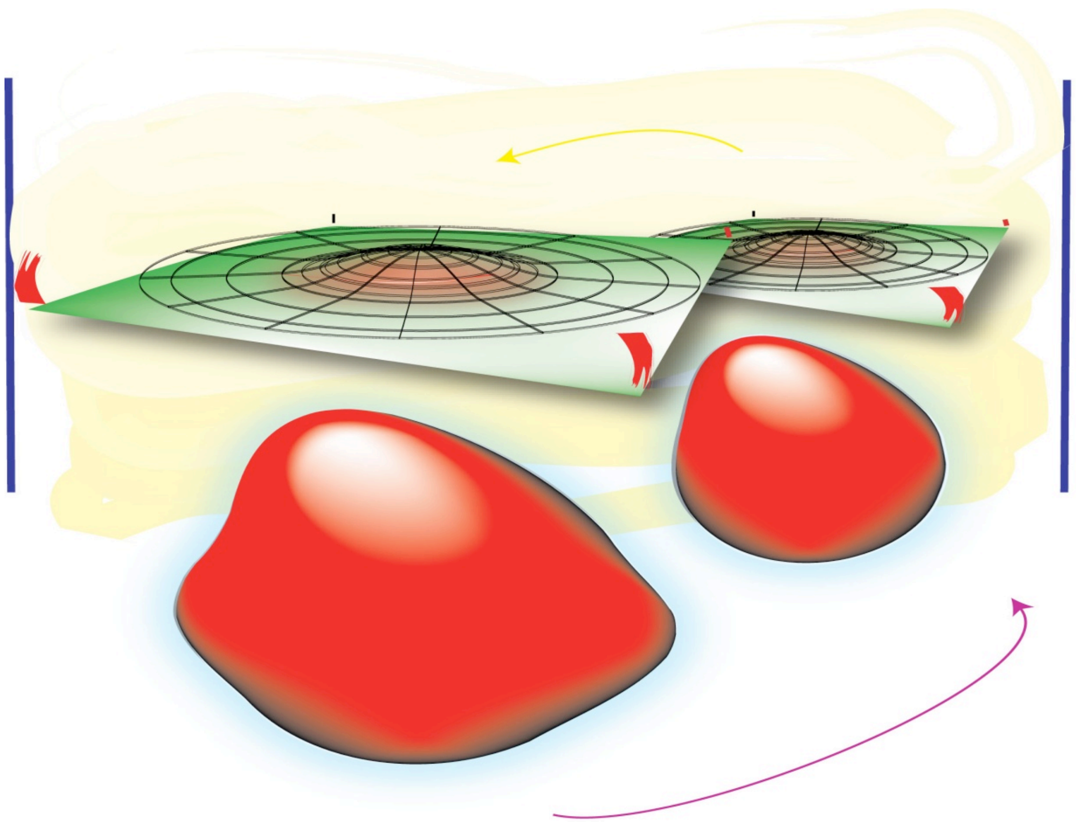
$$K^{ab}_{i'k'}(y, x; \eta = \beta^*) = L^{ab}_{i'k'}(x, y; \beta = \beta^*)$$



By applying Anderson and Chow's pinching estimate for the linearized flow it follows that if along the given Ricci Flow, we let $\rho \in [0, \infty)$ be such that $R_{min}(\beta = 0) + \rho > 0$, then there exists a constant $C = C(g_0, \rho, T_0)$ such that

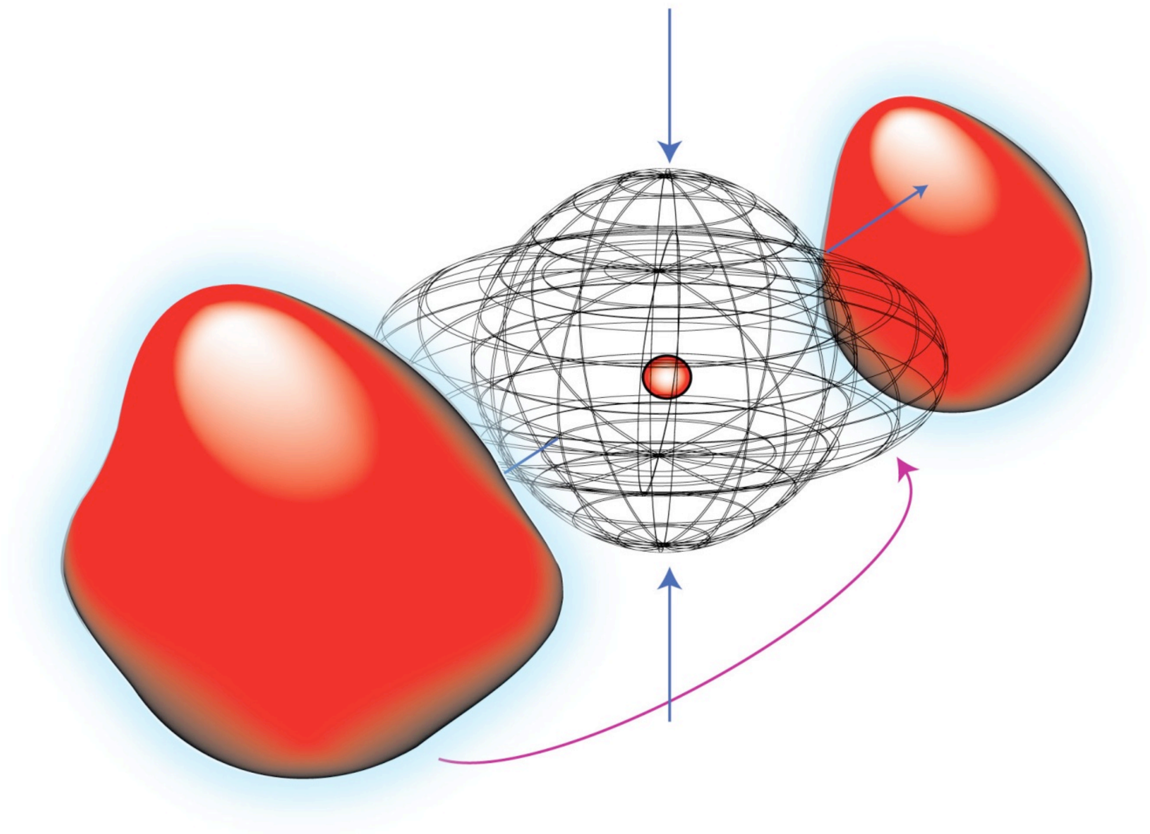
$$\left|K^{ab}_{i'k'}(y, x; \eta)\right| \doteq [K^{ab}_{i'k'}(y, x; \eta) K^{i'k'}_{ab}(y, x; \eta)]^{\frac{1}{2}} \leq C[R(\eta) + \rho] \ ,$$

for all $0 < \eta \leq \beta^*$.



Moreover, if $\eta \mapsto g_{ab}(\eta)$, $\eta \doteq \beta^* - \beta$, is a backward Ricci flow of bounded geometry on $\Sigma_\eta \times [0, \beta^*]$ with $\mathcal{R}(\eta) \geq 0$, $\eta \doteq \beta^* - \beta$, then.

$$\frac{d}{d\eta} \int_{\Sigma} \left| \nabla_a K^{ab}_{i'k'}(y, x; \eta) \right|^2 d\mu_{g(\eta)} \leq 0 \, ,$$

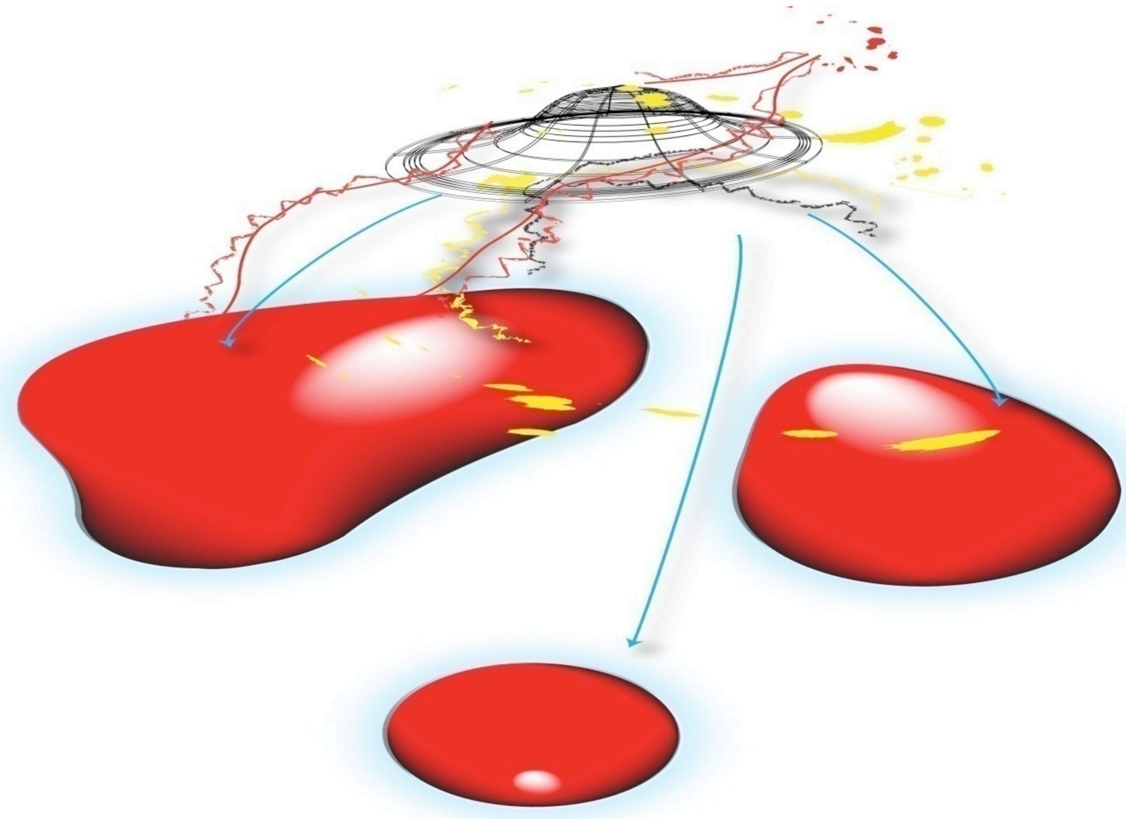


Finally, by applying Chow–Hamilton's linear Harnack inequality for the linearized flow, if $\beta \mapsto g_{ab}(\beta)$, is a Ricci flow of bounded geometry with non–negative curvature operator on $\Sigma_\eta \times [0, \beta^*]$, then, for $\eta = \beta^*$, we get by duality

$$K^{ab}_{i'k'}(y, x; \eta) \left[\frac{g_{ab}(x,\eta)}{2\beta^*} + \mathcal{R}_{ab}(x, \eta)\right] + \nabla_a \nabla_b K^{ab}_{i'k'}(y, x; \eta) +$$

$$+ 2 \nabla_a K^{ab}_{i'k'}(y, x; \eta) V_b + K^{ab}_{i'k'}(y, x; \eta) V_a V_b \Big|_{\eta = \beta^*} \geq 0 \,,$$

for any 1–form $V$.



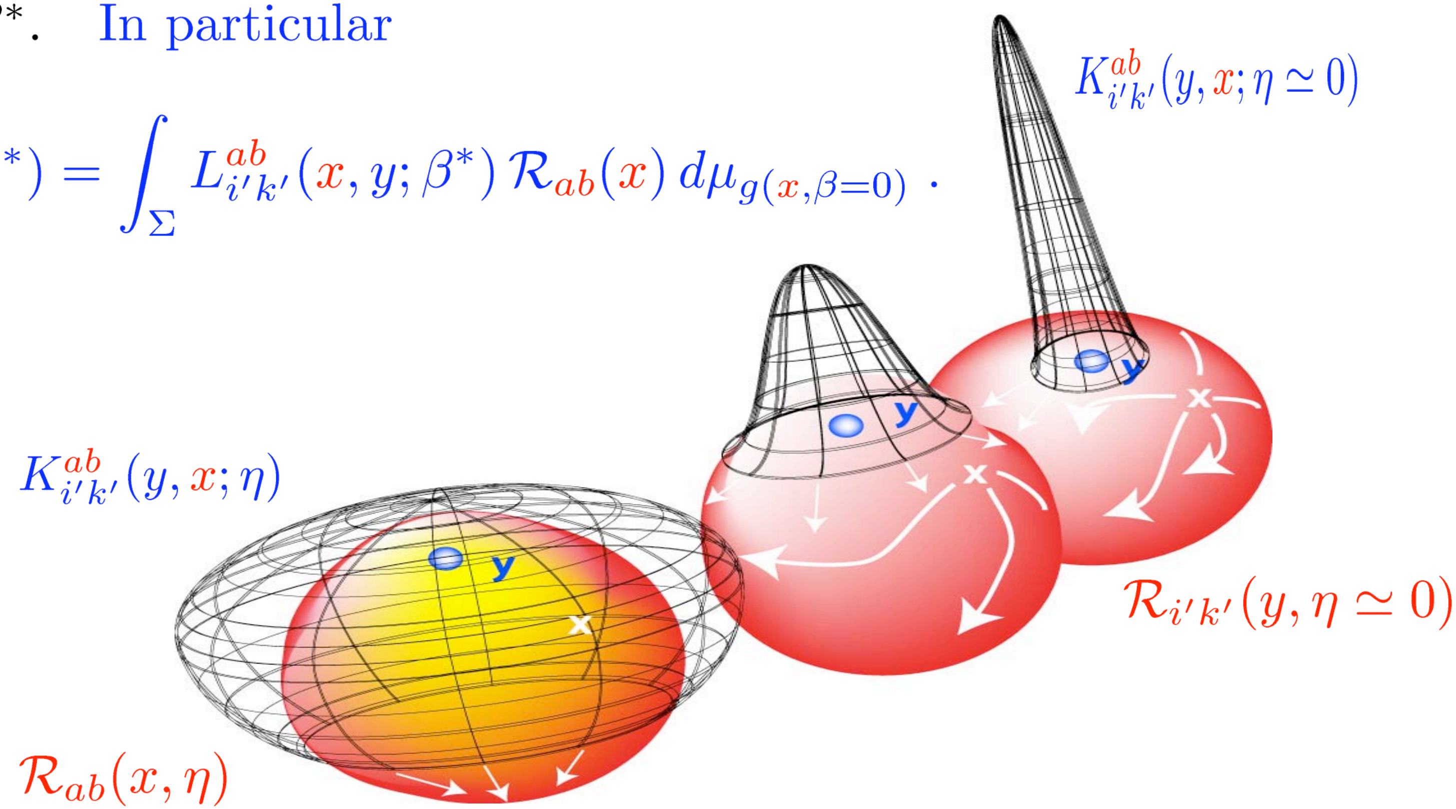
## $K^{ab}_{i'k'}(y,x;\eta)$ is the Heat Kernel for the Ricci Flow

Let $\eta \mapsto g_{ab}(\eta)$ be a backward Ricci flow with bounded geometry on $\Sigma_\eta \times [0,\beta^*]$, and let $K^{ab}_{i'k'}(y,x;\eta)$ be the (backward) heat kernel of the corresponding conjugate linearized Ricci flow, then

$$\mathcal{R}_{i'k'}(y,\eta=0) = \int_\Sigma K^{ab}_{i'k'}(y,x;\eta)\,\mathcal{R}_{ab}(x,\eta)\,d\mu_{g(x,\eta)}\ ,$$

for all $0 \le \eta \le \beta^*$. In particular

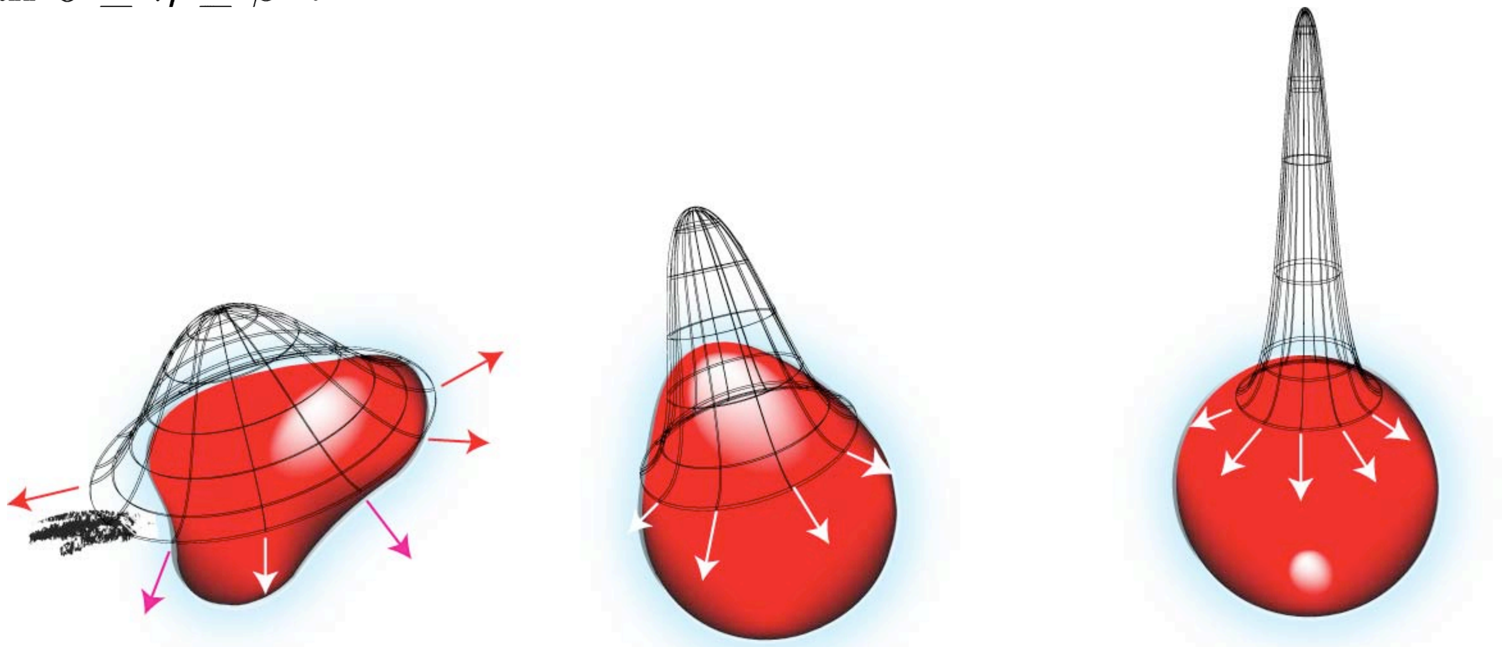
$$R_{i'k'}\,(y,\beta^*) = \int_\Sigma L^{ab}_{i'k'}(x,y;\beta^*)\,\mathcal{R}_{ab}(x)\,d\mu_{g(x,\beta=0)}\ .$$



and we have the following integral representation of the backward Ricci flow on $\Sigma_\eta \times (0, \beta^*]$

$$g_{i'k'}\,(y, \eta = 0) = \int_\Sigma K^{ab}_{i'k'}(y, x; \eta)\;[g_{ab}(x, \eta) - 2\eta\;\mathcal{R}_{ab}(x, \eta)]\;d\mu_{g(x,\eta)}\;.$$

for all $0 \leq \eta \leq \beta^*$.



or equivalently

$$g_{i'k'}\,(y, \beta^*) = \int_\Sigma L^{ab}_{i'k'}(x, y; \beta^*)\;[g_{ab}(x) - 2\beta^*\;\mathcal{R}_{ab}(x)]\;d\mu_{g(x,\beta=0)}\;.$$

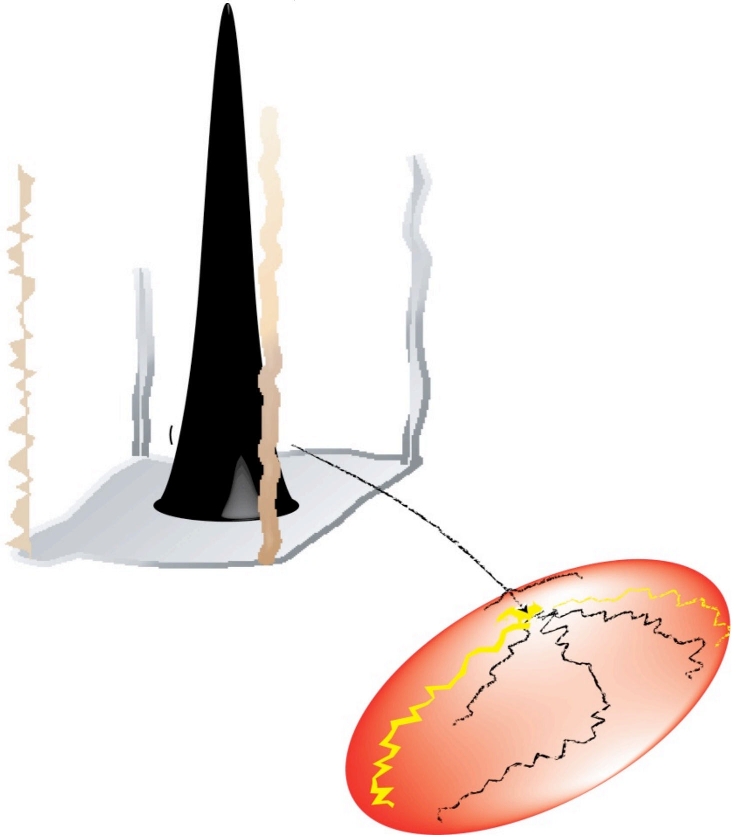
From which we also get

$$\int_\Sigma g^{i'k'}(y;\eta=0)\, K^{ab}_{i'k'}(y,x;\eta)\ [2\eta\,\mathcal{R}_{ab}(x,\eta)\ -g_{ab}(x,\eta)]\ d\mu_{g(x,\eta)} = -n\,(=3)\ .$$

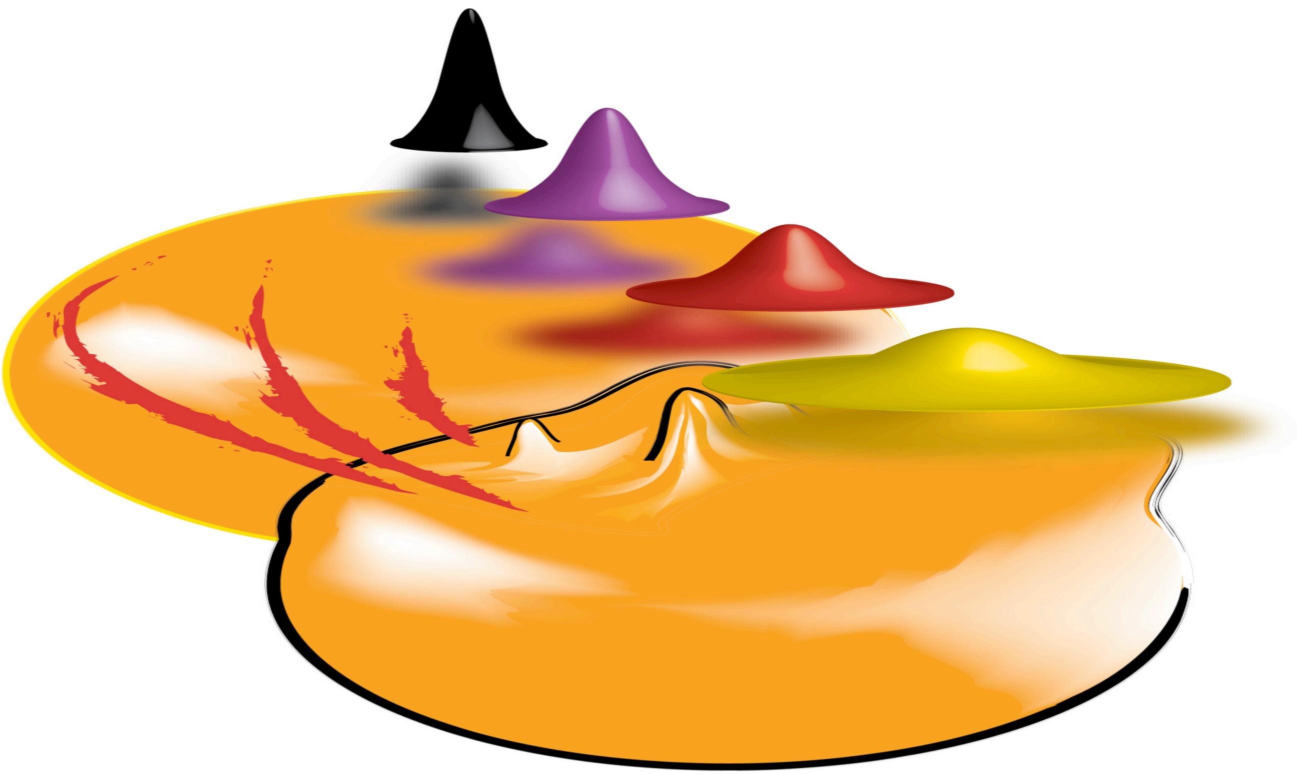
implying the point based $(y)$ monotonicity result

$$\frac{d}{d\eta}\int_\Sigma g^{i'k'}(y;\eta=0)\, K^{ab}_{i'k'}(y,x;\eta)\, g_{ab}(x,\eta)\, d\mu_{g(x,\eta)} = R(y;\eta=0)\ .$$

reminiscent of Perelman's
reduced volume monotonicity.



These integral representations are related with the Harnack estimate for the conjugate linearized flow, however with no a priori curvature restrictions. This suggests that for the heat kernel $K^{ab}_{i'k'}(y, x; \eta)$ may be possible to provide a Harnack inequality without positive curvature assumptions and construct an associated entropy functional.

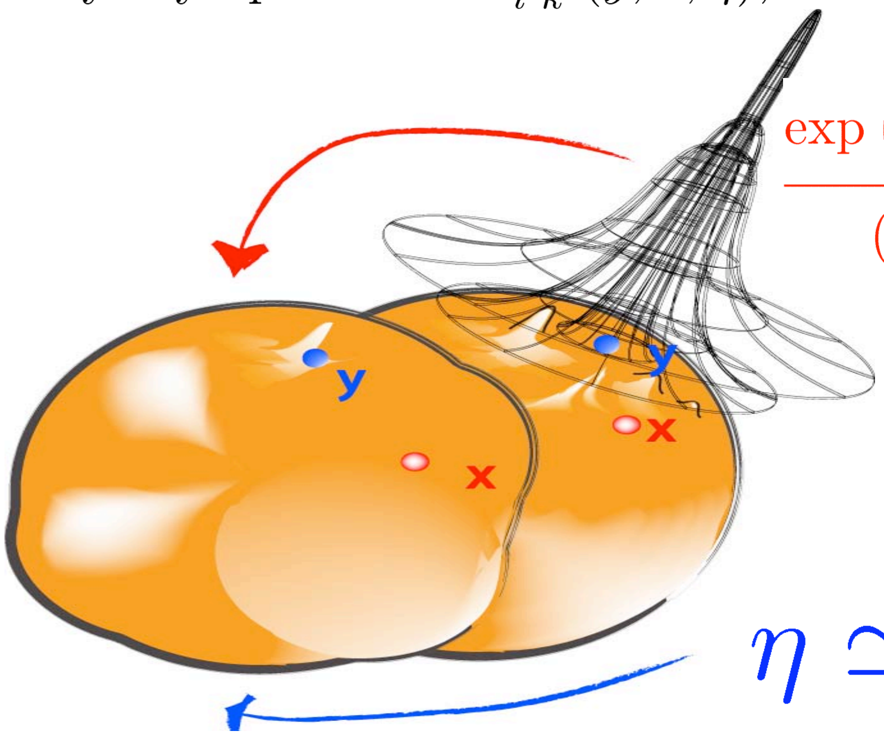


# Heat Kernel Asymptotics

As $\eta \searrow 0^+$, and for all $(y,x) \in \Sigma$ such that $d_0(y,x) < \mathit{inj}\,(\Sigma, g(0))$, there exists a sequence of smooth sections $\Phi[h]\,{}^{ab}_{i'k'}\,(y,x;\eta)$ with $\Phi[0]\,{}^{ab}_{i'k'}\,(y,x;\eta) = \tau^{ab}_{i'k'}\,(y,x;\eta)$, (the parallel transport operator), such that

$$\frac{\exp\left(-\frac{d_0^2(y,x)}{4\eta}\right)}{(4\pi\,\eta)^{\frac{3}{2}}}\,\sum_{h=0}^{N}\eta^h\,\Phi[h]\,{}^{ab}_{i'k'}\,(y,x;\eta)\;,$$

is uniformly asymptotic to $K^{ab}_{i'k'}(y,x;\eta)$,




$$\frac{\exp\left(-\frac{d_0^2(y,x)}{4\eta}\right)}{(4\pi\,\eta)^{\frac{3}{2}}}\;\tau\,{}^{ab}_{i'k'}\,(y,x;\eta) + \ldots\;,$$

$$\eta \simeq 0$$

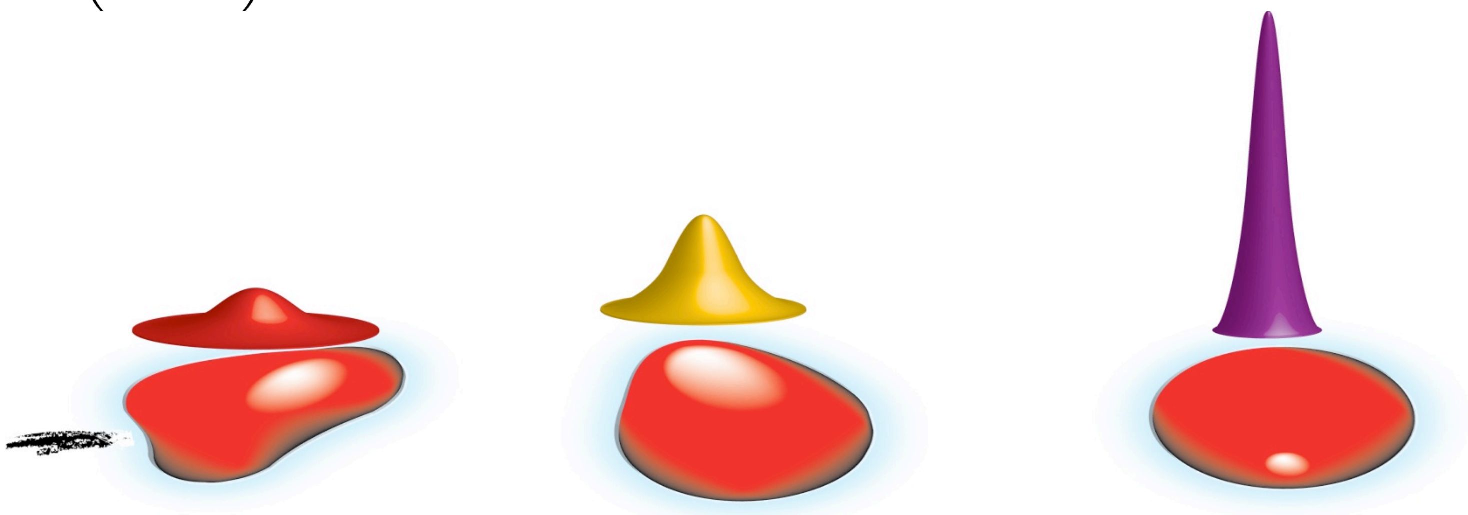
As $\eta \searrow 0^+$, we have the uniform asymptotic expansion

$$\mathcal{R}_{i'k'}(y, \eta = 0) = \tag{1}$$

$$\frac{1}{(4\pi\,\eta)^{\frac{3}{2}}} \int_\Sigma \exp\left(-\frac{d_0^2(y,x)}{4\eta}\right) \tau^{ab}_{i'k'}(y,x;\eta)\, \mathcal{R}_{ab}(x,\eta)\, d\mu_{g(x,\eta)}$$

$$+\sum_{h=1}^{N} \frac{\eta^h}{(4\pi\,\eta)^{\frac{3}{2}}} \int_\Sigma \exp\left(-\frac{d_0^2(y,x)}{4\eta}\right) \Phi[h]^{ab}_{i'k'}(y,x;\eta)\, \mathcal{R}_{ab}(x,\eta)\, d\mu_{g(x,\eta)}$$

$$+O\left(\eta^{N-\frac{1}{2}}\right) \,,$$



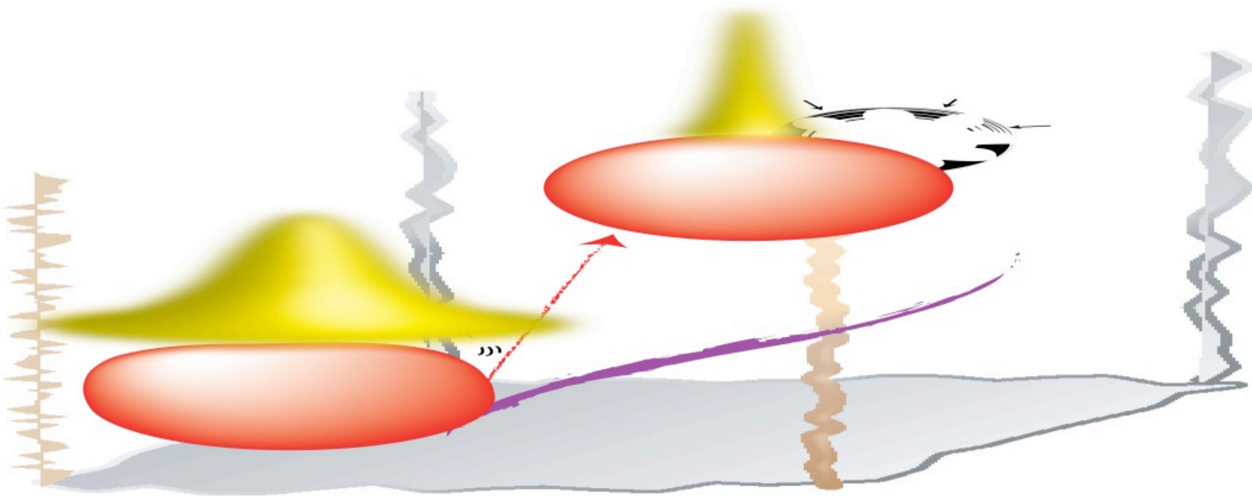
and

$$g_{i'k'}(y,\eta=0)= \tag{1}$$

$$\frac{1}{(4\pi\,\eta)^{\frac{3}{2}}}\int_{\Sigma}\exp\left(-\frac{d_0^2(y,x)}{4\eta}\right)\tau_{i'k'}^{ab}(y,x;\eta)\,\left[g_{ab}(x,\eta)-2\eta\mathcal{R}_{ab}(x,\eta)\right]\,d\mu_{g(x,\eta)}$$

$$+\sum_{h=1}^{N}\frac{\eta^h}{(4\pi\,\eta)^{\frac{3}{2}}}\int_{\Sigma}\exp\left(-\frac{d_0^2(y,x)}{4\eta}\right)\Phi[h]_{i'k'}^{ab}(y,x;\eta)\,\left[g_{ab}(x,\eta)-2\eta\mathcal{R}_{ab}(x,\eta)\right]\;d\mu_{g(x,\eta)}$$

$$+O\left(\eta^{N-\frac{1}{2}}\right)\;,$$



# Conclusions

Nice properties of the Heat Kernel of the conjugate linearized Ricci flow

It provides an integral representation of both the evolution of the Ricci tensor and of the metric itself

It provides non–trivial conserved quantities and relevant asymptotics useful also in physical applications of Ricci flow theory.

It also calls for a deeper analysis of its properties. They can be relevant in the study of singularities formation in Ricci flow theory.