Geometry of Ricci Solitons

H.-D. Cao, Lehigh University
LMU, Munich

November 25, 2008

Ricci Solitons

A complete Riemannian (M^n, g_{ij}) is a *Ricci soliton* if there exists a smooth function f on M such that

$$R_{ij} + \nabla_i \nabla_j f = \rho g_{ij}, \tag{1.1}$$

for some constant ρ. f is called a *potential function* of the Ricci soliton. $\rho = 0$: *steady* soliton; $\rho > 0$: *shrinking* soliton; $\rho < 0$: *expanding* soliton; $f =$ Const.: Einstein metric.

Ricci solitons are

- natural extension of Einstein manifolds;
- self-similar solutions to the Ricci flow
- possible singularity models of the Ricci flow
- critical points of Perelman's λ-entropy and μ-entropy.

Thus it is important to understand the geometry/topology of Ricci solitons and their classification.

Some Basic Facts about Ricci Solitions

- Compact steady or expanding solitons are Einstein in all dimensions;

- Compact shrinking solitons in dimensions $n = 2$ and $n = 3$ must be of positive constant curvature (by Hamilton and Ivey respectively);

- In dimension $n \geq 4$, there are compact non-Einstein gradient shrinking solitons;

- There is no non-flat complete noncompact shrinking soliton in dimension $n = 2$;

- The only three-dimensional complete noncompact non-flat gradient shrinking gradient solitons are quotients of round cylinder $\mathbb{S}^2 \times \mathbb{R}$.

- Ricci solitons exhibit rich geometric properties.

Examples of gradient Shrinking Ricci Solitions

- *Positive Einstein manifolds such as round spheres*

  Remark:

  (i) Suppose $R_{ij} = \frac{1}{2} g_{ij}$. Then under the Ricci flow, we have

  $$g_{ij}(t) = (1-t) g_{ij},$$

  which exists for $-\infty < t < 1$, and shrinks homothetically as t increases. Moreover, the curvature blows up like $1/(1-t)$. (This is an example of Type I singularity).

  (ii) Similarly, a shrinking gradient Ricci soliton satisfying the equation

  $$R_{ij} + \nabla_i \nabla_j f - \frac{1}{2} g_{ij} = 0$$

  corresponds to the self-similar Ricci flow solution $g_{ij}(t)$ of the form

  $$g_{ij}(t) = (1-t)\varphi_t^*(g_{ij}), \qquad t < 1,$$

  where φ_t are the 1-parameter family of diffeomorphisms generated by $\nabla f/(1-t)$.

- *Round cylinders* $\mathbb{S}^{n-1} \times \mathbb{R}$

- *Gaussian shrinking solitons*

  $(\mathbb{R}^n, g_0, f(x) = |x|^2/4)$ is a gradient shrinker:

$$Rc + \nabla^2 f = \frac{1}{2} g_0.$$

- *Gradient Kähler shrinkers on* $\mathbb{C}P^2\#(-\mathbb{C}P^2)$

  In early 90's Koiso, and independently by myself, constructed a gradient shrinking metric on $\mathbb{C}P^2\#(-\mathbb{C}P^2)$. It has $U(2)$ symmetry and positive Ricci curvature. More generally, they found $U(n)$-invariant Kähler-Ricci solitons on twisted projective line bundle over $\mathbb{C}P^{n-1}$ for all $n \geq 2$.

- *Gradient Kähler shrinkers on* $\mathbb{C}P^2\#2(-\mathbb{C}P^2)$

  In 2004, Wang-Zhu found a gradient Kähler-Ricci soliton on $\mathbb{C}P^2\#2(-\mathbb{C}P^2)$ which has $U(1) \times U(1)$ symmetry. More generally, they proved the existence of gradient Kähler-Ricci solitons on all Fano toric varieties of complex dimension $n \geq 2$ with non-vanishing Futaki invariant.

- *Noncompact gradient Kähler shrinkers*

  In 2003, Feldman-Ilmanen-Knopf found the first complete noncompact $U(n)$-invariant shrinking gradient Kähler-Ricci solitons, which are cone-like at infinity.

Examples of Steady Ricci Solitons

- *The cigar soliton* Σ

  In dimension $n = 2$, Hamilton discovered the *cigar soliton* $\Sigma = (\mathbb{R}^2, g_{ij})$, where the metric g_{ij} is given by

  $$ds^2 = \frac{dx^2 + dy^2}{1 + x^2 + y^2}$$

  with potential function

  $$f = -\log(1 + x^2 + y^2).$$

  The cigar has positive (Gaussian) curvature and linear volume growth, and is asymptotic to a cylinder of finite circumference at ∞.

- $\Sigma \times \mathbb{R}$: an 3-D steady Ricci soliton with nonnegative curvature.

- *The Bryant soliton on* $\mathbb{R}^n$

  In the Riemannian case, higher dimensional examples of noncompact gradient steady solitons were found by Robert Bryant on $\mathbb{R}^n$ $(n \geq 3)$. They are rotationally symmetric and have positive sectional curvature. The volume of geodesic balls $B_r(0)$ grow on the order of $r^{(n+1)/2}$, and the curvature approaches zero like $1/s$ as $s \to \infty$.

- *Noncompact steady Kähler-Ricci soliton on* $\mathbb{C}^n$

  I found a complete $U(n)$-symmetric steady Ricci soliton on $\mathbb{C}^n$ $(n \geq 2)$ with positive curvature. The volume of geodesic balls $B_r(0)$ grow on the order of r^n, n being the complex dimension. Also, the curvature $R(x)$ decays like $1/r$.

- *Noncompact steady Kähler-Ricci soliton on* $\widehat{\mathbb{C}^n/\mathbb{Z}_n}$

  I also found a complete $U(n)$ symmetric steady Ricci soliton on the blow-up of $\mathbb{C}^n/\mathbb{Z}_n$ at 0, the same underlying space that Eguchi-Hanson (n=2) and Calabi $(n \geq 2)$ constructed ALE Hyper-Kähler metrics.

Examples of 3-D Singularities in the Ricci flow.

- *3-manifolds with* $Rc > 0$.

  According to Hamilton, any compact 3-manifold (M^3, g_{ij}) with $Rc > 0$ will shrink to a point in finite time and becomes round.

- *The Neck-pinching*

  If we take a dumbell metric on topological $\mathbb{S}^3$ with a neck like $\mathbb{S}^2 \times I$, we expect the neck will shrink under the Ricci flow because the positive curvature in the $\mathbb{S}^2$ direction will dominate the slightly negative curvature in the direction of interval I. We also expect the neck will pinch off in finite time. (In 2004, Angnents and Knopf confirmed the neck-pinching phenomenon in the rotationally symmetric case.)

- *The Degenerate Neck-pinching*

  One could also pinch off a small sphere from a big one. If we choose the size of the little to be just right, then we expect a degenerate neck-pinching: there is nothing left on the other side. (This picture is confirmed by X.-P. Zhu and his student Gu in the rotationally symmetric case in 2006)

Singularities of the Ricci flow

In all dimensions, Hamilton showed that the solution $g(t)$ to the Ricci flow will exist on a maximal time interval [0, T), where either $T = \infty$, or $0 < T < \infty$ and $|Rm|_{\max}(t)$ becomes unbounded as t tends to T. We call such a solution a *maximal solution.* If $T < \infty$ and $|Rm|_{\max}(t) \to \infty$ as $t \to T$, we say the maximal solution $g(t)$ *develops singularities* as t tends to T and T is a *singular time.* Furthermore, Hamilton classified them into two types:

Type I: $\limsup_{t\to\infty} (T-t)\ |Rm|_{\max}(t) < \infty$

Type II: $\limsup_{t\to\infty} (T-t)\ |Rm|_{\max}(t) = \infty$

Determine the structures of singularities

Understanding the structures of singularities of the Ricci flow is the first essential step. The parabolic rescaling/blow-up method was developed by Hamilton since 1990s' and further developed by Perelman to understand the structure of singularities. We now briefly outline this method.

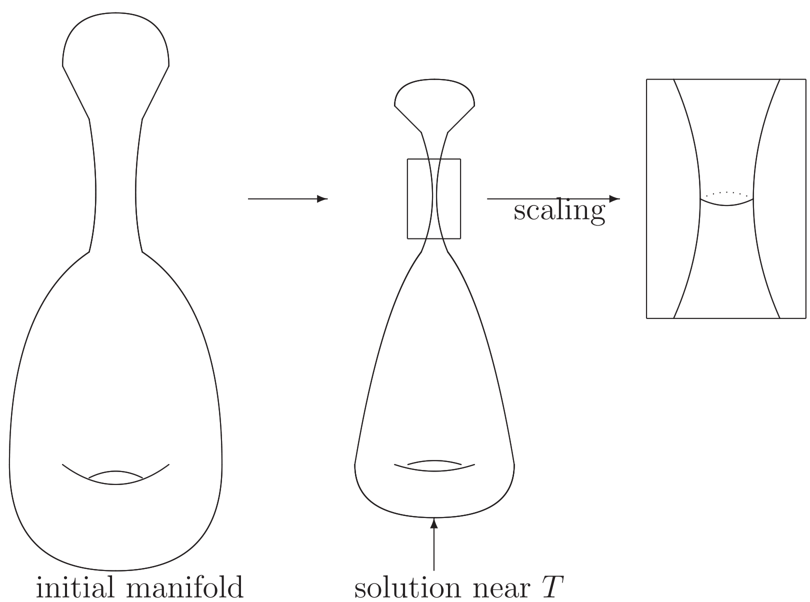

Figure 1: Rescaling

The Rescaling Argument:

- **Step 1**: Take a sequence of (almost) maximum curvature points (x_k, t_k), where $t_k \to T$ and $x_k \in M$, such that for all $(x,t) \in M \times [0, t_k]$, we have

$$|Rm|(x,t) \leq CQ_k, \qquad Q_k = |Rm|(x_k, t_k).$$

- **Step 2**: rescale $g(t)$ around (x_k, t_k) (by the factor Q_k and shift t_k to new time zero) to get the rescaled solution to the Ricci flow $\tilde{g}_k(t) = Q_k g(t_k + Q_k^{-1} t)$ for $t \in [-Q_k t_k, Q_k(T - t_k))$ with

$$|Rm|(x_k, 0) = 1, \quad \text{and} \quad |Rm|(x,t) \leq C$$

  on $M \times [-Q_k t_k, 0]$.

By Hamilton's **compactness theorem** and Perelman's **non-collapsing estimate**, rescaled solutions $(M^n, \tilde{g}_k(t), x_k)$ converges to $(\tilde{M}, \tilde{g}(t), \tilde{x})$, $-\infty < t < \Omega$, which is a complete *ancient solution* with bounded curvature and is κ*-noncollapsed on all scales.*

Hamilton's Compactness Theorem:

For any sequence of marked solutions $(M_k, g_k(t), x_k)$, $k = 1, 2, \ldots$, *to the Ricci flow on some time interval* $(A, \Omega]$, *if for all* k *we have*

- $|Rm|_{g_k(t)} \leq C$, *and*
- *inj* $(M_k, x_k, g_k(0)) \geq \delta > 0$,

then a subsequence of $(M_k, g_k(t), x_k)$ *converges in the* C^∞_{loc} *topology to a complete solution* $(M_\infty, g_\infty(t), x_\infty)$ *to the Ricci flow defined on the same time interval* $(A, \Omega]$.

Remark: In $n = 3$, by imposing an injectivity radius condition, Hamilton obtained the following structure results:

Type I: spherical or necklike structures;

Type II: either a steady Ricci soliton with positive curvature or $\Sigma \times \mathbb{R}$, *the product of the cigar soliton with the real line.*

Perelman's No Local Collapsing Theorem

Given any solution $g(t)$ on $M^n \times [0,T)$, with M compact and $T < \infty$, there exist constants $\kappa > 0$ and $\rho_0 > 0$ such that for any point $(x_0, t_0) \in M \times [0,T)$, $g(t)$ is κ-noncollapsed at (x_0, t_0) on scales less than ρ_0 in the sense that, for any $0 < r < \rho_0$, whenever

$$|Rm|(x,t) \leq r^{-2}$$

on $B_{t_0}(x_0, r) \times [t_0 - r^2, t_0]$, we have

$$Vol_{t_0}(B_{t_0}(x_0, r)) \geq \kappa r^n.$$

Corollary: *If $|Rm| \leq r^{-2}$ on $B_{t_0}(x_0, r) \times [t_0 - r^2, t_0]$, then*

$$inj(M, x_0, g(t_0) \geq \delta r$$

for some positive constant δ.

Remark: There is also a stronger version: only require the scalar curvature $R \leq r^{-2}$ on $B_{t_0}(x_0, r)$.

The Proof of Perelman's No Local Collapsing Theorems

Perelman proved two versions of "no local collapsing" property, one with a new entropy functional, and the other the reduced volume associated to a space-time distance function obtained by path integral analogous to what Li-Yau did in 1986.

- The $\mathcal{W}$-functional and μ-entropy:

  $$\mu(g,\tau) = \inf\left\{\mathcal{W}(g,f,\tau) \mid f \in C^\infty(M), \int_M (4\pi\tau)^{-\frac{n}{2}} e^{-f} dV = 1\right\},$$

  where $\mathcal{W}(g,f,\tau) = \int_M [\tau(R+|\nabla f|^2)+f-n](4\pi\tau)^{-\frac{n}{2}}e^{-f}dV$.

- Perelman's reduced distance l and reduced volume $\tilde{V}(\tau)$:

  For any space path $\sigma(s)$, $0 \le s \le \tau$, joining p to q, define the action $\int_0^\tau \sqrt{s}(R(\sigma(s),t_0-s)+|\dot{\sigma}(s)|^2_{g(t_0-s)})ds$, the L-length $L(q,\tau)$ from (p,t_0) to $(q,0)$, $l(q,\tau) = \frac{1}{2\sqrt{\tau}}L(q,\tau)$, and

  $$\tilde{V}(\tau) = \int_M (4\pi\tau)^{-\frac{n}{2}} e^{-l(q,\tau)} dV_\tau(q).$$

- Monotonicity of μ and $\tilde{V}$ under the Ricci flow:

  Perelman showed that under the Ricci flow $\partial g(\tau)/\partial\tau = 2Rc(\tau)$, $\tau = t_0 - t$, both $\mu(g(\tau),\tau)$ and $\tilde{V}(\tau)$ are nonincreasing in τ.

Remark: $\mathbb{S}^1 \times \mathbb{R}$ is **NOT** κ-noncollapsed on large scales for any $\kappa > 0$ and neither is the cigar soliton Σ, or $\Sigma \times \mathbb{R}$. In particular, $\Sigma \times \mathbb{R}$ cannot occur in the limit of rescaling! (However, $\mathbb{S}^2 \times \mathbb{R}$ is κ-noncollapsed on all scales for some $\kappa > 0$.)

Magic of 3-D: The Hamilton-Ivey Pinching Theorem

In dimension $n = 3$, we can express the curvature operator $Rm : \Lambda^2(M) \to \Lambda^2(M)$ as

$$Rm = \begin{pmatrix} \lambda & & \\ & \mu & \\ & & \nu \end{pmatrix},$$

where $\lambda \geq \mu \geq \nu$ are the principal sectional curvatures and the scalar curvature $R = 2(\lambda + \mu + \nu)$.

The Hamilton-Ivey pinching theorem *Suppose we have a solution $g(t)$ to the Ricci flow on a three-manifold M^3 which is complete with bounded curvature for each $t \geq 0$. Assume at $t = 0$ the eigenvalues $\lambda \geq \mu \geq \nu$ of Rm at each point are bounded below by $\nu \geq -1$. Then at all points and all times $t \geq 0$ we have the pinching estimate*

$$R \geq (-\nu)[\log(-\nu) + \log(1+t) - 3]$$

whenever $\nu < 0$.

Remark: This means in 3-D if $|Rm|$ blows up, the positive sectional curvature blows up faster than the (absolute value of) negative sectional curvature. As a consequence, **any limit of parabolic dilations at an almost maximal singularity has** $Rm \geq 0$

Ancient κ-Solution

An **ancient κ-solution** is a *complete ancient solution with nonnegative and bounded curvature, and is κ-noncollapsed on all scales.*

Recap: Whenever a maximal solution $g(t)$ on a compact M^n develop singularities, parabolic dilations around any (maximal) singularity converges to some limit ancient solution $(\tilde{M}, \tilde{g}(t), \tilde{x})$, which has bounded curvature and is κ-noncollapsed. Moreover, if $n = 3$, then the ancient solution has nonnegative sectional curvature, thus an ancient κ-solution.

Structure of Ancient κ-solutions in 3-D

Canonical Neighborhood Theorem (Perelman):

$\forall\ \varepsilon > 0$, *every point* (x_0, t_0) *on an orientable nonflat ancient* κ*-solution* $(\tilde{M}^3, \tilde{g}(t))$ *has an open neighborhood* B*, which falls into one of the following three categories:*

(a) B *is an* ε*-neck of radius* $r = R^{-1/2}(x_0, t_0)$*; (i.e., after scaling by the factor* $R(x_0, t_0)$*,* B *is* ϵ*-close, in* $C^{[\epsilon^{-1}]}$*-topology, to* $\mathbb{S}^2 \times [-\epsilon^{-1}, \epsilon^{-1}]$ *of scalar curvature 1.)*

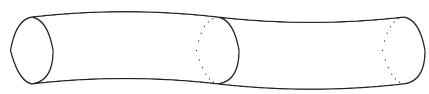

Figure 2: ε-neck

(b) B *is an* ε*-cap; (i.e., a metric on* $\mathbb{B}^3$ *or* $\mathbb{RP}^3 \setminus \bar{\mathbb{B}}^3$ *and the region outside some suitable compact subset is an* ε*-neck).*

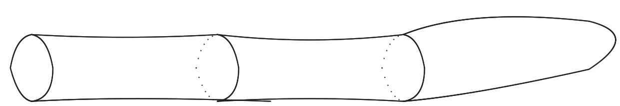

Figure 3: ε-cap

(c) B *is compact (without boundary) with positive sectional curvature (hence diffeomorphic to the 3-sphere by Hamilton).*

Classification of 3-d κ-shrinking solitons

- Classification of 3-d κ-shrinking solitons (Perelman): they are either round $\mathbb{S}^3/\Gamma$, or round cylinder $\mathbb{S}^2 \times \mathbb{R}$, or its $\mathbb{Z}_2$ quotients $\mathbb{S}^2 \times \mathbb{R}/\mathbb{Z}_2$. In particular, *there is no 3-d complete noncompact κ-shrinking solitons with $Rm > 0$.*

Improvements in 3-D

• Complete noncompact non-flat shrinking gradient soliton with bounded Rm and $Rc \geq 0$ are quotients of $\mathbb{S}^2 \times \mathbb{R}$. (Naber, 2007)

• Complete noncompact non-flat shrinking gradient soliton with $Rc \geq 0$ and with curvature growing at most as fast as $e^{ar(x)}$ are quotients of $\mathbb{S}^2 \times \mathbb{R}$. (Ni-Wallach, 2007)

• Complete noncompact non-flat shrinking gradient soliton are quotients of $\mathbb{S}^2 \times \mathbb{R}$ (Cao-Chen-Zhu, 2007).

Further Extensions

I. 4-D:

- Any complete gradient shrinking soliton with $Rm \geq 0$ and PIC, and satisfying additional assumptions, is either a quotient of $\mathbb{S}^4$ or a quotient of $\mathbb{S}^3 \times \mathbb{R}$. (Ni-Wallach, 2007)

- Any non-flat complete noncompact shrinking Ricci soliton with bounded curvature and $Rm \geq 0$ is a quotient of either $\mathbb{S}^3 \times \mathbb{R}$ or $\mathbb{S}^2 \times \mathbb{R}^2$. (Naber, 2007)

II. $n \geq 4$ with Weyl tensor $W = 0$: Let (M^n, g) be a complete gradient shrinking soliton with vanishing Weyl tensor.

- Ni-Wallach (2007): Assume (M^n, g) has $Rc \geq 0$ and that $|Rm|(x) \leq e^{a(r(x)+1)}$ for some constant $a > 0$, where $r(x)$ is the distance function to some origin. Then its universal cover is $\mathbb{R}^n$, $\mathbb{S}^n$ or $\mathbb{S}^{n-1} \times \mathbb{R}$. (also by X. Cao and B. Wang)

- Petersen-Wylie (2007): *Assume*

$$\int_M |Rc|^2 e^f dvol < \infty.$$

  Then (M^n, g) *is a finite quotient of* $\mathbb{R}^n$, $\mathbb{S}^n$ *or* $\mathbb{S}^{n-1} \times \mathbb{R}$.

- Z.-H. Zhang (2008): *Any complete gradient shrinking soliton with vanishing Weyl tensor must be a finite quotients of* $\mathbb{R}^n$, $\mathbb{S}^n$ *or* $\mathbb{S}^{n-1} \times \mathbb{R}$.