



# Collective neutrino flavor conversion: Recent developments

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## Abstract

Neutrino flavor evolution in core-collapse supernovae, neutron-star mergers, or the early universe is dominated by neutrino–neutrino refraction, often spawning “self-induced flavor conversion,” i.e., shuffling of flavor among momentum modes. This effect is driven by collective run-away modes of the coupled “flavor oscillators” and can spontaneously break the initial symmetries such as axial symmetry, homogeneity, isotropy, and even stationarity. Moreover, the growth rates of unstable modes can be of the order of the neutrino–neutrino interaction energy instead of the much smaller vacuum oscillation frequency: self-induced flavor conversion does not always require neutrino masses. We illustrate these newly found phenomena in terms of simple toy models. What happens in realistic astrophysical settings is up to speculation at present.

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## 1. Introduction

Neutrino dispersion in matter strongly modifies the flavor evolution caused by their masses and mixing parameters [1–3]. Moreover, in dense astrophysical environments, notably in core-

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collapse supernovae, merging neutron stars, or the early universe, neutrinos themselves provide a large refractive effect. Flavor evolution implies that the “background medium” also evolves, i.e., neutrino flavor evolution feeds back unto itself [4,5] and, among other effects, can produce collective run-away modes in flavor space. One consequence is “self-induced flavor conversion,” meaning that some modes of the neutrino mean field swap flavor with other modes [6–12]. To lowest order, neutrino–neutrino interactions are flavor blind [13], so collective effects alone do not change the global flavor content of the ensemble. Yet the reshuffling among modes can effectively engender flavor equilibration. The simplest example is a gas of equal densities of  $\nu_e$  and  $\bar{\nu}_e$  that would turn to an equal  $\nu$ – $\bar{\nu}$  mixture of all flavors, yet the overall flavor lepton numbers remain zero from beginning to end [14].

The neutrino mean field is described by flavor matrices  $\rho(t, \mathbf{r}, \mathbf{p})$  with elements of the type  $\langle a_i^\dagger a_j \rangle$  in terms of creation and annihilation operators with flavor index  $i$ . The diagonal elements are occupation numbers, the off-diagonal ones represent flavor correlations. The seven-dimensional phase space  $(t, \mathbf{r}, \mathbf{p})$  is not tractable and was always reduced by symmetry assumptions. For supernova neutrinos, one has usually assumed stationary solutions which depend only on radial distance and, in momentum space, on energy and zenith angle, reducing the problem to three dimensions. The emission region was often modeled as a “neutrino bulb,” meaning an emitting surface (“neutrino sphere”), where neutrinos of all flavors emerge with a blackbody-inspired zenith-angle distribution. Such models lead to sharp spectral features (“spectral splits”) caused by flavor swaps between different parts of the spectrum [15–18]. Being triggered by an instability which is sensitive to the neutrino mass ordering, these effects seemed to offer an opportunity to learn about the latter even for a very small  $\Theta_{13}$  mixing angle [19].

Meanwhile the situation has changed on several fronts. The mixing angle  $\Theta_{13}$  has been measured and is not very small, so the mass ordering will be experimentally accessible. Moreover, our ideas about the flavor evolution of supernova neutrinos had to be revised because it has dawned on us that the earlier symmetry assumptions had constrained the solutions in unphysical ways. Even within the bulb model of neutrino emission, axial symmetry [20–24], homogeneity [25–29], and stationarity [30,31] can be spontaneously broken, completely changing the stability conditions. It has been speculated that self-induced flavor conversion may commence in the decoupling region, much deeper than the usual “onset radius,” and that flavor equilibration could be a generic outcome instead of ordered spectral swaps.

Moreover, in a more realistic emission model,  $\nu_e$ ,  $\bar{\nu}_e$  and the other species have different zenith-angle distributions. Surprisingly, self-induced flavor conversion can then be “fast” in the sense that the evolution speed is of the order of the neutrino–neutrino interaction energy  $\mu = \sqrt{2}G_{FN\nu}$  instead of the much smaller vacuum oscillation frequency  $\omega = \Delta m^2/2E$  [9,32,33]. While this phenomenon had been noted a long time ago [9], its significance had eluded much of the community. Fast flavor conversion does not depend on neutrino masses,<sup>1</sup> except perhaps for providing initial disturbances to seed the run-away modes.<sup>2</sup> This counter-intuitive behavior owes to the character of self-induced flavor conversion as an instability and to its nature of flavor shuffling among modes which globally conserves flavor number.

<sup>1</sup> The Physics Nobel Prize 2015 was awarded “for the discovery of neutrino oscillations, which shows that neutrinos have mass.” Ironically, self-induced flavor conversion does not always depend on neutrino masses, although this connection exists, of course, in the context of vacuum oscillations and standard MSW conversion.

<sup>2</sup> One may speculate that initial disturbances could be provided even by quantum fluctuations of our classical mean-field quantities. However, since neutrinos are known to mix and to have small masses, in practice ordinary neutrino flavor oscillations are guaranteed to provide disturbances even on the mean-field level.

Many studies of supernova neutrino flavor evolution were restricted to two-flavor effects driven by the atmospheric mass difference and the small mixing angle  $\Theta_{13}$  because all effects driven by the solar mass difference require larger distances. Moreover, the  $\nu_\mu$ ,  $\bar{\nu}_\mu$ ,  $\nu_\tau$  and  $\bar{\nu}_\tau$  species, collectively called  $\nu_x$ , are produced with similar spectra, so flavor conversion among them may not be crucial. Even then, however, three-flavor effects can be important [34–40] and also allow new instabilities opened up by the additional degree of freedom [41]. Moreover, fast flavor conversion is independent of  $\Delta m^2$ , so realistically one needs to include three flavors.

Most of these effects have been recognized only very recently and we have no complete or even approximate picture yet of how flavor really evolves in realistic astrophysical environments. In particular, realistic numerical studies seem out of the question for the time being. If quick flavor decoherence is a generic outcome, or at least if one can develop criteria under which circumstances this will be the case, full-scale simulations may not be needed. For the moment, it may be more fruitful to take a reductionist approach and develop a better understanding of the various forms of behavior shown by an interacting neutrino gas. In this spirit, we discuss the recent developments here in the framework of simple toy models for illustration, restricting ourselves to the method of linearized stability analysis [42].

We begin in Sec. 2 with the equations of motion for the neutrino mean field on the refractive level and we explain the geometric structure of our problem relevant for compact astrophysical sources. In Sec. 3 we linearize the equation in terms of small off-diagonal elements of the  $\varrho$  matrices, assuming neutrinos were produced in weak-interaction states. In Sec. 4 we set up our main “gedanken experiment,” a neutrino gas consisting of a total of four modes: left- and right-moving neutrinos and antineutrinos, i.e., colliding beams of  $\nu_e$  and  $\bar{\nu}_e$ . We study the simple cases of spontaneous left–right symmetry breaking as well as the spontaneous breaking of homogeneity in the form of unstable spatial Fourier modes. In Sec. 5 we turn to examples with fewer initial symmetries and show how fast flavor conversion appears, i.e., instabilities with growth rates of order  $\mu$ , the neutrino–neutrino interaction energy, instead of the much smaller vacuum oscillation frequency  $\omega$ . In Sec. 6 we comment on the latest development: unstable time-varying modes from a stationary source. In Sec. 7 we close with a brief summary and outlook.

## 2. Equations of motion and adopted geometry

The flavor evolution of dense neutrino gases thus far has only been studied on the refractive level, i.e., neutrinos were always taken to be freely streaming without collisions. As a physical description one uses the neutrino mean field, i.e., “occupation numbers” of the type  $\langle a_i^\dagger a_j \rangle$  in terms of the usual creation and annihilation operators. They are assembled in the form of matrices in flavor space  $\varrho(t, \mathbf{r}, E, \mathbf{v})$ , essentially providing classical phase-space densities. This mean-field description ignores higher-order correlators, i.e., we work on the lowest level of the BBGKY hierarchy and in addition we ignore spin and pairing correlations [43–47]. It is most economical to include antineutrinos in the equations as modes with negative energy so that  $E$  can be both positive and negative. The direction of motion is represented by the velocity  $\mathbf{v} = \mathbf{p}/|E|$  and we approximate  $|\mathbf{v}| = 1$  for our ultra-relativistic neutrinos. The diagonal elements of  $\varrho$  for antineutrinos (negative  $E$  modes) are negative occupation numbers, which is the “flavor isospin convention.” It is sometimes more intuitive to represent antineutrinos by matrices  $\bar{\varrho}$  of positive occupation numbers and positive energies at the price of more cumbersome equations.

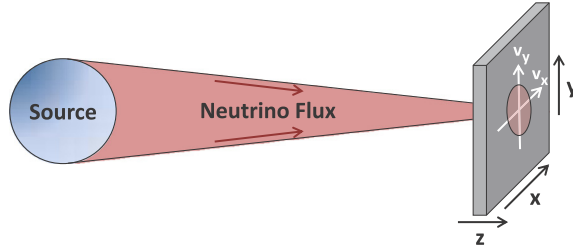


Fig. 1. Geometry for flavor evolution of neutrinos streaming from a compact source. The current–current nature of weak interactions implies that neutrino–neutrino refraction depends on relative velocities in the transverse plane ( $x$ – $y$ -plane) caused by the finite size of the source. In a linearized stability analysis we look for modes which grow exponentially in flavor space as a function of  $z$ .

The space–time evolution of  $\varrho(t, \mathbf{r}, E, \mathbf{v})$  is governed by free streaming and by local flavor evolution in the form of a Liouville equation [5]

$$i(\partial_t + \mathbf{v} \cdot \nabla_{\mathbf{r}}) \varrho = [H, \varrho]. \tag{1}$$

The “Hamiltonian matrix”  $H$  depends on the phase-space variables  $(t, \mathbf{r}, E, \mathbf{v})$ . It is explicitly

$$H = \frac{M^2}{2E} + \sqrt{2}G_F \left[ N_\ell + \int d\Gamma' (1 - \mathbf{v} \cdot \mathbf{v}') \varrho_{t,\mathbf{r},E',\mathbf{v}'} \right], \tag{2}$$

where  $M^2$ , the matrix of neutrino mass-squares, is what drives vacuum oscillations. The matrix of charged-lepton densities,  $N_\ell$ , provides the usual Wolfenstein matter effect, assuming the background medium is isotropic. The neutrino and antineutrino phase-space integration  $\int d\Gamma'$  is explicitly  $\int_{-\infty}^{+\infty} dE' E'^2 \int d\mathbf{v}' / (2\pi)^3$ , where the velocity integration is over the unit sphere. The factor  $(1 - \mathbf{v} \cdot \mathbf{v}')$  represents the current–current nature of low-energy weak interactions. As a consequence, collinear neutrino modes do not influence each other and neutrino–neutrino refraction involves direction of motion (“multi angle effects”) as a central ingredient.

Early studies of neutrino–neutrino refraction were motivated by cosmology and considered a homogeneous system evolving as a function of time. All directional effects in Eq. (1) were integrated out. The matter effect also dropped out, being common to all modes, leading to a much simpler equation. A homogeneous system evolving in time remains an important theoretical laboratory to develop an understanding of collective flavor evolution. We will use it later in its incarnation of a “colliding neutrino beam,” reducing it to one spatial dimension.

However, the main interest in this subject is provided by core-collapse supernovae or neutron-star mergers where neutrinos stream away from a compact source and we ask for flavor evolution as a function of distance. The current–current nature of weak interactions implies that collinear neutrinos do not affect each other. Therefore, neutrino–neutrino refraction depends on the finite size of the source which implies that at some distance, the local neutrino field has transverse velocity components corresponding to their origin at different source locations as shown in Fig. 1. If the source is spherical with radius  $R$ , the transverse velocities at the test distance  $r$  fill a circle with radius (in velocity space) of  $(v_x^2 + v_y^2)^{1/2} \leq r/R$  as shown in Fig. 1. The same applies to a non-spherical source if it is compact and we can think of  $R$  as an envelope.

When we perform a linearized stability analysis we consider a small volume at some distance  $r$  from the source and search for modes of the neutrino mean field with off-diagonal elements that grow exponentially as a function of distance. We identify the latter with the  $z$ -direction,

whereas  $x$  and  $y$  are cartesian coordinates in the transverse plane. Notice that this picture also applies to a non-spherical source geometry, where one would identify the local  $z$ -direction with the neutrino flux direction [48]. In this case, of course, the neutrino field in the transverse plane is a nontrivial function of  $x$  and  $y$  and of the transverse velocity components  $v_x$  and  $v_y$ . In the following, however, we will always consider a spherical source.

This or similar geometric arrangements have been used in all studies of supernova neutrino flavor evolution, yet may not be adequate to capture crucial aspects of the problem. One issue arises because neutrinos occasionally scatter on their way out, providing a small “halo flux.” In other words, the transverse velocities of the “neutrino beam” do not only form a compact disk as shown in Fig. 1, but in addition have a diffuse halo [49,50]. Even though this halo is orders of magnitude smaller than the primary beam, it has large angular leverage and thus a strong refractive impact. Notice that neutrinos in the main beam are nearly collinear so that  $(1 - \mathbf{v} \cdot \mathbf{v}')$  suppresses the effective interaction energy by an approximate factor  $(R/r)^2$  with distance, in addition to the geometric  $r^{-2}$  flux dilution, i.e.,  $\mu_{\text{eff}} \propto r^{-4}$  for the beam component [15].

Most studies have modeled the neutrino flux as a “compact beam,” but including broad tails in the transverse velocity distribution is a quantitative issue, not a conceptual one. However, the halo flux extends to all directions so that at the test location, there is also a “backward” or “inbound” neutrino flux which has never been included. If neutrinos move in all directions it is not obvious that we can describe the flavor evolution as a function of distance. In a linearized stability analysis, it is not obvious that we should look only for modes which grow exponentially with distance from the source. This issue is exacerbated if flavor instabilities occur in the decoupling or “neutrino sphere” region as has been speculated by several authors [9,26,27,31,32]. One may wonder if in this case we should think of flavor evolution as a collective phenomenon as a function of a single space or time variable. More plausibly, it represents a nontrivial space–time phenomenon, but has never been treated as such in the literature.

One may speculate that a linearized stability analysis remains useful in the sense of a self-consistency test. If one finds stability, self-induced flavor conversion likely do not occur. On the other hand, if instability is found in regions where neutrinos stream in all directions, what happens realistically is a matter of speculation at present. In this spirit we limit ourselves to the method of linearized stability analysis for the rest of our discussion.

### 3. Linearized equations of motion

As a first simplification we limit ourselves to two flavors  $\nu_e$  and  $\nu_x$ , where  $\nu_x$  is a suitable mixture of  $\nu_\mu$  and  $\nu_\tau$ . We use the unit vectors in flavor space  $\vec{B}$  pointing in the mass direction and  $\vec{L}$  in the weak-interaction direction.<sup>3</sup> Their relative angle is  $2\Theta$ , twice the mixing angle, which is taken to be small. We also use the vacuum oscillation frequency  $\omega = \Delta m^2/2E$ , where  $\Delta m^2$  is a positive number. Henceforth we label neutrino modes by  $\omega$  instead of  $E$ , where positive  $\omega$  is for neutrinos and negative  $\omega$  for antineutrinos. Flavor oscillations are driven by

$$\mathbb{H} = \frac{1}{2} \left( \omega \vec{B} + \lambda \vec{L} \right) \cdot \vec{\sigma} + \sqrt{2} G_F \int d\Gamma' (1 - \mathbf{v} \cdot \mathbf{v}') \varrho_{t,\mathbf{r},\omega',\mathbf{v}'} \quad (3)$$

It is understood that a Jacobian for the  $E \rightarrow \omega$  variable transformation has been included in the definition of  $\varrho_{t,\mathbf{r},\omega,\mathbf{v}}$  which is a phase-space density in the new variables. We use  $\lambda = \sqrt{2} G_F n_e$

<sup>3</sup> We denote vectors in flavor space with an arrow, vectors in coordinate and momentum space in boldface.

with  $n_e$  the net electron density (electrons minus positrons) and  $\vec{\sigma}$  is a vector of Pauli matrices. The matter effect is always large ( $\lambda \gg \omega$ ) and we can go to a frame rotating with frequency  $\lambda$  around the  $\vec{L}$  direction. In this frame,  $\vec{B}$  rotates fast and we assume its components transverse to  $\vec{L}$  average to zero. Equivalently, we can take the mixing angle in matter to be infinitely small. Either way, in effect  $\vec{B}$  and  $\vec{L}$  are taken to be collinear.

To determine stability against self-induced flavor conversion we study the evolution of small deviations from a system initially prepared in flavor eigenstates. To this end we express the mean-field matrices in terms of occupation numbers in the form

$$\varrho = \frac{f_{\nu_e} + f_{\nu_x}}{2} \mathbb{1} + \frac{f_{\nu_e} - f_{\nu_x}}{2} \begin{pmatrix} s & S \\ S^* & -s \end{pmatrix}, \tag{4}$$

where all quantities depend on the phase-space variables  $(t, \mathbf{r}, \omega, \mathbf{v})$ . For antineutrinos, the coefficients are  $-f_{\nu_e} - f_{\nu_x}$  and  $-f_{\nu_e} + f_{\nu_x}$ , respectively. The first term is not affected by flavor conversion. The matrix structure is encoded in the real number  $s$  and the complex number  $S$  with  $s^2 + |S|^2 = 1$ . Equivalently, the matrix structure can be expressed in terms of a normalized polarization vector  $\vec{P}$  as  $\vec{P} \cdot \vec{\sigma}$ , corresponding to  $S = P_1 + iP_2$  and  $s = P_3$ . We will linearize in  $S$ , the off-diagonal element of  $\varrho$ , and note that to this order  $s = 1$ .

As a further simplification we restrict ourselves to a neutrino gas with constant properties over the distances and time scales of flavor conversion, i.e., we assume that the initial occupation numbers  $f$  depend only on  $\omega$  and  $\mathbf{v}$ . The linearized equation of motion ( $|S| \ll 1$ ) is

$$i(\partial_t + \mathbf{v} \cdot \nabla_{\mathbf{r}}) S_{t,\mathbf{r},\omega,\mathbf{v}} = (\omega + \lambda) S_{t,\mathbf{r},\omega,\mathbf{v}} + \sqrt{2}G_F \int d\Gamma' (1 - \mathbf{v} \cdot \mathbf{v}') h_{\omega',\mathbf{v}'} (S_{t,\mathbf{r},\omega,\mathbf{v}} - S_{t,\mathbf{r},\omega',\mathbf{v}'}), \tag{5}$$

where  $h = f_{\nu_e} - f_{\nu_x}$  for  $\omega > 0$  and  $h = -(f_{\bar{\nu}_e} - f_{\bar{\nu}_x})$  for  $\omega < 0$ . Next we introduce an effective neutrino density  $n_\nu = \frac{1}{2}(n_{\nu_e} + n_{\bar{\nu}_e} - n_{\nu_x} - n_{\bar{\nu}_x})$  and the neutrino spectrum

$$g_{\omega,\mathbf{v}} = \frac{h_{\omega,\mathbf{v}}}{n_\nu} = \frac{1}{n_\nu} \times \begin{cases} f_{\nu_e}(\omega, \mathbf{v}) - f_{\nu_x}(\omega, \mathbf{v}) & \text{for } \omega > 0, \\ -f_{\bar{\nu}_e}(\omega, \mathbf{v}) + f_{\bar{\nu}_x}(\omega, \mathbf{v}) & \text{for } \omega < 0. \end{cases} \tag{6}$$

Our definitions imply the normalization  $\int d\Gamma \text{sign}(\omega) g_{\omega,\mathbf{v}} = 2$ . We also introduce the parameters  $\epsilon = \int d\Gamma g_{\omega,\mathbf{v}}$  and  $\phi = \int d\Gamma \mathbf{v} g_{\omega,\mathbf{v}}$  which express the asymmetry of the  $\nu$  vs.  $\bar{\nu}$  density and flux, respectively. In terms of these quantities, the linearized equation becomes

$$i(\partial_t + \mathbf{v} \cdot \nabla_{\mathbf{r}}) S_{t,\mathbf{r},\omega,\mathbf{v}} = \left[ \omega + \lambda + \mu (\epsilon - \mathbf{v} \cdot \phi) \right] S_{t,\mathbf{r},\omega,\mathbf{v}} - \mu \int d\Gamma' (1 - \mathbf{v} \cdot \mathbf{v}') g_{\omega',\mathbf{v}'} S_{t,\mathbf{r},\omega',\mathbf{v}'}, \tag{7}$$

where the effective neutrino–neutrino interaction energy is  $\mu = \sqrt{2}G_F n_\nu$ . The term  $\mu (\epsilon - \mathbf{v} \cdot \phi)$  is the ordinary matter effect caused by neutrinos on each other irrespective of collective effects. The matter background quantified by  $\lambda$  also provides a flux term if it is not isotropic.

Our equations are formulated for inverted neutrino mass ordering in our two-flavor system. Normal ordering corresponds to  $\vec{B} \rightarrow -\vec{B}$  in Eq. (3) and thus to a minus sign in front of the  $\omega$  term in square brackets in Eq. (7). In a stability analysis, one can instead flip the sign of  $\mu$  and  $\lambda$  and extend these parameters to the range  $-\infty < \mu, \lambda < +\infty$  to cover both cases of mass ordering.

On the level of a linearized stability analysis, all examples studied in the literature are special cases of Eq. (7) with different assumptions about the neutrino energy and velocity distribution, about the initial or boundary conditions, and about the symmetries of the solutions.

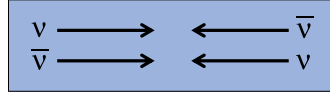


Fig. 2. Initially homogeneous ensemble of four neutrino modes (“colliding beams” of neutrinos and antineutrinos). The system is taken to be infinite in all directions. The normalized  $\nu$  flux is  $1 + a$ , the  $\bar{\nu}$  flux  $1 - a$  with the asymmetry parameter  $a$  in the range  $-1 \leq a \leq +1$ . The left–right asymmetry is parametrized by  $b$  such that the upper beam in this figure has normalized strength  $1 + b$ , the lower beam  $1 - b$  with  $-1 \leq b \leq +1$ .

## 4. Spontaneous symmetry breaking in one dimension

### 4.1. Colliding neutrino beam

The main conceptual point of our discussion is spontaneous symmetry breaking in collective flavor conversion, i.e., the solution does not respect the symmetry of the initial or boundary condition, leading to entirely new solutions than had been conceived of earlier. Assuming a stationary spherical source which emits neutrinos isotropically, the local neutrino field at our test position as in Fig. 1 will be axially symmetric (isotropic in the transverse plane). The solution can be axially symmetric, but can also break axial symmetry, which was termed the multi-azimuth angle (MAA) solution [20]. While the former solution appears for inverted neutrino mass ordering, the latter requires normal ordering. The symmetry-breaking solution varies as  $\cos \varphi$  or  $\sin \varphi$  in terms of the azimuth angle  $\varphi$  in the  $x$ – $y$ -plane, i.e., it has dipole structure, essentially as a consequence of the current–current nature of the interaction term.

In the geometry of Fig. 1, the crucial effects occur in the transverse plane, with distance  $z$  playing the role of a parameter that describes the evolution. Assuming a stationary source and solution, time drops out entirely and we may rename  $z$  as “time”, an affine parameter along the radial direction. In this sense, the stationary radial evolution of supernova neutrinos has often been described in terms of “time evolution.” The neutrino density decreases as a function of distance and transverse scales increase, so one can visualize the evolution of the stationary neutrino field as a function of distance instead as a time evolution in an expanding space, analogous to the evolution in the expanding universe [29]. In this sense the two cases bear many similarities.

Axial symmetry breaking of supernova neutrinos leads to a dipole variation in the transverse plane, which on the crudest level of approximation can be seen as two azimuth bins, corresponding to a beam of left-moving and a beam of right-moving neutrinos (in the transverse plane). This picture suggests to study simple toy cases of symmetry breaking in terms of a homogeneous system with two opposite-moving momenta and study it as a function of time [21] in the spirit of Fig. 2. One may consider this abstract system as a colliding neutrino beam, initially homogeneous and evolving in time, or as representing the transverse behavior of neutrinos streaming from a compact source. Other authors have constructed an analogous system as the “line model” of neutrino emission [26,28]. It has also been called a “two-dimensional model” [27], where one dimension is our beam, the other the  $z$ -direction which we here call “time.” Different authors prefer different visualizations of the same physical content.

We formulate this problem in terms of the colliding beam model of Fig. 2, where the possible velocities are  $v = \pm 1$ . (Of course, the transverse velocities in a geometry like Fig. 1 are very small, but we can renormalize them by rescaling the interaction parameter  $\mu$ .) In addition we use monochromatic neutrinos and antineutrinos with energies  $\pm E$  and thus with vacuum oscillation frequencies  $\pm \omega = \pm \Delta m^2 / 2E$ . Therefore, we represent the function  $S_{t,x,\omega,v}$  as a quartet of numbers  $(R, \bar{L}, L, \bar{R})$ , representing the off-diagonal elements of the left- and right-moving

neutrinos and antineutrinos, respectively, which all depend on  $t$  and one spatial coordinate  $x$ . Moreover, we denote the phase-space densities  $g_{\omega, \nu}$  for our four modes as  $(r, \bar{l}, l, \bar{r})$ . The linearized equations of motion can be transformed to Fourier space by considering solutions of the form  $S_{t, \mathbf{r}} = Q_{\Omega, \mathbf{k}} e^{-i(\Omega t - \mathbf{k} \cdot \mathbf{r})}$ , leading to

$$\Omega \begin{pmatrix} R \\ \bar{L} \\ L \\ \bar{R} \end{pmatrix} = \left[ \begin{pmatrix} \omega + k & 0 & 0 & 0 \\ 0 & -\omega - k & 0 & 0 \\ 0 & 0 & \omega - k & 0 \\ 0 & 0 & 0 & -\omega + k \end{pmatrix} + \lambda + 2\mu \begin{pmatrix} l + \bar{l} & -\bar{l} & -l & 0 \\ -r & r + \bar{r} & 0 & -\bar{r} \\ -r & 0 & r + \bar{r} & -\bar{r} \\ 0 & -\bar{l} & -l & l + \bar{l} \end{pmatrix} \right] \begin{pmatrix} R \\ \bar{L} \\ L \\ \bar{R} \end{pmatrix}. \tag{8}$$

The factor of 2 in the neutrino–neutrino interaction term arises from the factor  $(1 - \mathbf{v} \cdot \mathbf{v}')$  being 2 for opposite moving modes and 0 for parallel ones.

This is the most general one-dimensional four-mode case, which we simplify somewhat further. We assume the overall normalized neutrino density to be  $1 + a$ , the antineutrino density  $1 - a$  so that  $-1 < a < +1$  quantifies the  $\nu$ – $\bar{\nu}$ -asymmetry. Moreover, we picture the system as consisting of the beam  $R$  and  $\bar{L}$  (upper modes in Fig. 2) with overall strength  $1 + b$ , and the beam  $L$  and  $\bar{R}$  (lower modes in Fig. 2) with abundance  $1 - b$  so that  $-1 < b < +1$  describes an initial left–right asymmetry. Therefore, the phase-space densities of our four modes are taken to be

$$r = +\frac{1}{2}(1 + a)(1 + b), \tag{9a}$$

$$\bar{l} = -\frac{1}{2}(1 - a)(1 + b), \tag{9b}$$

$$\bar{r} = -\frac{1}{2}(1 - a)(1 - b), \tag{9c}$$

$$l = +\frac{1}{2}(1 + a)(1 - b). \tag{9d}$$

The overall normalization is indeed  $r + l - \bar{r} - \bar{l} = 2$  for all  $a$  and  $b$ . The neutrino–neutrino interaction matrix in Eq. (8) becomes

$$\mu \begin{pmatrix} 2(a - b) & (1 - a)(1 + b) & -(1 + a)(1 - b) & 0 \\ -(1 + a)(1 + b) & 2(a + b) & 0 & (1 - a)(1 - b) \\ -(1 + a)(1 + b) & 0 & 2(a + b) & (1 - a)(1 - b) \\ 0 & (1 - a)(1 + b) & -(1 + a)(1 - b) & 2(a - b) \end{pmatrix}. \tag{10}$$

Overall we have to deal with a quartic eigenvalue equation.

#### 4.2. Spontaneous left–right symmetry breaking

As a first explicit case we begin with a system which is prepared to be left–right symmetric ( $b = 0$ ). If the solution inherits this symmetry it should be left–right symmetric as well, so it makes sense to define left–right symmetric modes  $A_+ = \frac{1}{2}(R + L)$  and  $\bar{A}_+ = \frac{1}{2}(\bar{R} + \bar{L})$  as well as antisymmetric modes  $A_- = \frac{1}{2}(R - L)$  and  $\bar{A}_- = \frac{1}{2}(\bar{R} - \bar{L})$ . The eigenvalue equation then is



$$\Omega \begin{pmatrix} A_+ \\ \bar{A}_+ \\ A_- \\ \bar{A}_- \end{pmatrix} = \left[ \begin{pmatrix} \omega & 0 & k & 0 \\ 0 & -\omega & 0 & k \\ k & 0 & \omega & 0 \\ 0 & k & 0 & -\omega \end{pmatrix} + \lambda + 2a\mu + \mu \begin{pmatrix} -(1+a) & 1-a & 0 & 0 \\ -(1+a) & 1-a & 0 & 0 \\ 0 & 0 & 1+a & -(1-a) \\ 0 & 0 & 1+a & -(1-a) \end{pmatrix} \right] \begin{pmatrix} A_+ \\ \bar{A}_+ \\ A_- \\ \bar{A}_- \end{pmatrix}. \tag{11}$$

Assuming the solution to be homogeneous ( $k = 0$ ), the left–right symmetric and antisymmetric modes decouple and we find two pairs of equations.

The upper pair (symmetric solution) corresponds to the traditional flavor pendulum [10,11]. The eigenvalues are explicitly

$$\Omega = \lambda + a\mu \pm \sqrt{(a\mu)^2 + \omega(\omega - 2\mu)}. \tag{12}$$

The argument of the square root is negative, and thus we have a run-away solution, in the range

$$\frac{1}{1 + \sqrt{1 - a^2}} < \frac{\mu}{\omega} < \frac{1}{1 - \sqrt{1 - a^2}}. \tag{13}$$

The maximum growth rate arises for  $\mu = \omega/a^2$  and is  $\text{Im } \Omega|_{\text{max}} = \omega\sqrt{1 - 1/a^2}$ . For normal mass ordering, we should take  $\omega \rightarrow -\omega$  or equivalently  $\mu \rightarrow -\mu$  as explained earlier. In this case  $\Omega$  is always real and one finds no instability. This basic case illustrates what seemed to be a generic property of self-induced flavor conversion: there is an instability for inverted mass ordering and the growth rate is of order  $\Delta m^2/2E$ .

The lower pair of equations in Eq. (11) is the same as the upper one with  $\mu \rightarrow -\mu$ . Therefore, the left–right antisymmetric modes have the same instability as the symmetric case, but for normal neutrino mass ordering. This observation was the main point of Ref. [21], where a graphical explanation in terms of polarization vectors was given. The main lesson is that unstable solutions do not need to respect the symmetries of the initial condition, but of course the unstable mode requires a small seed (a small disturbance of the symmetric initial condition) to be able to grow.

### 4.3. Spontaneous breaking of homogeneity

We may next consider the evolution of a spatial Fourier mode with nonvanishing wave number  $k$ , keeping the system initially left–right symmetric ( $b = 0$ ). The idea to consider the evolution and stability of spatial Fourier modes was first proposed rather recently [25,26] and more detailed studies followed soon [27–29]. In our simple beam case, the left–right symmetric and antisymmetric modes are coupled so that it is simpler to return to the original version Eq. (8) together with Eq. (10). With  $b = 0$ , the eigenvalue equation is

$$\det \left[ \lambda + 2a\mu - \Omega + \begin{pmatrix} \omega + k & (1-a)\mu & -(1+a)\mu & 0 \\ -(1+a)\mu & -\omega - k & 0 & (1-a)\mu \\ -(1+a)\mu & 0 & \omega - k & (1-a)\mu \\ 0 & (1-a)\mu & -(1+a)\mu & -\omega + k \end{pmatrix} \right] = 0. \tag{14}$$

The explicit solutions of this quartic equation are too complicated to be informative. However, in the limit  $k \rightarrow \pm\infty$  it can be simplified. In this limit, the diagonal elements of the matrix strongly dominate and we expect the solution to be of the form  $\Omega = \tilde{\Omega} + \lambda + 2a\mu \pm k$ , where  $\tilde{\Omega}$  is a frequency which is small compared with  $k$ . We first consider  $\Omega = \tilde{\Omega} + \lambda + 2a\mu - k$ , leading to

$$\det \begin{pmatrix} 2k + \omega - \tilde{\Omega} & (1-a)\mu & -(1+a)\mu & 0 \\ -(1+a)\mu & -\omega - \tilde{\Omega} & 0 & (1-a)\mu \\ -(1+a)\mu & 0 & \omega - \tilde{\Omega} & (1-a)\mu \\ 0 & (1-a)\mu & -(1+a)\mu & 2k - \omega - \tilde{\Omega} \end{pmatrix} = 0. \quad (15)$$

In the limit  $k \rightarrow \infty$  we may approximate the upper left and lower right diagonal elements with  $2k$ , leading to a quadratic equation for the eigenvalues  $\tilde{\Omega}$ . The expressions simplify considerably if we write  $\mu = m\sqrt{\omega k}$  in terms of a dimensionless parameter  $m$ , leading to

$$\tilde{\Omega} = \left( -a^2 m^2 \pm \sqrt{1 - 2am^2 + a^4 m^4} \right) \omega. \quad (16)$$

For this expression to have an imaginary part, we need  $a > 0$ , corresponding to an excess of neutrinos over antineutrinos, and in a range of  $m$  values corresponding to

$$a \left( 1 - \sqrt{1 - a^2} \right) < \frac{k\omega}{\mu^2} < a \left( 1 + \sqrt{1 - a^2} \right). \quad (17)$$

For  $k \rightarrow -\infty$  and  $a > 0$  the system is stable. Unstable solutions also exist when both  $a$  and  $k$  are negative, i.e., the system has unstable modes if  $a$  and  $k$  have equal sign. The unstable  $\mu$  range scales with  $\sqrt{\omega k}$ , in agreement with previous findings [26,29].

Notice that the sign of  $k$  distinguishes between left- or right-moving spatial disturbances, i.e., the spontaneous breaking of homogeneity by the exponential growth of  $k$  modes also breaks the initial left–right symmetry of the system.

The Fourier mode with the maximum growth rate is  $k = a^3 \mu^2 / \omega$  and provides  $\text{Im } \Omega|_{\max} = \omega (a^{-2} - 1)^{1/2}$ . For any  $k$  and taking  $\omega = 0$ , the eigenvalue equation (14) has only real solutions. The speed of growth of the unstable modes in this system is always of order  $\omega$ .

Going beyond linear order, the equations of motion couple different spatial Fourier modes because in Fourier space, the rhs of Eq. (1) becomes a convolution. Therefore, spatial inhomogeneities will quickly lead to kinematical decoherence [27]. One may think that this outcome is inevitable because for any neutrino density (any value of  $\mu$ ) there is a range of unstable  $k$ -modes. On the other hand, in the geometry of Fig. 1 we also need to include the multi-angle matter effect and study the instability condition in the parameter space spanned by  $\mu$ ,  $\lambda$  and  $k$ . In this geometry, the meaning of  $\mathbf{k}$  is a Fourier mode in the transverse plane and the spontaneous breaking of homogeneity actually amounts to the breaking of global spherical symmetry. As it has been seen several times [50–54], the inclusion of multi-angle matter effects tends to suppress instabilities when the matter potential is large. For the inhomogeneous instabilities this also turns out to be true [29] (see also Fig. 3), and the unstable region of parameter space, the instability “footprint”, is pushed to larger values of  $\mu$  for a large  $\lambda$ . Another interesting consequence of the multi-angle matter effect seems to be that the largest spatial scales (small  $k$ ) in supernovae are the most unstable ones. This, in the sense that larger values of  $k$  give rise to instability footprints that are further away from the density profile of a realistic supernova than for  $k = 0$ , meaning that all transverse  $k$  modes are stable if the homogeneous mode is stable. On the other hand, if an instability is encountered, flavor decoherence may be a generic outcome.

## 5. Fast flavor conversion

### 5.1. Asymmetric beam

So far the growth rate of unstable modes was found to be proportional to the vacuum oscillation frequency. On the other hand, Sawyer has shown that one can construct cases where in

our notation the growth rate is proportional to  $\mu$ , not to  $\omega$  [9,32]. Our colliding beam model provides perhaps the simplest possible example for this phenomenon of “fast flavor conversion,” but it requires explicit left–right symmetry breaking by the initial condition [33]. Setting  $b = 1$  in Eq. (9) implies that in Fig. 2 only the upper modes are occupied, the lower ones are empty: we have only right-moving  $\nu$  and only left-moving  $\bar{\nu}$ . The equation of motion for the two occupied modes following from Eqs. (8) and (10) is

$$\Omega \begin{pmatrix} R \\ \bar{L} \end{pmatrix} = \left[ \begin{pmatrix} \omega + k & 0 \\ 0 & -\omega - k \end{pmatrix} + 2\mu \begin{pmatrix} -1 + \tilde{a}, & 1 - a \\ -1 - a, & 1 + a \end{pmatrix} \right] \begin{pmatrix} R \\ \bar{L} \end{pmatrix}. \quad (18)$$

We notice immediately that the role  $\omega$  is here played by  $\tilde{\omega} = \omega + k$ . Even for vanishing  $\omega = \Delta m^2/2E$ , we find unstable solutions in the form of spatial Fourier modes with  $k \neq 0$ . The reason for this behavior is that neutrinos (vacuum frequency  $+\omega$ ) move right so that the spatial Fourier term  $\mathbf{v} \cdot \mathbf{k}$  enters as  $+k$ , and the other way round for the beam of left-moving antineutrinos.

The explicit eigenvalues are found to be  $\Omega = 2a\mu \pm [(2a\mu)^2 + \tilde{\omega}(\tilde{\omega} - 4\mu)]^{1/2}$ , i.e., for  $\tilde{\omega} = 0$  they are purely real. One finds an imaginary part for  $1 - (1 - a^2)^{1/2} < \tilde{\omega}/2\mu < 1 + (1 - a^2)^{1/2}$ . Because  $\mu$  is defined to be positive, we have unstable solutions only for  $\tilde{\omega} > 0$  or  $-\omega < k$ . For a fixed  $\mu$  value, the maximum growth rate occurs for  $\tilde{\omega} = 2\mu$  and is

$$\text{Im } \Omega|_{\text{max}} = 2\mu\sqrt{1 - a^2}. \quad (19)$$

This rate is indeed “fast” in the sense that it is proportional to  $\mu = \sqrt{2}G_{\text{F}}n_{\nu}$ .

The explicit left–right symmetry breaking of the initial system need not be maximal to obtain this effect. The stability in the entire parameter space  $-1 < a < 1$  and  $-1 < b < 1$  was studied in Ref. [33]. A certain range of unstable  $k$  modes exists if  $a^2 < b^2$ , i.e., the left–right asymmetry must exceed the neutrino–antineutrino asymmetry. Details aside, the main point is that a sufficiently asymmetric system can be unstable in flavor space even if neutrinos were exactly massless and even if vacuum flavor oscillations would not exist.

## 5.2. Different zenith angle distributions for neutrinos and antineutrinos

In the supernova geometry of Fig. 1, the “left–right asymmetry” of the previous section would translate into significant azimuthal asymmetries of the neutrino and antineutrino velocities in the transverse plane. However, fast flavor conversion can also appear in a different case of explicit symmetry breaking, i.e., when the  $\nu_e$  and  $\bar{\nu}_e$  zenith angle distributions are sufficiently different [32]. These species have rather different interaction rates with a nuclear medium and thus decouple in different regions. Typically, one expects a larger flux of  $\nu_e$  because of depletion, and a broader zenith-angle distribution for  $\nu_e$  because of their larger interaction rate. However, in different phases or regions of a core-collapse supernova or in neutron-star mergers, a variety of distributions may occur. We here worry only about the conceptual issue of fast flavor conversion, not about realistic quantitative scenarios.

This system can be mimicked with a small variation of our colliding beam model. The zenith-angle distribution in the geometry of Fig. 1 translates into a distribution of velocities in the transverse plane. Neutrinos emitted in the radial direction have vanishing velocity in the observation plane, whereas those emitted from the limb of the source have a maximal transverse velocity of  $v_{\text{max}} = R/r$  (source radius  $R$ , distance  $r$ ). Therefore, the zenith-angle distribution can be represented by a colliding beam with different velocities. We can use any convenient range of velocities, absorbing an overall scale in the definition of the interaction strength  $\mu$ . So we consider a colliding beam as in Fig. 2, the crucial ingredient being a different velocity for  $\nu_e$  and  $\bar{\nu}_e$ ,

while maintaining perfect left–right symmetry. This toy example mimics the idea that  $\nu_e$  and  $\bar{\nu}_e$  are emitted from neutrino spheres with different radius, which is the original model studied by Sawyer [32].

Specifically we use  $r = l = \frac{1}{2}(1 + a)$  and  $\bar{r} = \bar{l} = -\frac{1}{2}(1 - a)$  for the occupations of the neutrino and antineutrino modes with the same asymmetry parameter  $-1 < a < 1$  as before. Moreover, we use the velocities  $v_R = -v_L = 1 + b$  and  $v_{\bar{R}} = -v_{\bar{L}} = 1 - b$  so that  $-1 < b < 1$  expresses the different “zenith angles.” The left–right symmetric setup and using the homogeneous case  $k = 0$  implies that the left–right symmetric and antisymmetric modes decouple. In the limit  $\omega = 0$ , the eigenvalues for the latter are purely real and thus do not show fast flavor conversion. The symmetric modes, on the other hand, produce the eigenvalues

$$\Omega/\mu = a - 2b - ab^2 \pm \sqrt{4ab(1 + b^2) + a^2(1 + 6b^2 + b^4)}. \quad (20)$$

A necessary condition for a nonvanishing imaginary part is  $ab < 0$ , meaning that the species  $\nu_e$  or  $\bar{\nu}_e$  with the larger abundance must have the smaller “zenith angle” for a fast instability to exist. The main point is that we can easily construct a homogeneous and left–right symmetric beam model that shows fast flavor conversion.

More physical examples in the spirit of Fig. 1 were studied in Refs. [32,33], assuming neutrino “bulbs” with different radii for  $\nu_e$  and  $\bar{\nu}_e$ . The exact conditions where fast flavor conversion appears [33] depends on the matter effect  $\lambda$  besides the neutrino zenith-angle distributions. The appearance of this effect does not seem to require extreme parameters.

## 6. Temporal instabilities

In principle it is straightforward to solve the general equation of motion Eq. (1) or its linearized version Eq. (7) in Fourier space, although it took a long time until it dawned on our community that the nature of self-induced flavor conversion as an instability implies that, for example, an initially homogeneous system can grow large inhomogeneities in flavor space if some spatial Fourier modes are unstable. A similar observation applies to a Fourier transform in the time domain. We can imagine neutrinos streaming from a stationary source and we may ask for the flavor evolution as a function of distance, for the moment assuming perfect spherical symmetry of the source and the solution.

As a first case we may imagine that the source has some small periodic time variation with frequency  $\Omega$  imprinted on it [30]. Then, instead of solving the stationary version of Eq. (1) without time variation of the source or solution, we should instead consider a mode varying as  $e^{-i\Omega t}$ . In particular in the linearized two-flavor form of Eq. (7), the only consequence is that  $\lambda \rightarrow \lambda - \Omega$ . As a next step, we should proceed as before and find, for example, growing modes as a function of radial distance. This approach was taken one step further in Ref. [31] where the authors argued that we do not need a driving frequency  $\Omega$  provided by some supernova physics, but that a small temporal disturbance was enough to trigger a growing mode of that frequency.

The crucial physical impact of the modification  $\lambda \rightarrow \lambda - \Omega$  is that the matter effect is reduced. As we already discussed, the presence of  $\lambda$  has a strong impact on the flavor evolution as a function of distance through the multi-angle matter effect [50–54]. As a simple case we consider plane-parallel geometry and study the evolution as a function of  $z$ , assuming translational sym-

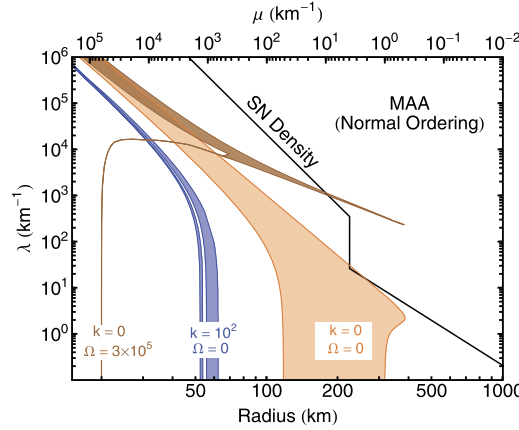


Fig. 3. Instability footprints of the MAA instability for temporal or spatial variations for a model with  $R = 15$  km. The homogeneous and time-independent instability ( $k = 0, \Omega = 0$ ) is shown for reference. In addition, an inhomogeneous mode with wave number  $k = 10^2$  (in units of the vacuum oscillation frequency) and a temporal mode with frequency  $\Omega = 3 \times 10^5$  are shown. The shaded areas are defined by  $\text{Im } \Omega > 10^{-2}$ . The black line indicates a schematic density profile of a supernova, where the decrease in  $\lambda$  just above  $r = 200$  km corresponds to the shock wave front. Notice that  $\mu \propto r^{-4}$  which gives a one-to-one correspondence between  $\mu$  and radius. (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

metry in the  $x$ - $y$ -plane. In this case  $i\mathbf{v} \cdot \nabla$  in Eq. (7) becomes  $i v_z \partial_z$  and the linearized equation of motion becomes

$$i v_z \partial_z S_{\Omega, z, \omega, \mathbf{v}} = \left[ \omega + \lambda - \Omega + \mu (\epsilon - v_z \phi_z) \right] S_{\Omega, z, \omega, \mathbf{v}} - \mu \int d\Gamma' (1 - \mathbf{v} \cdot \mathbf{v}') g_{\omega', \mathbf{v}'} S_{\Omega, z, \omega', \mathbf{v}'}. \tag{21}$$

The impact of  $v_z$  appearing on the lhs is most easily understood in the limit where the neutrino flux is nearly collinear, i.e., at a large distance from the source. We may describe all modes by the small velocity vector  $\boldsymbol{\beta}$  in the transverse plane and we bring  $v_z$  to the rhs in the form  $v_z^{-1} = (1 - \boldsymbol{\beta}^2)^{-1/2} \approx 1 + \frac{1}{2} \boldsymbol{\beta}^2$ . On the rhs all terms except  $\omega + \lambda - \Omega$  are already of order  $\boldsymbol{\beta}^2$  and thus to order  $\boldsymbol{\beta}^2$  remain unaffected by this factor. The mode frequencies  $\omega$  already vary over a broad range and are small compared with  $\lambda$ , so we may approximate  $(1 + \frac{1}{2} \boldsymbol{\beta}^2) \omega \approx \omega$ . To solve Eq. (21) we are looking for solutions proportional to  $e^{iKz}$  with  $K$  a complex wavenumber where an imaginary part of  $K$  represents an exponentially growing mode. Therefore, the eigenvalue equation involves a term  $\omega + (1 + \frac{1}{2} \boldsymbol{\beta}^2)(\lambda - \Omega) + K = \omega + \frac{1}{2} \boldsymbol{\beta}^2(\lambda - \Omega) + \tilde{K}$  with  $\tilde{K} = K + \lambda - \Omega$ . In other words, the real part of  $K$  gets shifted, corresponding to solving the equation in a frame in flavor space which rotates with frequency  $\lambda - \Omega$  as a function of  $z$ . What remains is a term  $\frac{1}{2} \boldsymbol{\beta}^2(\lambda - \Omega)$  which is different for every velocity, i.e., the impact of  $\lambda$  depends on direction. This “multi-angle matter effect” originates from the simple observation that a mode with general  $\mathbf{v}$  covers a larger distance to reach the same  $z$  than a mode which moves along the  $z$ -direction and thus acquires a larger matter phase.

Neutrinos streaming from a SN core encounter a certain trajectory of  $(\lambda, \mu)$  values and may never encounter any instability if  $\lambda$  is large, depending on the “footprint” of the unstable region in the  $\lambda$ - $\mu$ -plane [29,50,52,55]. In Fig. 3, we show such footprints, following the approach from Ref. [29] for the MAA instability. The footprints correspond to solutions of Eq. (21)

with  $\text{Im } K > 10^{-2}$  for different initial instability seeds. Compared to the notation in Eq. (21), we have absorbed a factor of  $\beta^2$  into the definitions of  $\mu$  and  $\lambda$ , which means that we redefine  $\lambda = \sqrt{2}G_F N_e(r)(R/r)^2$  and  $\mu = \sqrt{2}G_F n_{\nu_e}(r)(R/r)^2$  where  $r$  is the radius and  $R$  is the neutrino-sphere radius. The red-orange footprint is for the stationary, homogeneous mode ( $\Omega = 0, k = 0$ ), and all the stationary, smaller scale modes fill out the region below this footprint as demonstrated with the blue footprint for  $k = 10^2$  which we also described in Sec. 4. However, with the modification  $\lambda \rightarrow \lambda - \Omega(R/r)^2$ , at any distance from the source there are some  $\Omega$  modes that are unstable<sup>4</sup> and grow exponentially [31]. The brown footprint shows the homogeneous case with a frequency  $\Omega = 3 \times 10^5$ . Here the cancellation between  $\Omega$  and  $\lambda$  is obvious as the unstable region is lifted from small values of  $\lambda$  to much larger values. However, it is also evident that it is only relevant for the supernova profile in a very narrow distance range. Therefore, the huge growth factors envisioned in Ref. [31] may not necessarily materialize. On the other hand, one also needs to worry about the impact of spatial Fourier modes in the transverse direction. Moreover, the question of fast-growing modes for nontrivial angle distributions as proposed by Sawyer [32] have not yet been studied in this context.

## 7. Conclusions and outlook

The topic of collective neutrino flavor conversion has undergone a shift of paradigm over the past couple of years. Neutrinos are produced at the source far from flavor equilibrium due to their different low-energy interaction channels in stellar core collapse or in neutron-star mergers. Collective flavor effects seemed to imprint interesting features on the flavor-dependent spectral fluxes, yet preserve the flavor dependence. However, many of the old results were based on too many symmetry assumptions about the emission characteristics and the solutions. Meanwhile it has become clear that the story of collective neutrino oscillations is a complicated saga of instabilities in flavor space caused in particular by the spontaneous breaking of space–time symmetries. Moreover, depending on the angle distributions, collective flavor conversion does not even depend on neutrino masses and mixing parameters, a phenomenon dubbed “fast flavor conversion” because the exponential growth rate is of order the neutrino interaction energy  $\mu$  instead of the much smaller vacuum oscillation frequency  $\omega$ . Several authors have speculated that these different instabilities and their nonlinear coupling with each other might push the system close to kinematical decoherence in flavor space. In this sense, the final outcome might be simple.

However, this development poses new questions. If self-induced flavor conversion and concomitant decoherence already commence in the neutrino decoupling region, the traditional treatment in the spirit of Fig. 1 would not be appropriate. One would have to deal with an environment where neutrinos stream in all directions, yet with a net outward flux. Probably one then needs to worry about Eq. (1) as a full space–time problem, and not as collective evolution along the outward radial direction, but also not as a simple time evolution of a quasi-homogeneous system akin to the early universe. Presumably, exponential growth can happen in both space and time, and moreover may not need to follow symmetries of the macroscopic system. Moreover, non-forward collisions would have to be taken into account on some level.

Neutrino flavor evolution under the influence of neutrino–neutrino refraction remains a challenging subject. As of late, many questions have come into much clearer focus, yet the full picture has not yet entirely emerged.

<sup>4</sup> The explicit dependence on  $R$  in the cancellation condition means that the footprint will depend on the neutrino-sphere radius. This behavior is in contrast to the stationary case ( $\Omega = 0$ ) where the footprints are independent of  $R$ .

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