I. INTRODUCTION

The neutrino flux streaming off a collapsed supernova (SN) core is an intriguing astrophysical case where flavor transformations depend sensitively on some of the unknown elements of the leptonic mixing matrix \([1,2]\). Since the conversion probabilities depend also on the time-dependent matter profile, a high-statistics SN neutrino observation may also reveal, for example, signatures for shock-wave propagation \([3–9]\). While galactic SNe are rare, various ongoing and future experimental programmes depend on large detectors that, besides their main purpose, are also sensitive to SN neutrinos \([10]\). Therefore, understanding the flavor evolution of a SN neutrino signal remains of topical interest.

The flavor transformation probabilities not only depend on the matter background, but also on the neutrino fluxes themselves: neutrino-neutrino interactions provide a nonlinear term in the equations of motion \([11,12]\) that causes collective flavor transformations \([13–22]\). Only recently has it been fully appreciated that in the SN context these linear term in the equations of motion \([11,12]\) that causes collective flavor transformations not only depend on the time-dependent matter profile, a high-statistics SN neutrino observation may also reveal, for example, signatures for shock-wave propagation \([3–9]\). While galactic SNe are rare, various ongoing and future experimental programmes depend on large detectors that, besides their main purpose, are also sensitive to SN neutrinos \([10]\). Therefore, understanding the flavor evolution of a SN neutrino signal remains of topical interest.

The flavor transformation probabilities not only depend on the matter background, but also on the neutrino fluxes themselves: neutrino-neutrino interactions provide a nonlinear term in the equations of motion \([11,12]\) that causes collective flavor transformations \([13–22]\). Only recently has it been fully appreciated that in the SN context these collective effects give rise to qualitatively new phenomena \([23–36]\).

One peculiar aspect of the expected SN neutrino fluxes is the hierarchy \(F_{\nu_e} > F_{\nu_\mu} > F_{\bar{\nu}_e} = F_{\bar{\nu}_\mu} = F_{\nu_x} = F_{\bar{\nu}_x}\), so that there is an excess flux of \(\nu_x\bar{\nu}_x\) pairs over those of the other flavors. The nonlinear terms cause a collective transformation \(\nu_x\bar{\nu}_x \rightarrow \nu_x\bar{\nu}_x\), where \(\nu_x\) is a specific linear combination of \(\nu_\mu\) and \(\nu_x\). The detailed dynamics of this transition is complicated, and several important aspects are only numerically observed, not analytically understood. Still, the most crucial point is that the pair transformation \(\nu_x\bar{\nu}_x \rightarrow \nu_x\bar{\nu}_x\) proceeds collectively much faster than ordinary pair annihilation, so we have to contend with a "speed-up phenomenon" \([21,22]\). The pair process does not violate flavor-lepton number. Being an instability in flavor space, it proceeds efficiently even for a very small mixing angle.

A very different situation prevails in the interior of a SN core where the \(\nu_x\) distribution is determined by a large chemical potential that enhances the \(\nu_e\) density and suppresses the \(\bar{\nu}_x\) density relative to that of \(\nu_x\) and \(\bar{\nu}_x\) so that collective pair conversions are not possible. Neutrino-neutrino interactions are strong, but their only impact is to synchronize the flavor oscillations ("self-maintained coherence"). Significant flavor transformation here requires a violation of flavor-lepton number. However, the high density of ordinary matter suppresses the effective mixing angles between \(\nu_e\) and the other flavors. Therefore, in the interior of a SN core the individual flavor-lepton numbers are almost perfectly conserved \([37]\).

Immediately after collapse, the SN emits a prompt \(\nu_x\) burst that arises from the deleptonization (neutronization) of the outer layers of the collapsed core. Once more, we have a strongly enhanced \(\nu_x\) and a suppressed \(\bar{\nu}_x\) flux relative to the other flavors \([38]\). Once more, collective pair transformations are not possible: efficient flavor conversion requires a large violation of flavor-lepton number and is not possible if the ordinary matter density is large. At some distance from the neutrino sphere, the \(\nu_x\) flux encounters the usual Mikheyev-Smirnov-Wolfenstein (MSW) level crossings driven by the atmospheric neutrino mass difference \(\Delta m^2_{\text{atm}}\) (H crossing) and by the solar mass difference \(\Delta m^2_{\odot}\) (L crossing), leading to the usual resonant transformations \([1,38]\).

An interesting new case is motivated by the insight that SNe with the lowest progenitor masses of \(8–10\,M_\odot\), encompassing perhaps 30% of all cases, collapse before producing an iron core. These stars form the class of O-Ne-Mg core collapse SNe \([39–42]\). In state-of-the-art numerical simulations these SNe explode even in a spherically symmetric treatment (convection plays no role),...
largely because their envelope mass is very small. By the same token, the matter density profile above the core is very steep even at the time of core bounce. In this case the H and L level crossings occur very close to the neutrino sphere and may well lie deeply within the collective neutrino region. This is illustrated in Fig. 1 where we show \( \lambda(r) = \sqrt{2} G_F n_e(r) \) of an O-Ne-Mg core progenitor star [39–41]. We also show \( \omega_H = \langle \Delta m_{\text{atm}}^2/2E \rangle \) and \( \omega_L = \langle \Delta m_{\text{sol}}^2/2E \rangle \) as horizontal lines, where the average is over the Fermi-Dirac spectrum of neutrino energies described below. The intersection of \( \lambda(r) \) with these lines indicates the locations of the H and L level crossings.

In Fig. 1 we also show the effective neutrino-neutrino interaction potential \( \mu = \sqrt{2} G_F F_{\nu_e} \langle 1 - \cos \theta \rangle_{\text{eff}} \), where \( \theta \) is the angle between different neutrino trajectories and \( \langle \ldots \rangle_{\text{eff}} \) stands for a suitable average. At large distances, \( \mu \) scales approximately as \( r^{-3} \). Collective neutrino effects driven by \( \Delta m_{\text{atm}}^2 \) are important for \( \mu(r) \simeq \omega_H \) and driven by \( \Delta m_{\text{sol}}^2 \) for \( \mu(r) \simeq \omega_L \).

Duan et al. [34] have recently shown that in this case the interplay of ordinary MSW conversions with collective effects leads to interesting effects. We start with a pure \( \nu_e \) flux with a Fermi-Dirac spectrum \( \langle E_{\nu_e} \rangle = 11 \) MeV, degeneracy parameter \( \eta = 3 \), and numerically calculate the mass eigenstate fractions \( P_{\nu_1} \) and the \( \nu_e \) survival probabilities \( P_{\nu_e} \) far away from the star, as shown in Fig. 2. These plots are in qualitative agreement with the corresponding curves in Fig. 2 of Ref. [34]. However, our \( P_{ee} \) is constructed as an incoherent sum of the mass fractions, thus representing the physical situation far away from the star, where the oscillatory features seen in Duan et al.’s \( P_{ee} \) have disappeared.

In inverted hierarchy, one observes that the neutrinos emerging from the star are in the \( \nu_2 \) state at low energies and in the \( \nu_1 \) state at high energies, with the transition taking place around \( E = 12 \) MeV. This results in a step function in energy for \( P_{ee} \).

In the normal hierarchy, the neutrinos emerging from the star are in the \( \nu_1 \) state for \( E \simeq 17 \) MeV, in the \( \nu_2 \) state for \( 15 \text{ MeV} \leq E \leq 17 \) MeV, and in the \( \nu_3 \) state for \( E \lesssim 15 \) MeV. The bump seen around 5 MeV is due to an abrupt change in the matter density profile used for the computation (see [34] for details), and we do not address it here. The transition at \( E \simeq 15 \) MeV is rather sharp; however, the one at \( E = 17 \) MeV is not as abrupt. This results in a two-step function for \( P_{ee} \), with the step at \( E = 17 \) MeV somewhat smoothened out.

Broadly, this is an example of a “MSW-prepared spectral split.” In a two-flavor language, it is explained as follows. The strong neutrino-neutrino interactions lead to a synchronization of the neutrino oscillations. The flavor polarization vector of the ensemble begins at high density essentially aligned with the \( \nu_e \) direction in flavor space. After passing the MSW region, the polarization vector emerges with a significant transverse component relative to the mass direction because the MSW transition is not fully adiabatic. Subsequently this MSW-prepared initial condition is subject to collective effects only. As the effective neutrino-neutrino interaction becomes weaker, the modes above a certain energy \( E_{\nu} \) orient themselves along the mass direction, while those with smaller energies in the opposite direction. This is precisely the “neutrino only” case studied in Refs. [28,29] as a generic case for a spectral split, a case that requires one to prepare the polarization vector with a large transverse component. (For neutrinos plus antineutrinos, the MSW preparation is not necessary because the collective pair transformations alone engineer a split.)

Our main goal here is to use the picture of a MSW-prepared spectral split to derive analytically the main
features seen in Duan et al.'s numerical study [34], viz. the existence and the positions of the spectral splits. The numerical model in Fig. 1 shows that the MSW resonances are within the collective neutrino region, but not deep inside. Therefore it is not a priori obvious if the “synchronized MSW preparation” and the subsequent split can be clearly separated. For our discussion we therefore adopt a more schematic model. We artificially increase the neutrino-neutrino interaction strength (raise the $\mu$ profile in Fig. 1) such that the MSW region and the spectral-split regions are clearly separate. In this framework we first calculate the MSW preparation analytically and then study a three-flavor treatment of the spectral split, based on the machinery recently developed by two of us [36]. We find that our analytic treatment reproduces the numerical results surprisingly well. It also reproduces all relevant features of the realistic case, i.e. with the $\mu$ profile of Fig. 1, justifying our simplifying assumptions and verifying our general interpretation.

We first set up in Sec. II our schematic SN model that captures the features relevant for our treatment, and continue in Sec. III with the equations of motion. In Sec. IV we derive analytically the MSW-prepared three-flavor state that serves as input for our three-flavor spectral-split study in Sec. V. We conclude in Sec. VI.

II. SIMPLIFIED SUPERNOVA MODEL

A. Realistic scenario

We take the electron density profile $n_e$ obtained from numerical studies of O-Ne-Mg core SNe [39-41], providing $\lambda(r)$ as shown in our Fig. 1. The neutrino luminosity is assumed to be $L_{\nu_e} = 10^{53}$ erg s$^{-1}$ with a Fermi-Dirac spectrum with the average energy $\langle E_{\nu_e} \rangle = 11$ MeV and a degeneracy parameter $\eta = 3$, implying a temperature $T = 2.76$ MeV. The neutrino sphere is taken at the radius $R = 60$ km, implying an effective neutrino-neutrino interaction strength at large distances of

$$\mu(r) = \mu_0 \left(\frac{R}{r}\right)^3$$

(1)

with $\mu_0 = 8.6 \times 10^{11}$ km$^{-1}$. This is the profile shown in Fig. 1. Based on this model we have solved the equations of motion numerically and found the electron neutrino survival probability shown in Fig. 2, in agreement with the results of Duan et al. [34].

In this realistic situation, the decrease of the effective neutrino-neutrino interaction with radius is such that the spectral split is essentially adiabatic. Quantitatively, the length scale $\ell_{\mu} = |d\ln\mu(r)/dr|^{-1}$ is large enough to satisfy the adiabaticity condition (see Sec. V of [28]) during the spectral split. The very development of a steplike feature in $P_{ee}$ in Fig. 2 is an expression of the adiabaticity. The “sharpness” of the step in $P_{ee}$ is a measure of the degree of adiabaticity; an extremely slowly decreasing neutrino-neutrino interaction corresponds to a perfect step function in $P_{ee}$ [28,29].

B. Analytical treatment

Our analytic treatment is based on a schematic representation of the essentials of this realistic case. In our picture the synchronized MSW effect occurs first and factorizes from the collective oscillations that lead to the spectral split. To this end, we consider the limiting case where $\mu_0$ in Eq. (1) is arbitrarily large. This makes the H and L level crossings more synchronized, and pushes the regions where the spectral splits occur, i.e. the regions where $\mu(r)$ becomes comparable to $\omega_H$ or $\omega_L$, to much larger radii. This also renders the spectral splits even more adiabatic, making the steps in $P_{ee}$ sharper.

It is also important that the MSW transition is not perfectly adiabatic. In our schematic model, we use the power-law profile for the matter potential,

$$\lambda(r) = \lambda_0 \left(\frac{r_0}{r}\right)^{\alpha}$$

(2)

with $\lambda_0 = 10^3$ km$^{-1}$, $r_0 = 900$ km, and $\alpha = 50$. This approximates reasonably the numerical $\lambda(r)$ profile of Fig. 1 in the neighborhood of the H and L crossings. This allows an analytic estimate of the level crossing probabilities at the two resonances.

Thus, we obtain the analytic results in the limit of $\mu_0 \rightarrow \infty$ and a power-law profile for $\lambda(r)$, where (i) the MSW transitions are perfectly synchronized but semiadiabatic, (ii) the spectral-split regions are well separated from the MSW region, and (iii) the spectral splits are perfectly adiabatic, making the steps in $P_{ee}$ infinitely sharp. Apart from these minor changes, the physical pictures with the realistic profile and our schematic profile are identical.

C. Numerical treatment

We also study the spectral splits numerically, assuming the analytic MSW-prepared spectra as input. For such a numerical illustration we employ

$$\mu_0 = 10^{15} \text{ km}^{-1},$$

(3)

much larger than the realistic value shown in Fig. 1. The analytic results match the numerical ones in all details, and reproduce the main features of the realistic situation, with a minor difference that the spectral splits appear sharper. This is because the net effect of our approximations is only to make the MSW resonances more synchronized, and the spectral splits more adiabatic.

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1Since $\mu(r)$ is a power law, the length scale $\ell_{\mu}$ increases with increasing $r$. As a result, a larger radius for the spectral split implies larger $\ell_{\mu}$ and more adiabaticity.
III. EQUATIONS OF MOTION

A. Matrices of density

Mixed neutrinos are described by matrices of density $\varrho_p$ for each momentum. The diagonal entries are the usual occupation numbers, whereas the off-diagonal terms encode phase information. The equations of motion (EOMs) are

$$i \frac{d}{dt} \varrho_p = [H_p, \varrho_p],$$

where the Hamiltonian is [12]

$$H_p = \Omega_p + V + \sqrt{2}G_F \int \frac{d^3q}{(2\pi)^3} (\varrho_q - \varrho_q) (1 - \varrho_q \cdot \varrho_p),$$

$\varrho_p$ being the velocity. The matrix of vacuum oscillation frequencies is $\Omega_p = \text{diag}(m^2_1, m^2_2, m^2_3)/2|\mathbf{p}|$ in the mass basis. The matter effect is represented, in the weak interaction basis, by $V = \sqrt{2}G_F n_e \text{diag}(1,0,0)$. While, in general, there is a second-order difference between the $\nu_\mu$ and $\nu_\tau$ refractive index [43] that can be important for collective neutrino oscillations [35]; for the low matter densities relevant in our case, this “mu-tau matter term” is irrelevant.

The factor $(1 - \varrho_q \cdot \varrho_p)$ in $H_p$ implies “multiangle effects” for neutrinos moving on different trajectories [21,22,24]. However, for realistic SN conditions the modifications are small, allowing for a single-angle approximation [24,30]. In the strongly synchronized regime this is not surprising: the strong neutrino-neutrino interaction causes self-maintained coherence not only between different energy modes, but also between different angular modes. It has not been explained why the single-angle approximation remains good even when the neutrino-neutrino interaction becomes weak, although numerically this is observed to be the case [24,30,33].

We are studying the spatial evolution of the neutrino fluxes in a quasistationary situation. Therefore, the matrices $\varrho_p$ do not depend on time explicitly, so that the total time derivative in the EOMs reduces to the Liouville term involving only spatial derivatives. (See Ref. [44] for a recent comprehensive discussion of the role of the Liouville term in Boltzmann collision equations involving oscillating neutrinos.) Moreover, we consider a spherically symmetric system so that the only spatial variable is the radial coordinate $r$. In the single-angle approximation we finally need to study the simple EOMs,

$$i \frac{d}{dr} \varrho_\omega = [H_\omega, \varrho_\omega],$$

where we now classify different modes by the variable

$$\omega = \frac{\Delta m^2_{\text{atm}}}{2E}.$$ 

The single-mode Hamiltonians are

$$H_\omega = \Omega_\omega + \lambda(r) L + \mu(r) Q.$$ 

Here we have introduced $\lambda(r) = \sqrt{2}G_F n_e(r)$ and $L = \text{diag}(1,0,0)$ in the weak interaction basis. The matrix of the total density $Q = \int d\omega \varrho_\omega$ is normalized such that at the neutrino sphere it is $Q = \text{diag}(1,0,0)$ in the weak interaction basis. It is conserved except for oscillation effects, i.e., the physical neutrino density has been absorbed in the coefficient $\mu(r)$ which measures a suitable angular average of the neutrino-neutrino interaction energy. The radial variation of $\mu(r)$ is proportional to $r^{-4}$, where a factor $r^{-2}$ comes from the geometric flux dilution, another approximate factor $r^{-2}$ from the fact that neutrinos become more collinear with distance from the source.

B. Mixing parameters

Since $H_\omega$ appears in a commutator, we may arbitrarily add terms proportional to the $3 \times 3$ unit matrix. It will prove convenient to express the matrix of vacuum oscillation frequencies in the form

$$\Omega_\omega = \omega \text{diag}(-\frac{1}{2}\alpha, +\frac{1}{2}\alpha, \pm 1),$$

where the mass hierarchy parameter is

$$\alpha = \frac{\Delta m^2_{\odot}}{\Delta m^2_{\text{atm}}} = \frac{1}{30}.$$ 

A positive sign in the third component of $\Omega_\omega$ signifies the normal mass hierarchy, and a negative sign signifies the inverted hierarchy. For the mass differences themselves we use [45]

$$\Delta m^2_{\text{atm}} = 2.4 \times 10^{-3} \text{ eV}^2, \quad \Delta m^2_{\odot} = 8 \times 10^{-5} \text{ eV}^2.$$ 

For the mixing angles we use

$$\theta_{12} = 0.6, \quad \theta_{23} = \pi/4, \quad \theta_{13} = 0.1.$$ 

Of course, for $\theta_{13}$ only upper limits exist. In this context the CP phase $\delta_{CP}$ can be ignored, since it does not influence the relevant probabilities for equal $\nu_\mu$ and $\nu_\tau$ fluxes [46,47].

C. Bloch vectors

In the two-flavor context it is well known that the matrices of density can be expressed in terms of Bloch vectors, leading to EOMs that resemble the precession of a gyroscope around an external force field. This picture helps to recognize properties of the EOMs that are difficult to fathom in the commutator form of the EOMs. Therefore, we follow a recent paper by two of us [36] and note that every Hermitian $3 \times 3$ matrix $X$ can be expressed in the form

$$X = \frac{1}{2} X_0 + \frac{1}{2} \mathbf{x} \cdot \mathbf{a},$$

where $X_0 = \text{Tr}(X)$, $X$ is an eight-dimensional Bloch vec-
The vector on the right-hand side is orthogonal to both basis one concludes that lepton numbers that are separately conserved. In the mass other words, in the three-flavor context we have two flavor- structures, respectively.

In the mass basis, the “magnetic field” components are ignoring the ordinary matter term, we write the single-mode EOMs as

where the cross product is understood in the SU(3) sense: 

With the global polarization vector \( \mathbf{P} = \int d\omega \mathbf{P}_\omega \) and ignoring the ordinary matter term, we write the single-mode EOMs in the form

In the mass basis, the “magnetic field” components are

representing the “atmospheric” and “solar mass directions,” respectively.

In the absence of ordinary matter, the EOM for the global polarization vector is

where the “magnetic moment” of the system is \( \mathbf{M} = \int d\omega \mathbf{P}_\omega \). In the mass basis this is

The vector on the right-hand side is orthogonal to both \( \mathbf{e}_3 \) and \( \mathbf{e}_8 \). The reason is that the matrices \( \Lambda_3 \) and \( \Lambda_8 \) commute or, in other words, that \( f_{ijk} = 0 \) for \( i, j, k = 1, \ldots, 8 \) and the same for all permutations. As a consequence, the vector \( \mathbf{P} \) has no \( \mathbf{e}_3 \) or \( \mathbf{e}_8 \) component so that \( P_3 = 0 \) and \( P_8 = 0 \). In a general basis this implies

This is the equivalent of “flavor-lepton-number conservation” \( d_\lambda (\mathbf{P} \cdot \mathbf{B}_H) = 0 \) in the two-flavor context [25,28,29]. In other words, in the three-flavor context we have two flavor-lepton numbers that are separately conserved. In the mass basis one concludes that

are conserved.

IV. SYNCHRONIZED MSW EFFECT

As a first step in our analytic study we consider the matter-induced conversion of the initial \( \nu_e \) flux as it passes the H and L level crossings, assuming that both lie deep in the region where the neutrino-neutrino interaction is strong. As a result, the flavor oscillations are synchronized, meaning that all \( \varrho_\omega \) stay pinned to each other. In other words, it is enough to study the evolution of the matrix of the total density \( \varrho \). It obeys the EOM

where in the mass basis

and \( \omega_0 = (\Delta m^2_{23}/2E) \).

For the Fermi-Dirac spectrum described in Sec. II, we find \( \omega_0 = 0.710 \text{ km}^{-1} \). We will also use the notation

The matrix of vacuum oscillation frequencies thus can also be written as \( \Omega = \text{diag}(\frac{1}{2} \omega_0, \frac{1}{2} \omega_0, \pm \omega_0) \).

The assumed perfect synchronization implies that we can treat this system as an equivalent system with a single energy or rather with two fixed vacuum frequencies \( \omega_H \) and \( \omega_L \) determining \( \Omega \). It is most useful to study its evolution in the basis of instantaneous propagation eigenstates, where we denote the total matrix of density as \( \tilde{\varrho} \).

Since the only effect of the neutrino-neutrino interactions is to synchronize the oscillations and to reduce the system to an equivalent single-energy case, the propagation eigenstates are defined by the ordinary matter term for a monochromatic neutrino beam, whereas the neutrino-neutrino interaction plays no further role. The propagation basis coincides with the weak interaction basis when the matter density is large, so that our initial state is \( \tilde{\varrho} = \text{diag}(0, 0, 1) \) in the normal hierarchy and \( \text{diag}(0, 1, 0) \) in the inverted hierarchy. If the subsequent evolution were perfectly adiabatic, the system would remain in this state so that the \( \nu_e \) survival probability after the H and L crossings would be given by well-known results [1].

We assume indeed that the evolution is adiabatic, except near the H and L level crossings where in our system the jumping probabilities \( P_H \) and \( P_L \) need not be small. The neutrino mass-gap hierarchy ensures that the two crossings factorize with good approximation. In the normal hierarchy, the system encounters both crossings, and the final occupation of the propagation eigenstates is given by the products of probabilities shown in Table I. In the inverted hierarchy, only the L crossing is encountered because the H level crossing is now in the antineutrino sector which is irrelevant in our case. Again, the final mass-state occupations are shown in Table I.

\(^2\)An MSW transition in the presence of a dense neutrino gas was first treated in this way in the context of early-universe neutrino oscillations by Wong [18] and by Abazajian, Beacon, and Bell [19]. They used the terminology “collective MSW-like transformation” and “synchronized MSW effect,” respectively.
The jumping probability for incomplete adiabaticity is given to a good approximation by the so-called double-expontential formula [48–50]

\[
P_H = \exp\left(2\pi R_H \omega_H \sin^2 \theta_{13}\right) - 1, \tag{24}
\]

where the scale height is

\[
R_H = \left[\frac{d \ln \lambda(r)}{dr}\right]^{-1}_{r = r_1}. \tag{25}
\]

It has to be evaluated at the point of maximum violation of adiabaticity [51–53], given by \(\omega_H = \lambda(r_H)\) [50]. The assumed power-law profile of Eq. (2) and our choices for \(\omega_H\) and \(\theta_{13}\) imply \(r_H = 1089\) km, \(R_H = 21.8\) km, and \(\Gamma_H = 0.38\). Analogous results pertain to \(P_L\) with \(H \rightarrow L\) everywhere and the substitution \(\theta_{13} \rightarrow \theta_{12}\). We find \(r_L = 1166\) km, \(R_L = 23.3\) km, and \(\Gamma_L = 0.31\). These numerical results for \(P_H\) and \(P_L\) imply the numerical results for the final occupations shown in Table 1.

In the steep density profile used here, the H and L resonances look spatially very close (Fig. 1) so that one may worry if they indeed factorize in the usual way. We stress that the two resonances do not overlap, but one may still worry about possible interference effects. However, we can compare the analytic results with a numerical three-flavor system is simplified by the mass-gap hierarchy [28,29]. We can follow the previous two-flavor treatment almost step by step because the present three-flavor system is simplified by the mass-gap hierarchy \(\alpha = 1/30 \ll 1\). While the two conserved flavor-lepton numbers present in the three-flavor case lead to two spectral splits, these will occur in sequence and their dynamics factorizes in practice.

The first split to develop is driven by the atmospheric mass difference and thus can be called the H split. As in Refs. [28,29] we go to a rotating frame, at first rotating around the \(B_H\) direction. The single-mode Hamiltonians in this corotating frame are

\[
H_\omega = (\omega - \omega_H)B_H + \mu P, \tag{26}
\]

neglecting for now the much smaller term \(\omega B_L\). This is justified because, when \(\mu \gg \omega\) and \(\mu \gg \alpha \omega\), the ensemble of neutrinos is in a regime where we expect spectral splitting along \(e_3\) and synchronized oscillations along \(e_1\). This factorization has been explicitly shown in [36]. Flavor conversion is thus driven primarily by \(B_H\), while \(B_L\) gives subleading corrections due to the synchronized oscillations. Similarly, when \(\mu \sim \alpha \omega\), flavor conversion proceeds efficiently via a spectral split along \(e_3\) and is driven by \(B_L\), while \(B_H\) drives vacuum oscillations along \(e_3\).

Now, as \(\mu\) adiabatically goes to zero, the corotation frequency \(\omega_{H}^2\) approaches the final split frequency \(\omega_{H}^2\). Meanwhile, the modes with \(\omega > \omega_{H}^2\) will orient themselves along \(B_H\) and those with \(\omega < \omega_{H}^2\) in the \(-B_H\) direction. The value of \(\omega_{H}^2\) is fixed by the conservation of \(\rho_s\). Since the evolution associated with \(B_H\) has saturated, we can next go into a frame rotating around \(B_L\) where

\[
H_\omega = (\omega - \omega_L^2)B_L + \mu P, \tag{27}
\]

and repeat the analogous argument.

To illustrate the dynamics of the split in a form similar to Refs. [28,29], we consider an explicit example with an initial “box spectrum” at high density of the form

\[
\rho_{e\nu}(\omega) = \begin{cases} (2\omega_0)^{-1} \text{ for } 0 \leq \omega \leq 2\omega_0, \\ 0 \text{ otherwise.} \end{cases} \tag{28}
\]

At high densities \(\rho_{e\nu}(\omega)\) coincides with \(\tilde{\rho}_{33}(\omega)\) in the

### Table 1. Final occupations of the mass eigenstates after passing both MSW level crossings as described in the text.

<table>
<thead>
<tr>
<th>Normal hierarchy</th>
<th>Inverted hierarchy</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\hat{\rho}_{11})</td>
<td>(P_H P_L)</td>
</tr>
<tr>
<td>(\hat{\rho}_{22})</td>
<td>(P_H (1 - P_L))</td>
</tr>
<tr>
<td>(\hat{\rho}_{33})</td>
<td>((1 - P_H))</td>
</tr>
</tbody>
</table>

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normal hierarchy and with $\tilde{Q}_{22}(\omega)$ in the inverted hierarchy. After the MSW crossings the spectrum is still of box shape because of the assumed strong neutrino-neutrino interaction, but now has the $Q_{11}$, $Q_{22}$, and $Q_{33}$ components shown in Table I. Note that after the MSW transitions we neglect ordinary matter so that the propagation eigenstates are identical with the mass eigenstates and $\tilde{Q} = \tilde{Q}$.

A. Normal hierarchy

At first we study the evolution caused by the neutrino-neutrino interactions numerically. To this end we use the usual $\mu(r) \propto r^{-4}$ profile of Eq. (1) with the coefficient $\mu_0$ of Eq. (3). In this way the evolution is strongly but not perfectly adiabatic. Of course, the analytic results, based on the conservation of flavor-lepton number, apply in the perfectly adiabatic limit which here requires $\mu_0 \to \infty$.

In Fig. 4 (left column) we show the evolution in the mass basis of the diagonal elements of $\rho_\omega$ for 50 modes as a function of $\omega_0/\mu$. All modes start with the same initial condition prepared by the MSW transitions. Around $\omega_0/\mu = 1$ one recognizes a first H split that affects all components. Around $\omega_0/\mu = \alpha^{-1}$ one recognizes a second split, the L split, that affects only $\rho_{11}$ and $\rho_{22}$. During the H split, the modes with $\omega < \omega_H^\rho$ tend to $\rho_{33} \to 0$, while those with $\omega > \omega_H^\rho$ approach $\rho_{33} \to 1$. Then $\rho_{11}$ and $\rho_{22}$ modes with $\omega > \omega_H^\rho$ go to 0, as implied by the conservation of the trace of $\rho$. While the modes with $\omega < \omega_H^\rho$ rise towards higher values of $\rho_{11}$ and $\rho_{22}$, they encounter the L split at a frequency $\omega_L^\rho < \omega_H^\rho$. At this split, for $\omega < \omega_L^\rho$, the $\rho_{11}$ approach 1 and the $\rho_{22}$ approach 0, and vice versa for $\omega_L^\rho < \omega < \omega_H^\rho$. As a result of imperfect adiabaticity, some modes do not reach these extreme values, but get frozen earlier.

In Fig. 5 (left column) we show the same case in terms of the mass-basis $P_{\omega,3}$ and $P_{\omega,8}$ components. We observe that in $P_{\omega,8}$ only the H split operates, whereas in $P_{\omega,3}$ the H and L splits operate in sequence.

The situation can be visualized in terms of the $e_3$-$e_8$ triangle diagram [36] shown in Fig. 6. Each point in the interior and on the boundary of the triangle represents the projection of the polarization vector $P_\omega$ in the $e_3$-$e_8$ plane. Neutrinos from the $\nu_e$ burst start in the state $\nu_e = \tilde{\nu}_3$, where by the “tilde,” we represent the instantaneous mass eigenstates. The H crossing shifts the neutrino state from the $\tilde{\nu}_3$ vertex towards the $\tilde{\nu}_2$ state, but only partially, due to the semiadiabatic nature of the transition. After that crossing, all neutrinos find themselves at the point $A'$ inside the triangle. The L crossing further transports the state along a line parallel to the $\tilde{\nu}_2$-$\tilde{\nu}_1$ edge towards $\tilde{\nu}_1$, again only partly due to the semiadiabaticity. Before the split, all the neutrinos are thus at a point A in the interior of the triangle.

The H split takes the $\omega > \omega_H^\rho$ modes towards the $\tilde{\nu}_3$ state ($P_{\omega,3} = 0$, $P_{\omega,8} = -2/\sqrt{3}$) and the modes $\omega < \omega_H^\rho$ towards some combination of $\tilde{\nu}_1$ and $\tilde{\nu}_2$, while conserving the total $P_3$ and $P_8$. Since $\alpha \ll 1$, the H and L splits are well separated and the high-$\omega$ modes reach the $\tilde{\nu}_3$ vertex, i.e. the H split saturates, before the L split begins. The low-$\omega$ modes propagating towards the $P_{\omega,8} = 1/\sqrt{3}$ line encounter the L split that tends to take the $\omega > \omega_L^\rho$ modes towards $\tilde{\nu}_2$ ($P_{\omega,3} = -1$, $P_{\omega,8} = 1/\sqrt{3}$) and the $\omega < \omega_L^\rho$ modes towards $\tilde{\nu}_1$ ($P_{\omega,3} = 1$, $P_{\omega,8} = 1/\sqrt{3}$). In the adiabatic limit, given sufficient time to propagate, the H and L

![FIG. 4. Evolution of the diagonal elements of $Q_\omega$ for our box-spectrum example. The mode density is increased around the splits.](image)

![FIG. 5. Evolution of the 3 and 8 components of $P_\omega$ for our box-spectrum example.](image)
diagonal elements of $\rho$. In our example, $P_3$ and $P_8$ of Eq. (20), one can evaluate the split frequencies $\omega^2_H$ and $\omega^2_L$. For $\omega < \omega^2_H$, we have $P_{\omega,8} \rightarrow 1/\sqrt{3}$, while for $\omega > \omega^2_H$ they reach $-2/\sqrt{3}$. In the limit of perfect adiabaticity, the conservation of $P_8$ implies

$$2\omega_0P^0_{\omega,8} = \frac{1}{\sqrt{3}}\omega^2_H - \frac{2}{\sqrt{3}}(2\omega_0 - \omega^2_H),$$

where $P^0_{\omega,8}$ is the common value of $P_{\omega,8}$ before the split begins. In our example, $P^0_{\omega,8} = -0.50$, leading to $\omega^2_H = 0.76\omega_0$.

When the H split saturates, all modes with $\omega > \omega^2_H$ have $P_{\omega,8} = -2/\sqrt{3}$, and hence $P_{\omega,3} = 0$ due to the conservation of the norm of $\mathbf{P}$. These modes have reached the bottom vertex of the $\mathbf{e}_3\mathbf{-e}_8$ triangle and hence cannot split further due to the L split. On the other hand, for modes with $\omega < \omega^2_H$ a second split in $P_{\omega,3}$ happens. These modes approach $P_{\omega,3} = +1$ for $\omega < \omega^2_L$ and $P_{\omega,3} = -1$ for $\omega > \omega^2_L$. Applying the conservation law for $P_3$ gives us

$$2\omega_0P^0_{\omega,3} = \omega^2_L - (\omega^2_H - \omega^2_L).$$

In our example $P^0_{\omega,3} = -0.14$ so that $\omega^2_L = 0.24\omega_0$.

We show in Figs. 7 and 8 the mass-basis spectra of the diagonal elements of $\mathcal{Q}$ and of $P_3$ and $P_8$. Thin lines are the MSW-prepared initial spectra. Thick lines show the numerical end states, corresponding to the split diagrams of Figs. 4 and 5. Dotted lines show the adiabatic limiting behavior based on the lepton-number conservation laws. Once more, the agreement is striking. Imperfect adiabaticity leads to a smoothing of the splits which otherwise are sharp spectral steps.

Finally, in Fig. 9 we show the $\nu_e$ spectrum after the two splits. The solid curve is the numerical result, where the vacuum oscillations between the SN and the observer have been averaged (kinematical decoherence between different $\omega$ modes), i.e.,

FIG. 6 (color online). Projection of the polarization vectors $\mathbf{P}_\omega$ on the $\mathbf{e}_3\mathbf{-e}_8$ plane for our box example. The vertices of the triangle represent pure (instantaneous) mass eigenstates. After both MSW transitions, the system is at the point A in the interior of the triangle. (See the text for details.)

splits result in all neutrinos reaching one of the three vertices of the $\mathbf{e}_3\mathbf{-e}_8$ triangle.

Using the conservation law for $P_3$ and $P_8$ of Eq. (20), one can evaluate the split frequencies $\omega^2_H$ and $\omega^2_L$. For $\omega < \omega^2_H$, we have $P_{\omega,8} \rightarrow 1/\sqrt{3}$, while for $\omega > \omega^2_H$ they reach $-2/\sqrt{3}$. In the limit of perfect adiabaticity, the conservation of $P_8$ implies

$$2\omega_0P^0_{\omega,8} = \frac{1}{\sqrt{3}}\omega^2_H - \frac{2}{\sqrt{3}}(2\omega_0 - \omega^2_H),$$

where $P^0_{\omega,8}$ is the common value of $P_{\omega,8}$ before the split begins. In our example, $P^0_{\omega,8} = -0.50$, leading to $\omega^2_H = 0.76\omega_0$.

When the H split saturates, all modes with $\omega > \omega^2_H$ have $P_{\omega,8} = -2/\sqrt{3}$, and hence $P_{\omega,3} = 0$ due to the conservation of the norm of $\mathbf{P}$. These modes have reached the bottom vertex of the $\mathbf{e}_3\mathbf{-e}_8$ triangle and hence cannot split further due to the L split. On the other hand, for modes with $\omega < \omega^2_H$ a second split in $P_{\omega,3}$ happens. These modes approach $P_{\omega,3} = +1$ for $\omega < \omega^2_L$ and $P_{\omega,3} = -1$ for $\omega > \omega^2_L$. Applying the conservation law for $P_3$ gives us

$$2\omega_0P^0_{\omega,3} = \omega^2_L - (\omega^2_H - \omega^2_L).$$

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Finally, in Fig. 9 we show the $\nu_e$ spectrum after the two splits. The solid curve is the numerical result, where the vacuum oscillations between the SN and the observer have been averaged (kinematical decoherence between different $\omega$ modes), i.e.,


$$\mathcal{Q}_{ee} = U_{e1}^2\mathcal{Q}_{11} + U_{e2}^2\mathcal{Q}_{22} + U_{e3}^2\mathcal{Q}_{33}. \quad (31)$$

The dotted curve is our analytic result in the adiabatic limit. This result is easily explained if we observe that in Eq. (31), in the case of maximal 23 mixing, one has $U_{e1}^2 = \cos^2\theta_{13}\cos^2\theta_{12}, \ U_{e2}^2 = \cos^2\theta_{13}\sin^2\theta_{12},$ and $U_{e3}^3 = \sin^2\theta_{13}$. Therefore,

FIG. 8. The 3 and 8 components of $\mathbf{P}_\omega$. Convention for lines is the same as in Fig. 7.
split. This is because in agreement with the numerical result.

Again the agreement between the analytic and numerical results is very good.

B. Inverted hierarchy

For the inverted hierarchy we show the analogous information in the right-handed columns of Figs. 4–9. The initial state here is \( \nu_2 \). The nonadiabatic L crossing takes the neutrino states partly towards \( \nu_1 \). After the L crossing and before the split, the neutrino state for all modes is along the \( \nu_1-\nu_2 \) edge, at A, as shown in Fig. 6 (right column), where \( P_{o,8} = 1/\sqrt{3} \). Since all neutrinos are already in one of the extreme values of \( P_{o,8} \), the H split is inoperational. This corresponds to \( \rho_{13} \) remaining in its MSW-prepared initial value of 0. The L split takes \( \rho_{11} \to +1 \) for \( \omega < \omega_1^L \) and \( \rho_{11} \to 0 \) for \( \omega > \omega_1^L \), and vice versa for \( \rho_{22} \). In the inverted hierarchy we have an effective two-flavor case in the \( \nu_1-\nu_2 \) subsector. This is a consequence of the MSW-prepared initial condition. Initially \( P_{o,8} = 1/\sqrt{3} \). Applying now the conservation of \( P_8 \), we obtain \( \omega_1^H = 2\omega_0 \), i.e., the split occurs at the edge of the box and thus is not visible. The conservation law for \( P_3 \) and using in our case \( P_{o,3} = -0.38 \), one obtains \( \omega_1^L = 0.62\omega_0 \). For the electron flavor we predict for the final spectrum

\[
\rho_{ee} \approx \begin{cases} 
\cos^2\theta_{12} & \text{for } \omega < \omega_1^L, \\
\sin^2\theta_{12} & \text{for } \omega_1^L < \omega < \omega_1^H, \\
\sin^2\theta_{13} & \text{for } \omega_1^H < \omega.
\end{cases}
\]  

(32)

in agreement with the numerical result.

Note that, in the inverted hierarchy, there is only one split. This is because \( P_{o,8} \) is already at an extreme value before the split can begin. In general, there are two splits if the neutrino state before the split is in the interior of the \( e_3-e_8 \) triangle, one split if it is along one of the edges of the triangle (as in this case), and no split occurs if the neutrino state is at any of the three vertices (see Fig. 6).

C. Fermi-Dirac spectrum

It is straightforward to extend these arguments to a general spectrum, e.g. a Fermi-Dirac spectrum \( f(\omega) \). The conservation laws imply for the normal hierarchy

\[
\sqrt{3}P_{o,8} = \int_0^{\omega_1^H} d\omega f(\omega) - 2 \int_0^{\omega_1^H} d\omega f(\omega),
\]

(34)

\[
P_{o,3} = \int_0^{\omega_1^L} d\omega f(\omega) - \int_0^{\omega_1^L} d\omega f(\omega).
\]

(35)

On the other hand, for the inverted hierarchy we find

\[
P_{o,3} = \int_0^{\omega_1^L} d\omega f(\omega) - \int_0^{\omega_1^L} d\omega f(\omega).
\]

These relations allow us to calculate \( \omega_1^H \) and \( \omega_1^L \). Note that these results are exact only in the limit of an infinite mass-gap hierarchy, i.e., for \( \alpha \to 0 \), where the H and L splits perfectly factorize. Once the split frequencies have been found, the corresponding energies are \( E_1^H = \Delta m_{23}^2/\omega_1^H \) and \( E_1^L = \Delta m_{32}^2/2\omega_1^L \).

For our schematic SN model where the MSW level crossings and the spectral-split region are widely separated, we find \( E_1^H = 11.9 \text{ MeV} \) and \( E_1^L = 16.9 \text{ MeV} \) in the normal hierarchy, and \( E_1^H = 12.7 \text{ MeV} \) in the inverted hierarchy. We show the initial and final \( \nu_e \) spectra for the Fermi-Dirac case in Fig. 10. Once more, we have coarse-grained over neighboring modes, representing the effect of kinematical decoherence.

VI. CONCLUSIONS

We have studied the three-flavor evolution of a \( \nu_\tau \) burst that first undergoes two MSW level crossings driven by \( \Delta m_{23}^2 \) and \( \Delta m_{32}^2 \), respectively, and then undergoes spectral splits by the adiabatically decreasing strength of the neutrino-neutrino interaction. This case study of an MSW-prepared spectral split serves as a proxy for the recent numerical study of the prompt \( \nu_\tau \) burst in an O-Ne-Mg core collapse SN. Here, the matter density profile is
so steep that the sequence between MSW crossings and collective neutrino oscillations is reversed from what would be expected in a traditional iron-core SN.

First we have analytically estimated the population of the propagation eigenstates after the MSW transformations. Because of the sharply falling matter density at the edge of the core of the star, the MSW transitions are not completely adiabatic, which helps in satisfying a precondition for the spectral splits to take place. We have used the well-known analytic double-exponential formula for calculating the jump probabilities.

We have studied numerically the subsequent spectral splits. We have analytically determined the split frequencies on the basis of two conserved flavor-lepton-number combinations that supersede the single conservation law encountered in a two-flavor situation. The neutrino mass-gap hierarchy allows for a factorization of an H split and an L split, similar to the factorization of the MSW effect into an H and an L crossing. The dynamics of the split evolution is clearly seen to be a two-step process.

The dynamics of the two spectral splits can be understood in terms of the motion of the neutrino state in the $e_3 - e_8$ triangle diagram, which can explain many of the features of neutrino evolution qualitatively. The number of possible splits can be deduced by the location of the neutrino state inside the triangle. We have also shown how the positions of the splits can be calculated accurately given the initial neutrino spectra, and we have calculated the $\nu_e$ survival probability analytically, which matches the numerical computations.

Our analytic treatment accounts very nicely for the numerical findings of Duan et al. [34]. In their case the MSW conversion and the spectral splits are spatially very close so that it is not a priori obvious that our schematic model would be a good representation. In our treatment we have enforced a clear separation between the MSW region and the spectral-split region by assuming the limit of large neutrino-neutrino interactions, a limit that also ensures that the MSW transition is perfectly synchronized and that the spectral split is perfectly adiabatic. A posteriori, however, our interpretation as a MSW-prepared spectral split appears nicely justified and quantitatively appropriate. Our treatment is also a useful application of the three-flavor oscillation machinery developed by two of us recently [36].

It appears that the impact of collective neutrino oscillations on the propagation of the prompt $\nu_e$ burst is conceptually and quantitatively well under control. Characteristic signatures of these flavor transitions in large underground detectors have also been recently investigated [54]. What remains is to observe these features in the neutrino signal of the next galactic SN.

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*Note added.—*After our manuscript was completed, a paper by Duan, Fuller, and Qian appeared that treats three-flavor split phenomena [55]. The results partly overlap with our work.