Neutrinos with magnetic moment: Depolarization rate in plasma

Per Elmfors a,1, Kari Enqvist b,2, Georg Raffelt c,3, Günter Sigl c,d,4

a Fysikum, Box 6730, S-113 85 Stockholm, Sweden
b Physics Department, P.O. Box 9, FIN-00014 University of Helsinki, Finland
c Max-Planck-Institut für Physik, Föhringer Ring 6, D-80805 Munich, Germany
d Department of Astronomy and Astrophysics, The University of Chicago, Chicago, IL 60637-1433, USA

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Abstract

Neutrinos with a magnetic moment \( \mu \) change their helicity when interacting with an electromagnetic field. Various aspects of this effect have been described as spin precession, spin-flip scattering, and magnetic Cherenkov radiation. These perspectives are unified in an expression for the \( \nu_L \rightarrow \nu_R \) transition rate which involves the correlators of the electromagnetic field distribution. Our general formula corrects a previous result and generalizes it to the case where the fields cannot be viewed as classical and where the momentum transfers need not be small. We evaluate our result explicitly for a relativistic QED plasma and determine the depolarization rate to leading order in the fine structure constant. Assuming that big-bang nucleosynthesis constraints do not allow a right-handed neutrino in equilibrium we derive the limit \( \mu < 6.2 \times 10^{-11} \mu_B \) on the neutrino magnetic moment. Bounds on \( \mu \) from a possible large scale magnetic fields are found to be more stringent even for very weak fields. © 1997 Elsevier Science B.V.

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1. Introduction

A neutrino or other neutral particle with a magnetic moment \( \mu \) gets depolarized when traversing a random distribution of electromagnetic fields as, for example, in a plasma of

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1 E-mail: elmfors@physto.se
2 E-mail: enqvist@pcu.helsinki.fi
3 E-mail: raffelt@mppmu.mpg.de
4 E-mail: sigl@mppmu.mpg.de

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charged particles. This effect could be important in stars or in the early universe where the standard weak interactions produce only left-handed neutrinos. More than twenty years ago the depolarization effect was considered as a possibility to solve the solar neutrino problem, but for plausible values of $\mu$ the rate was found to be too small [1]. In supernova cores, the left-handed neutrinos are trapped while the helicity-flipped states could escape more easily because they scatter only by the assumed magnetic-dipole interaction. The depolarization effect would thus disturb the standard picture of supernova theory so that the known properties of supernovae and the observed SN 1987A neutrino signal allow one to derive certain limits on Dirac magnetic moments [2,3].

In the early universe, the radiation density and thus expansion rate would increase due to the new thermally excited degree of freedom, modifying the predicted primordial light-element abundances. A comparison with observations again allows one to derive limits on $\mu$ [4-6].

In most of these studies the depolarization rate was calculated from the spin-flip scattering process $\nu_L + X \rightarrow X + \nu_R$ where $X$ represents electrons, positrons, or other charged particles and the interaction is due to the assumed neutrino magnetic moment. In addition there may be electromagnetic modes with a "space-like" dispersion relation which allow for the Cherenkov emission $\gamma + \nu \rightarrow \nu + \gamma$ and absorption $\gamma + \nu_L \rightarrow \nu_R$ and thus contribute to the depolarization rate [1,3]. This is the case in supernova cores where the photon dispersion relation is dominated by the nucleon magnetic moments or in non-relativistic plasmas, but it is not the case in the relativistic $e^+e^-$ plasma of the early universe. The rate for higher-order scattering processes such as magnetic Compton scattering $\gamma + \nu \rightarrow \nu + \gamma$ is proportional to $\mu^4$ and is thus ignored in the present discussion which is limited to $\mu^2$ effects. To order $\mu^2$ right-handed neutrinos can also be produced by pair processes of the type $\gamma \rightarrow \nu_L \bar{\nu}_R$ (plasmon decay) or $e^+e^- \rightarrow \nu_L \bar{\nu}_R$ (pair annihilation). In supernova cores or the early universe with a large population of left-handed neutrinos the $\nu_L \rightarrow \nu_R$ reactions always seem to be more important than the pair processes which, however, dominate in normal stars which do not have a population of trapped left-handed neutrinos.

Loeb and Stodolsky [6] were the first to look at the depolarization effect from a more classical perspective. They argued that in a macroscopic magnetic field, left-handed neutrinos simply spin-precess and that even in a microscopic distribution of random fields the depolarization rate should be calculable as a suitable ensemble average over the spin-precession formula. They found a result which represents the depolarization rate in terms of certain correlation functions of the electromagnetic fields. In principle, this correlator approach incorporates all electromagnetic effects to order $\mu^2$ which lead to depolarization such as spin-flip scattering and the Cherenkov processes.

These different approaches have co-existed in the literature with no apparent attempt to compare them, to understand their relationship, or to check their mutual consistency. One problem is that Loeb and Stodolsky's approach is entirely classical so that it is not obvious if and how their formula is limited when applied to a plasma which involves non-classical (quantum) excitations of the electromagnetic field.

We thus reconsider neutrino depolarization both classically in the spirit of Loeb
and Stodolsky (Section 2) and from a quantum-kinetic perspective (Section 3). Put differently, we derive the depolarization rate in terms of correlation functions of the electromagnetic fields which may be quasi-classical or true quantum variables. Even in the classical limit our general result involves terms which are absent in Loeb and Stodolsky's formula.

We also compare with the imaginary part of the neutrino self-energy (Section 4) which has a simple relation to the depolarization rate. This calculation amounts to a determination of the dominant spin-flip scattering rate, but even to lowest order in the fine-structure constant $\alpha$ the correct screening prescription has to be incorporated by resumming the photon propagator. Then it automatically includes all order $\mu^2$ contributions to the production rate of right-handed neutrinos, i.e. it includes $\nu_L \to \nu_R$ transitions from spin-flip scattering or from Cherenkov processes as well as plasmon decay and $e^+e^-$ annihilation.

In Section 5 we evaluate the depolarization rate explicitly by using the correlators for a relativistic QED plasma and compare the classical and quantum treatments. In Section 6 we apply this result to the depolarization of neutrinos in the early universe, allowing us to derive a limit on the magnetic dipole moment from considerations of big-bang nucleosynthesis. We also compare this limit to previous bounds obtained from assuming a large-scale background magnetic field. In Section 7 we summarize our findings.

2. Classical trajectory approach

2.1. Depolarization rate

A non-relativistic neutral particle with a magnetic moment $\mu$ and an intrinsic angular momentum (spin) precesses in the presence of an external magnetic field $B$ according to $\dot{P} = g\mu B \times P$. Here, $P$ is the spin polarization vector which, for a spin-$\frac{1}{2}$ particle, parametrizes the spin density matrix in the usual form $\rho = \frac{1}{2} (1 + P_i \sigma_i)$. Further, $g$ is the gyromagnetic ratio. Loeb and Stodolsky [6] used $g = 1$, i.e. a classical particle, while we always study neutrinos with spin $\frac{1}{2}$ so that $g = 2$. If the particle moves relativistically, the main modification is that in the laboratory frame only the magnetic field transverse to the direction of motion contributes, and that a transverse electric field is also important because in the particle's rest frame it manifests itself as a magnetic field. Altogether, the precession formula for an ultrarelativistic ($v = 1$) spin-$\frac{1}{2}$ particle is

$$\dot{P} = 2\mu (B_\perp - \hat{p} \times E_\perp) \times P,$$

(2.1)

where the subscript $\perp$ refers to the field vectors transverse to the direction of motion and $\hat{p}$ is a unit vector in the direction of the neutrino momentum.

The same equation of motion is obtained if one begins directly with the covariant Lagrangian which describes the coupling of a Dirac neutrino with magnetic moment $\mu$ to the electromagnetic field tensor $F^{\mu\nu}$.
$L_{\text{int}} = -\frac{1}{2} \mu \bar{\Psi} \sigma_{\mu \nu} F^{\mu \nu} \Psi = \mu \bar{\Psi} (\Sigma \cdot B - i \alpha \cdot E) \Psi , \quad (2.2)$

where $\Psi$ is the neutrino Dirac field. In terms of the usual Dirac matrices we have $\Sigma_i = \gamma_5 \gamma^0 \gamma_i$ and $\alpha_i = \gamma^0 \gamma_i$. This Lagrangian shows that the magnetic-moment interaction indeed only couples neutrinos of opposite chirality. As we are concerned only with ultrarelativistic neutrinos this is tantamount to a coupling of opposite helicities.

In the presence of other forces than those represented by the electromagnetic fields, the off-diagonal elements of the density matrix can be damped by collisions which "measure" the helicity content of a neutrino state. Standard weak interactions would have this property because only left-handed neutrinos are affected by collisions. This damping effect is important when the spin precesses in macroscopic magnetic fields [7,8], and can be taken into account by adding a term $-D \sigma_i$ to the right-hand side of the equation of motion for $P_i$, $i = 1, 2$, in Eq. (2.1). For depolarization in stochastic electromagnetic fields one can speak of a coherent spin precession only for a time period which represents the correlation time of the fields. The damping effect would be important if the average time between weak collisions were shorter than this coherence time scale. An example is a putative horizon-scale magnetic field in the early universe.

We always assume ultrarelativistic (i.e. effectively massless) neutrinos. However, in a medium the dispersion relations for left- and right-handed states are different because only the left-handed ones feel the weak potential produced by the background medium. We may write the refractive energy difference between left- and right-handed states in the form $\omega_{\text{refr}} = 2\mu B_{\text{refr}}$ in terms of an effective magnetic field $B_{\text{refr}}$ which points in the neutrino's direction of motion. Therefore, the precession formula $\dot{P} = 2\mu B_{\text{eff}} \times P$ finally involves an effective magnetic field with $B_{\perp}^{\text{eff}} = B_{\perp} - \hat{\sigma} \times E_{\perp}$ and $B_{\parallel}^{\text{eff}} = (\omega_{\text{refr}}/2\mu) \hat{\sigma}$.

In general, the electric and magnetic fields depend on location and time in arbitrary ways. Therefore, a neutrino with a magnetic moment will be deflected. However, the deflecting forces on a magnetic dipole are proportional to the field gradients while the precession effect depends on the fields directly. Therefore, if the spatial variations are small the neutrinos can still be assumed to move on a straight line which can be taken to be the $z$-direction. Moreover, the neutrino can be taken to "see" only the fields at a specific location which is assumed to vary with time as $z = t$ because of the assumed propagation with the speed of light and because we take $z = 0$ at $t = 0$. Therefore, through the condition $z = t$ the fields $B_{\text{eff}}$ are to be viewed as functions of time alone. We call this the "classical trajectory approach" to the problem of neutrino spin depolarization.

In order to derive the equation of motion of the polarization vector in a stochastic field distribution it proves useful to write the precession equation in the form

$$\dot{P}(t) = M(t) \ P(t) , \quad (2.3)$$

where the time-dependent matrix $M$ is explicitly given by

$$M_{ij} = \begin{pmatrix} -D & -\omega_{\text{refr}} & 2\mu(B_y - E_x) \\ \omega_{\text{refr}} & -D & -2\mu(B_x + E_y) \\ -2\mu(B_y - E_x) & 2\mu(B_x + E_y) & 0 \end{pmatrix} \quad (2.4)$$
if the neutrinos are ultrarelativistic and move in the z-direction.

In general we must view the matrix $M$ as consisting of one part $\langle M \rangle$ which is non-zero on the average plus stochastic fluctuations around that average. Even if there are no large-scale magnetic fields, the refractive effect provides a non-vanishing average contribution. In order to derive the depolarization rate we eliminate $\langle M \rangle$ by going to an "interaction picture" with $Q(t) \equiv e^{-(M)t} P(t)$ so that we are left with the equation of motion

$$\dot{Q}(t) = m(t)Q(t)$$

(2.5)

with

$$m(t) \equiv e^{-(M)t} (M - \langle M \rangle) e^{(M)t}.$$  

(2.6)

The formal solution to Eq. (2.5) is

$$Q(t) = \sum_{n=0}^{\infty} \int_0^t \int_0^{t_{n-1}} dt_1 \ldots \int_0 dt_n m(t_1) \ldots m(t_n) Q(0).$$

(2.7)

This is the solution we need to average over a statistical ensemble of field configurations. We shall assume that the field fluctuations are Gaussian so that the n-point correlation functions can be reduced to products of pair correlation functions. Let us observe that in QED the high temperature effective action is in fact Gaussian [9]. (This is not a general feature and not true e.g. for QCD.) We seek the solution only for times $t$ much larger than the correlation time of the stochastic fields. In the integral in Eq. (2.7) all time arguments have to occur in close pairs for the integrand to be non-negligible. Since the time arguments are ordered it is only the permutation where adjacent matrices are contracted that gives any contribution in the leading large-time limit. It can also be checked at the end that this asymptotic region is reached long before one decay time in the present case of neutrino spin oscillations. We can therefore approximate $\langle Q(t) \rangle$ by

$$\langle Q(t) \rangle \approx \sum_{n=0}^{\infty} \int_0^t \int_0^{t_{n-1}} dt_1 \ldots \int_0 dt_n \langle m(t_1)m(t_2) \rangle \ldots \langle m(t_{n-1})m(t_n) \rangle Q(0).$$

(2.8)

After computing the pair correlation matrix $\langle m(t_1)m(t_2) \rangle_{ij}$ with zero average field strength we find that there is no mixing between the third component and the rest so that it is useful to define

$$n(t_1 - t_2) \equiv -\langle m(t_1)m(t_2) \rangle_{33}$$

$$= (2\mu)^2 e^{-D(t_1-t_2)} \left\{ \cos[\omega_{\text{refr}}(t_1 - t_2)] \langle B_{\perp}(t_1) \cdot B_{\perp}(t_2) \rangle$$

$$+ \langle E_{\perp}(t_1) \cdot E_{\perp}(t_2) \rangle + \hat{\rho} \cdot \left[ B(t_1) \times E(t_2) - E(t_1) \times B(t_2) \right] \right\}$$

$$+ \sin[\omega_{\text{refr}}(t_1 - t_2)] \langle E_{\perp}(t_1) \cdot B_{\perp}(t_2) - B_{\perp}(t_1) \cdot E_{\perp}(t_2) \rangle$$
Of course, in the terms involving cross products of fields we could have equally used the transverse field components.

The depolarization rate is given by the shrinking rate of \( \langle P_3(t) \rangle \). If we carry out all the integrals in Eq. (2.8), keeping only the leading term at large \( t \), we finally find that the ensemble-averaged evolution of \( P_3 \) is given by

\[
\langle P_3(t) \rangle = \exp \left[ -t \int_0^\infty dt' n(t') \right] P_3(0) \equiv e^{-\Gamma_{\text{depol}} t} P_3(0).
\]

If \( \omega_{\text{refr}} \) is much larger than the inverse correlation time of the electromagnetic fields, the oscillating \( \cos(\omega_{\text{refr}} t) \) and \( \sin(\omega_{\text{refr}} t) \) terms under the integral in Eq. (2.9) would suppress the depolarization effect. Likewise, a very large helicity-measuring damping coefficient \( D \) would prevent oscillations and suppress the depolarization. In stars, even supernovae, and in the early universe these suppression effects are insignificant because the inverse of the electromagnetic correlation length is much larger than both the re- 

\[ \text{fractive energy difference between those states and the damping rate. Therefore, for all situations which are of astrophysical interest we can put } \omega_{\text{refr}} = D = 0. \]

In the presence of a large-scale background field there is an additional component to \( \Gamma_{\text{depol}} \) from \( \langle M \rangle \) causing a coherent spin precession. Of course, this additional term would depend on both \( \omega_{\text{refr}} \) and \( D \); see Section 6.2.) The depolarization rate from stochastic fields is then found to be

\[
\Gamma_{\text{depol}} = (2\mu)^2 \int_0^\infty dt \left[ B_\perp(t) \cdot B_\perp(0) + E_\perp(t) \cdot E_\perp(0) \right. \\
+ \left. \hat{p} \cdot \left[ B(t) \times E(0) - E(t) \times B(0) \right] \right).
\]

This result agrees with that of Loeb and Stodolsky \[6\] except for the cross term. They agree that it should be there and stress that it can be derived rather easily by beginning with the spin-precession formula in the neutrino's rest frame where only the \( B_\perp^2 \) correlator appears, and express it by the Lorentz transformed laboratory fields.\[5\]

Two remarks about the interpretation of Eq. (2.11) are in order. First, a term like \( B(t) \) really means \( B(t, r) \) with \( r = \hat{p} t \) where \( \hat{p} \) is the neutrino velocity vector. Therefore, even in an isotropic medium a vectorial quantity like \( \langle E(t) \times B(0) \rangle \) need not vanish because it depends on the external vector \( \hat{p} \). Second, the integrand is an even function of \( t \) because the fields are classical variables so that their ordering is arbitrary, and because of time translational invariance of the statistical ensemble in equilibrium. Therefore, the time integral may be extended to \(-\infty\) if a compensating factor \( 1/2 \) is introduced. We will always use this more symmetric form which is easier to handle in Fourier space,

\[ \text{\textsuperscript{5} L. Stodolsky, private communication.} \]
and which allows for a direct transition to the quantum case (Section 3) where the ordering of the fields is important.

2.2. Isotropic medium

In a homogeneous and isotropic plasma or other medium, correlator expressions like the ones appearing in Eq. (2.11) are conveniently calculated in Fourier space. To this end we introduce the usual notation

\[ \langle X_i Y_j \rangle_K \equiv \int dt e^{i k_0 t} \langle X_i(t, k) Y_j(0, -k) \rangle, \]  

(2.12)

where \( X_i \) and \( Y_j \) each represent a component of the \( E \)- or \( B \)-fields with \( X_i(t, k) \) the spatial Fourier transform of \( X_i(t, r) \) and so forth. Therefore, the depolarization rate Eq. (2.11) can be expressed as

\[ \Gamma_{\text{dep}}^\circ = \frac{(2\mu)^2}{2} \int_{-\infty}^{+\infty} dt \int \frac{d^3 k}{(2\pi)^3} \int_{-\infty}^{+\infty} \frac{dk_0}{2\pi} e^{i(\hat{p} \cdot \hat{k} - k_0 t)} S_P(K), \]  

(2.13)

where the \( P \)-dependent "dynamical structure function" is

\[ S_P(K) \equiv \left\langle \left. \left( B_{\perp i}^2 + E_{\perp i}^2 + \hat{p} \cdot (B \times E - E \times B) \right) \right|_K \right\rangle. \]  

(2.14)

It will turn out that \( \mu^2 S_P(K) \) is to be interpreted as the probability for a neutrino of four momentum \( P \) to transfer the four-momentum \( K \) to the medium. The \( dt \) integration yields \( 2\pi\delta(\hat{p} \cdot \hat{k} - k_0) \) which allows us to perform the \( dk_0 \) integration trivially. Altogether we find

\[ \Gamma_{\text{dep}}^\circ = 2\mu^2 \int \frac{d^3 k}{(2\pi)^3} S_P(K), \]  

(2.15)

where \( K \) is restricted by the condition \( k_0 = \hat{p} \cdot \hat{k} \).

The structure of \( S_P \) still depends on the field components transverse to the neutrino momentum. We stress that contrary to Ref. [6] the assumed isotropy of the medium does not imply that \( \langle B_{\perp}^2 \rangle_K \) equals \( \frac{2}{3}\langle B^2 \rangle_K \) because \( \langle B_{\perp}^2 \rangle_K \) depends on the external vector \( \hat{p} \) and thus is not a scalar. In order to use isotropy properly we observe that

\[ B_T = B - (\hat{p} \cdot B) \hat{p} \] so that

\[ S_P(K) = \langle B_i B_j + E_i E_j \rangle_K (\delta_{ij} - \hat{p}_i \hat{p}_j) + \langle B_i E_j - E_i B_j \rangle_K \epsilon_{ij} \epsilon_{k_l} \hat{p}_k \hat{p}_l. \]  

(2.16)

We further note that in an isotropic medium any correlator expression of the form \( \langle X_i Y_j \rangle_K \) can be proportional only to \( \delta_{ij}, \hat{k}_i \hat{k}_j, \) or \( \epsilon_{ij} \epsilon_{k_l} \hat{k}_k \) because only the vector \( \hat{k} \) is available to construct spatial tensor structures. The most general structure is thus found to be

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6 We use \( K = (k_0, k) \), \( P = (p_0, p) \) etc. for four-vectors and \( k = |k|, p = |p| \) etc. for the corresponding three-vectors.
\[
\langle X, Y \rangle_K = \langle X \cdot Y \rangle_K \frac{\delta_{ij} - 3\hat{k}_i\hat{k}_j}{2} + \langle (X \cdot \hat{k})(Y \cdot \hat{k}) \rangle_K \frac{3\hat{k}_i\hat{k}_j - \delta_{ij}}{2} + \langle X \times Y \rangle_K \cdot \hat{k} \frac{\epsilon_{ijk}\hat{k}_l}{2}.
\]

(2.17)

It is then straightforward to show that

\[
S_p(K) = \langle B^2 + E^2 \rangle_K \frac{1 + (\hat{k} \cdot \hat{p})^2}{2} + \langle (B \cdot \hat{k})^2 + (E \cdot \hat{k})^2 \rangle_K \frac{1 - 3(\hat{k} \cdot \hat{p})^2}{2} + (\hat{k} \cdot \hat{p}) \langle B \times E - E \times B \rangle_K \cdot \hat{k}.
\]

(2.18)

This expression simplifies further if we observe that \( E \) and \( B \) must obey Maxwell’s equations. Because the magnetic field is divergence free we have \( B \cdot k = 0 \). Further, \( \langle (E \cdot k)^2 \rangle_K = \langle E^2k^2 - (k \times E)^2 \rangle_K \). Maxwell’s equations give us \( k \times E = k_0B \) so that \( \langle (E \cdot \hat{k})^2 \rangle_K = \langle E^2 \rangle_K - \langle (k_0/k)^2 \langle B^2 \rangle_K \). Next, \( \langle B \times E \rangle_K \cdot k = \langle B \cdot (E \times k) \rangle_K \), allowing us again to apply \( k \times E = k_0B \) so that \( \langle B \times E - E \times B \rangle_K \cdot k = -2\langle k_0/k \rangle \langle B^2 \rangle \). Altogether we thus find

\[
S_p(K) = \langle B^2 \rangle_K \left( \frac{1 + (\hat{p} \cdot \hat{k})^2}{2} - \frac{2k_0}{k} \hat{p} \cdot \hat{k} - \frac{k_0^2}{k^2} \frac{1 - 3(\hat{p} \cdot \hat{k})^2}{2} \right) + \langle E^2 \rangle_K \left[ 1 - (\hat{p} \cdot \hat{k})^2 \right].
\]

(2.19)

In an isotropic medium the \( \langle \ldots \rangle_K \) expressions depend only on \( (k_0, k) \) and not on the direction \( \hat{k} \).

In order to present our final result we note that the depolarization rate studied thus far represents the rate by which a fixed ensemble of neutrinos gets depolarized. In the early universe or in supernovae, however, a more relevant quantity is the spin-flip rate which measures the speed by which the sea of right-handed neutrinos gets populated if the number density of left-handed neutrinos is held fixed at its thermal equilibrium value because they are replenished by other reactions. Evidently, since the number of right- and left-handed neutrinos is \( n_{R,L} = \frac{1}{2} (1 \pm P_3) \) we have that \( n_R/n_L = \Gamma_{\text{flip}} = \Gamma_{\text{depol}}/2 \) so that

\[
\Gamma_{\text{flip}} = \mu^2 \int \frac{d^3k}{(2\pi)^3} \left[ 1 - (\hat{p} \cdot \hat{k})^2 \right] \left( \langle E^2 \rangle_K + \frac{1 - 3(\hat{p} \cdot \hat{k})^2}{2} \langle B^2 \rangle_K \right),
\]

(2.20)

where, again, \( K \) is constrained by \( k_0 = \hat{p} \cdot k \).

3. Quantum kinetic approach

3.1. Relaxation rate

The spin-flip rate derived by the classical-trajectory approach in the previous section is valid only in the approximation that the momentum transfer \( k \) is always much smaller
than the neutrino momentum $p$ so that it is justified to view the neutrino as propagating on a straight line during a typical field correlation time. Further, the electric and magnetic fields were taken to be classical variables, ignoring the quantized nature of the exchange of energy and momentum $(k_0, k)$ between the neutrino and the medium.

In a quantum-kinetic approach the relaxation of the neutrino helicity by electromagnetic interactions is described by a Boltzmann collision equation of the form

$$\partial_t f^R_p = \int \frac{d^3q}{(2\pi)^3} \left[ W_{L \to R}(Q, P) f^L_q (1 - f^R_p) - W_{R \to L}(P, Q) f^R_p (1 - f^L_q) + \ldots \right],$$

(3.1)

where $f^L,R_p$ are the occupation numbers for left- and right-handed neutrinos of momentum $p$, respectively. Here, $W_{R \to L}(P, Q)$ is the transition rate for a right-handed neutrino of four-momentum $P$ to a left-handed one of $Q$ under the influence of the electromagnetic fields of the medium. There are also terms representing pair production and annihilation processes which we will discuss below. If there are large-scale magnetic fields the Boltzmann equation includes an oscillation term analogous to the simultaneous treatment of collisions and flavor oscillations in Ref. [10].

The neutrino interaction with the electromagnetic fields is of the current-current form. It is well known from linear-response theory that in this situation the transition probability is given essentially by the dynamical structure function of the medium (in our case the electromagnetic fields), i.e. by correlator expressions like the classical ones discussed in the previous section. Therefore, all that remains to be done to derive $W(P, Q)$ is to determine the required tensorial contraction between the neutrino momenta and the electromagnetic field correlators.

To this end we begin with the interaction Lagrangian Eq. (2.2) and consider the matrix element for the transition of a left-handed neutrino with four-momentum $P$ to a right-handed one with $Q$. With the massless neutrino Dirac spinors $u_{L,R}$ which involve the appropriate chirality projections one finds

$$\mathcal{M} = \mu \bar{u}_{PL} \left( B \cdot \Sigma - iE \cdot \alpha \right) u_{Q,R}.$$

Of course, because the interaction couples only states of opposite chirality it would have been enough to include one chirality projector. Taking the square of this matrix element and carrying out the Dirac traces we find

$$\mu^{-2} W_{L \to R}(P, Q) = \langle B_i B_j + E_i E_j \rangle_K \left( \frac{1}{2} \hat{p}_i \hat{q}_j - \frac{\hat{q}_i \hat{q}_j}{2} \right) + \langle E_i E_j \rangle_K \frac{\hat{p}_i + \hat{q}_i}{2},$$

(3.2)

where $K = P - Q$. In the limit of small $k$ we may set $\hat{p} = \hat{q}$, taking us back to Eq. (2.16) of the classical-trajectory approach apart from the new term which involves $(E \times E)_K$. It is easy to show that in the classical limit where the fields are commuting variables this term disappears under the $d^3k$ phase-space integration so that our present result leads to the same classical spin-flip rate.

In the general case the status of the $E \times E$ term is more subtle. It has the opposite sign for an $R \to L$ transition. On the other hand it is easy to show that $W_{L \to R}(P, Q) =$
$W_{R \rightarrow L}(P, Q)$ if the medium is invariant under parity. Therefore, in such a medium the $\langle E \times E \rangle_K$ term must vanish even in the general case. This is not surprising in the sense that the electromagnetic properties of a parity invariant medium are characterized by precisely two independent "material constants" which can be chosen to be the dielectric permittivity and the magnetic permeability. Another pair of equivalent parameters are the longitudinal and transverse polarization functions $\Pi_{TL}(K)$ which we will use below. Therefore, there are only two independent field correlator expressions, e.g. $\langle E^2 \rangle_K$ and $\langle B^2 \rangle_K$. They are related to $\Pi_{TL}(K)$ by the fluctuation–dissipation theorem. A parity-non-invariant medium, on the other hand, is characterized by three independent parameters—there are two different transverse polarization functions. The third function gives rise to a non-vanishing $\langle E \times E \rangle_K$ field correlator.

For the neutrino spin relaxation problem we inevitably have a medium which is not parity invariant because initially it involves only left-handed neutrinos. By their assumed magnetic moments they are expected to give rise to a non-vanishing $\langle E \times E \rangle$ term, i.e. $L \rightarrow R$ and $R \rightarrow L$ collisions are not expected to occur with the same transition probability. However, in terms of the relaxation rate this would be an order $\mu^4$ effect so that to order $\mu^2$ we may ignore electromagnetic neutrino–neutrino interactions. In the cases of interest to us we may thus ignore the $\langle E \times E \rangle_K$ term and use $W(P, Q) \equiv W_{L \rightarrow R}(P, Q) = W_{R \rightarrow L}(P, Q)$ for the transition rate.

We proceed by applying Maxwell's equations in the same way as in the previous section, never changing the order in which the non-commuting field operators appear. Since we never changed this order even in the classical case, we find the same simplifications as there. The final contraction of indices then yields

$$ W(P, Q) = \mu^{-2} \langle B^2 \rangle_K \left( \frac{1 + (\hat{p} \cdot \hat{k})(\hat{q} \cdot \hat{k})}{2} - 2k_0 \frac{\hat{p} \cdot \hat{k} + \hat{q} \cdot \hat{k}}{2} - \frac{k_0^2}{k^2} \left( \frac{\hat{p} \cdot \hat{q}}{2} - 3(\hat{p} \cdot \hat{k})(\hat{q} \cdot \hat{k}) \right) \right) + \langle E^2 \rangle_K \left( \frac{1 + \hat{p} \cdot \hat{q}}{2} - (\hat{p} \cdot \hat{k})(\hat{q} \cdot \hat{k}) \right). $$

This complicated looking expression can be transformed to

$$ W(P, Q) = -\mu^2 \frac{K^2(p_0 + q_0)^2}{8k^2p_0q_0} \left[ 2\langle E^2 \rangle_K + \langle B^2 \rangle_K \left( 1 - \frac{3k_0^2}{k^2} + \frac{k^2}{(p_0 + q_0)^2} \right) \right]. $$

Note that $q_0 = p_0 - k_0$ and $p_0 = p$ and that $-K^2 = -(P - Q)^2 = 2PQ = 2(p_0q_0 - p \cdot q)$ is a positive number.

We now return to the Boltzmann collision equation (3.1) for a parity invariant medium where we need only one function $W(P, Q)$. The transition rate for the pair production and annihilation processes is given by the same function with "crossed" four-momenta, i.e. $P \rightarrow -P$ or $Q \rightarrow -Q$. The complete collision equation is then
Here, the first term in the collision integral represents gain by \( L \rightarrow R \) spin-flip scattering, the second loss by \( R \rightarrow L \) scattering, the third gain by pair production, and the fourth loss by pair annihilation.

In the astrophysical applications that we are interested in (supernovae, early universe) the left-handed neutrinos are and remain in thermal equilibrium so that we may replace \( f_{L} \) by the Fermi–Dirac distribution \( f_{F} \) at the appropriate temperature and chemical potential. The collision equation is then a linear differential equation of the form

\[
\partial_{t} f_{p}^{R} = \Gamma_{\text{gain}}(p)(1 - f_{p}^{R}) - \Gamma_{\text{loss}}(p)f_{p}^{R} \tag{3.6}
\]

with

\[
\Gamma_{\text{gain}}(p) = \int \frac{d^{3}q}{(2\pi)^{3}} \left[ W(Q, P) f_{q}^{F} + W(-Q, P)(1 - f_{q}^{F}) \right],
\]

\[
\Gamma_{\text{loss}}(p) = \int \frac{d^{3}q}{(2\pi)^{3}} \left[ W(P, Q)(1 - f_{q}^{F}) + W(Q, -P)f_{q}^{F} \right]. \tag{3.7}
\]

Further, in equilibrium we have \( \partial_{t} f_{p}^{R} = 0 \) and \( f_{p}^{R} = f_{p}^{F} \). Inserting this in Eq. (3.6) reveals that \( \Gamma_{\text{gain}} = \Gamma_{\text{tot}} f_{p}^{F} \) with \( \Gamma_{\text{tot}} = \Gamma_{\text{gain}} + \Gamma_{\text{loss}} \) so that we may write

\[
\partial_{t}(f_{p}^{R} - f_{p}^{F}) = -\Gamma_{\text{tot}}(p)(f_{p}^{R} - f_{p}^{F}). \tag{3.8}
\]

Therefore, \( \Gamma_{\text{tot}}(p) \) is the appropriate rate that measures the exponential approach of the \( p \) mode to helicity equilibrium.

### 3.2. Correlation functions

In order to evaluate the relaxation rate we need to know the electric and magnetic field fluctuations in a given medium. By virtue of the fluctuation–dissipation theorem they are related to the imaginary part of the medium's dielectric response functions. One way of expressing these quantities is in terms of the longitudinal and transverse photon spectral functions \( A_{T,L} \) which are the coefficients in the decomposition \( A_{\mu\nu} = -P_{\mu\nu} A_{T} - Q_{\mu\nu} A_{L} \) in the Landau gauge. The photon spectral function \( A_{\mu\nu}(K) \) is related to the retarded and advanced Green's functions through \( A_{\mu\nu}(K) = \text{Im} \left[ \frac{iD_{\mu\nu}(k_{0} + i\epsilon, k) - iD_{\mu\nu}(k_{0} - i\epsilon, k)}{2\pi} \right] \).

The transverse and longitudinal projection operators have the non-vanishing components \( P_{ij}(K) = -\delta_{ij} + (k_{i}k_{j})/k^{2} \) and \( Q_{\mu\nu}(K) = -(k^{2}k^{2})^{-1}(k_{2}, -k_{0}k)_{\mu}(k^{2}, -k_{0}k)_{\nu} \). In terms of the photon polarization functions we have

\[
A_{T,L}(K) = -\frac{1}{\pi} \frac{\text{Im} \Pi_{T,L}}{|K^{2} - \text{Re} \Pi_{T,L}|^{2} + |\text{Im} \Pi_{T,L}|^{2}}. \tag{3.9}
\]

The analytic continuation in the imaginary part is the retarded \( \text{Im} \Pi_{T,L}(k_{0} + i\epsilon) \). Note that the photon polarization tensor is given by \( \Pi_{\mu\nu} = P_{\mu\nu} \Pi_{T} + Q_{\mu\nu} \Pi_{L} \).
In terms of the spectral functions the field fluctuations are found to be [11]

\[
\langle B^2 \rangle_K = \frac{2\pi}{1 - e^{-\beta_0}} 2k^2 A_T(K),
\]

\[
\langle E^2 \rangle_K = \frac{2\pi}{1 - e^{-\beta_0}} \left[ 2k^2_0 A_T(K) + K^2 A_L(K) \right],
\]

(3.10)

where \( \beta = 1/T \) is the inverse temperature. Note that for a positive \( k_0 \) (energy given to the medium) the thermal factor is identical with \( 1 + (e^{\beta k_0} - 1)^{-1} \), i.e. it is understood as a Bose stimulation factor for exciting a quantum \((k_0, k)\) of the medium. Conversely, if \( k_0 < 0 \) (energy lost by the medium) this factor is \( -(e^{\beta |k_0|} - 1)^{-1} \). Apart from the minus sign it is the occupation number for such an excitation. The functions \( A_{T,L}(K) \) are odd in \( k_0 \) so that the negative sign of the thermal factor is automatically absorbed. Put differently, the correlators obey detailed-balance conditions of the form \( \langle B^2 \rangle_{-K} = \langle B^2 \rangle_K e^{-\beta k_0} \).

4. Imaginary part of the neutrino self-energy

An alternative way of calculating the rate of populating right-handed neutrinos is through the imaginary part of the neutrino self-energy [12]. The relaxation rate \( \Gamma_{\text{tot}} \) introduced in the previous section is directly related to the imaginary part of the neutrino self-energy through

\[
\Gamma_{\text{tot}}(p) = \frac{2 \text{Im} \langle p | P_{PR} \Sigma(p_0 + i\epsilon) P_{PR} \rangle}{\langle P_{PR} | U_{PR} | P_{PR} \rangle},
\]

(4.1)

where \( U_{PR} \) is the Dirac spinor of a right-handed neutrino with four-momentum \( P \).

The self-energy to one-loop order is a bubble diagram with a left-handed neutrino and a photon propagator connected via the interaction vertex \( \mu K_\alpha \sigma^{\alpha\mu} \). Its time ordered imaginary part can be shown to be [13]

\[
\text{Im} \Sigma(P) = -\frac{\mu^2}{\sin 2\phi_P} \int \frac{d^4 K}{(2\pi)^4} \epsilon(p_0 + k_0) \epsilon(k_0) \frac{1}{2} \sin 2\phi_{P+K} \frac{1}{2} \sinh 2\theta_K \times K_\alpha \sigma^{\alpha\mu}(P + K) \frac{1}{2} (1 - \gamma_5) K_\beta \sigma^{\beta\nu}(2\pi)^2 \delta((P + K)^2) A_{\mu\nu}(K),
\]

(4.2)

where \( \epsilon(x) = \pm 1 \) depending on the sign of \( x \), the photon spectral function \( A_{\mu\nu} \) was defined in Section 3.2 above, and

\[
\frac{1}{2} \sin 2\phi_K = \frac{e^{\beta |k_0|/2}}{e^{\beta |k_0|} + 1}, \quad \frac{1}{2} \sinh 2\theta_K = \frac{e^{\beta |k_0|/2}}{e^{\beta |k_0|} - 1}.
\]

(4.3)

The neutrino inside the loop is necessarily left handed if the external one is right handed since the interaction flips chirality, so we do not need to include the \( \frac{1}{2} (1 + \gamma_5) \) part of the propagator to compute the right-handed self-energy. The fermion distribution function in Eq. (4.3) should therefore describe a fully populated equilibrium ensemble for which the chemical potential is taken to vanish.
The results of Refs. [12,13] differ by the overall factor $\epsilon(p_0)$ which is related to the retarded [12] or time ordered [13] prescription for $\text{Im } \Sigma$. The physical sign is however obvious since $\Gamma_{\text{tot}}$ must be positive so we can concentrate on $p_0 > 0$. Explicitly we find for an on-shell $\nu_R$ with momentum $p$

$$
\Gamma_{\text{tot}}(p) = \frac{\mu^2}{2\pi} \int_0^\infty dk \int_{-\infty}^\infty dk_0 \theta\left(-K^2(K^2 + 4pk_0 + 4p^2)\right)
$$

\hspace{1cm} \times \left[\frac{\epsilon(k_0)}{e^{\beta|p_0 + k_0|} + 1} + \frac{\epsilon(p_0 + k_0)}{e^{\beta|k_0|} - 1} + \epsilon(p_0 + k_0)\theta(-k_0) - \epsilon(k_0)\theta(-p_0 - k_0)\right]

\hspace{1cm} \times \frac{K^4}{k^2} \left[(1 + \frac{k_0}{p} + \frac{K^2}{4p^2}) A_T(K) - \left(1 + \frac{k_0}{2p}\right)^2 A_L(K)\right] \epsilon(k_0), \tag{4.4}
$$

where the medium was taken to be parity invariant so that there is only one transverse polarization function. There are contributions to $\Gamma_{\text{tot}}$ both from creation and annihilation of $\nu_R$ through several processes. Depending on the signs of $k_0$ and $p + k_0$ these processes can be divided into pair creation/annihilation, Cherenkov radiation and scattering off charged particles through virtual photon exchange [12]. Calculating the imaginary part of the self-energy is thus a convenient way of obtaining all processes that are allowed at the one-loop level with a interacting photon correlation function.

As expected, this relaxation rate is identical to $\Gamma_{\text{gain}} + \Gamma_{\text{loss}}$ from Eq. (3.7) if we use a neutrino distribution at zero chemical potential and if we express the field correlators by virtue of Eq. (3.10) in terms of $A_T, A_L$.

5. Depolarization in a relativistic plasma

In order to evaluate the depolarization rate explicitly for a specific physical system we consider a relativistic QED plasma of the sort encountered in the early universe where the chemical potentials of the charged fermions are negligibly small. This system is characterized by the temperature $T$ alone which is taken to be much larger than the electron mass, but small enough that muons or pions are essentially absent.

Even in such a relatively simple system, the lowest-order polarization functions $\Pi_{T,L}$ depend on $K$ in very complicated ways. In order to arrive at a first estimate we recall that Ref. [5] implies that the main contribution to the neutrino spin relaxation rate arises from spin-flip scattering $\nu_L + e^\pm \rightarrow e^\pm + \nu_R$. The cross section for this process has an infrared divergence which is regularized by screening with a scale of order the photon plasma mass $M = eT/3$. As a first estimate it is thus enough to obtain expressions for $\Pi_{T,L}$ which are precise for $k_0, k \lesssim eT$ for the leading contribution. In the high-temperature limit ($m_e \ll T$) they depend only on the variable $k_0/k$. Explicitly one finds [14]

$$
\Pi_T(K) = \frac{3M^2}{2} \left[\frac{k_0^2}{k^2} + \left(1 - \frac{k_0^2}{k^2}\right) \frac{k_0}{2k} \ln \left(\frac{k_0 + k}{k_0 - k}\right)\right],
$$
\[ \Pi_L(K) = 3\mathcal{M}^2 \left( 1 - \frac{k_0^2}{k^2} \right) \left[ 1 - \frac{k_0}{2k} \ln \left( \frac{k_0 + k}{k_0 - k} \right) \right]. \] (5.1)

The real part of these functions is obtained by taking the modulus of the argument of the logarithms. Below the light cone \((k_0^2 < k^2)\) the logarithms yield the imaginary parts

\[ \text{Im} \Pi_T(K) = \frac{3\mathcal{M}^2}{2\pi} \left( 1 - \frac{k_0^2}{k^2} \right) \frac{k_0}{2k} \theta(k^2 - k_0^2), \]

\[ \text{Im} \Pi_L(K) = \frac{3\mathcal{M}^2}{2\pi} \left( 1 - \frac{k_0^2}{k^2} \right) \frac{k_0}{2k} \theta(k^2 - k_0^2). \] (5.2)

Above the light cone the only imaginary part of \(\Pi_T, L\) comes from the \(i\varepsilon\) in the retarded prescription. Put another way, above the light cone this approximation for \(\Pi_T, L\) provides support only on the discrete branches which correspond to transverse and longitudinal plasmons.

In the early universe, a typical thermal neutrino momentum is around \(3T\). Therefore, if indeed a typical momentum transfer is of order \(eT\) we are in a situation where the classical-trajectory approach should be justified where it was assumed that \(k_0, k \ll p\) and also \(k_0 \ll T\). Therefore, in Eq. (4.4) the infrared sensitive Bose-Einstein term dominates which expands as \(e(k_0)/(e|\varepsilon|) - 1 \approx T/k_0\). Altogether we thus find for the relaxation rate in the classical limit

\[ \Gamma_{\text{tot}} = \frac{\mu^2}{2\pi} \int_0^\infty dk \int_{-k}^k dk_0 \frac{T}{k_0} \frac{K^4}{k^2} [A_T(K) - A_L(K)]. \] (5.3)

If for the moment we ignore the resummation terms \(\text{Re}\Pi_T, L\) and \(\text{Im}\Pi_T, L\) in the denominator of Eq. (3.9), the classical depolarization rate is found to be

\[ \Gamma'_{\text{tot}} = \alpha \mu^2 T^3 \frac{2}{3} \int_0^\infty \frac{dk}{k} \] (5.4)

with \(\alpha = e^2/4\pi \approx 1/137\) the fine-structure constant. Because this integral diverges it needs to be cut off by minimum and maximum momentum transfers, leading to \(\Gamma_{\text{tot}} = \alpha \mu^2 T^3 (2/3) \ln(k_{\text{max}}/k_{\text{min}})\). Clearly, the infrared cutoff is provided by the plasma mass \(\mathcal{M} = eT/3\) while \(k_{\text{max}}\) is given by the neutrino momentum itself which is typically \(3T\). With these numbers we find \((2/3) \ln(k_{\text{max}}/k_{\text{min}}) \approx 2\). Because \(\Gamma_{\text{tot}}\) depends only logarithmically on the assumed cutoffs we expect this simple estimate to be already rather precise.

We stress, however, that the initial assumption that only small momentum transfers of order \(eT\) were important was not justified. In the classical approximation the momentum transfers are distributed as \(1/k\) so that the average is \(\langle k \rangle = (k_{\text{max}} - k_{\text{min}})/\ln(k_{\text{max}}/k_{\text{min}})\). Therefore, with \(k_{\text{max}} \gg k_{\text{min}}\) the scale for a typical momentum transfer is set by \(k_{\text{max}}\) and not by \(k_{\text{min}}\). Thus, even though the distribution of momentum transfers peaks around \(eT\), the average is still of order \(T\) and thus not small.
A numerically precise calculation of $\Gamma_{\text{tot}}$ thus requires the full quantum expressions. Moreover, while the approximate expressions (5.1) and (5.2) for the polarization functions are sufficient to implement the screening cutoff at small $k_0$ and $k$ and thus are sufficiently precise in the denominator of Eq. (3.9), they are not guaranteed to be accurate enough in the numerator of Eq. (3.9). While it is well known that the approximate expressions are surprisingly accurate even for $|K|$ of order $T$, we still use the full one-loop expressions which are given in Ref. [15]. The full one-loop expressions also have the advantage of providing continuous support for $A_{\tau \ell}$ above the light cone which corresponds to pair processes. Therefore, we are able to compare directly the contribution to $\Gamma_{\text{tot}}$ from pair processes (above the light cone) with that from spin-flip transitions (below the light cone).

The numerical integration of the full quantum expression in Eq. (4.4) yields the momentum dependent depolarization rate shown in Fig. 1. When we average these rates over a thermal neutrino distribution we obtain

$$\langle \Gamma_{\text{tot}} \rangle = 1.81 \alpha \mu^2 T^3,$$

where only 1.5% of the coefficient arise from pair processes. They are indeed significantly subdominant as predicted in Ref. [5]. Moreover, our simple estimate derived from the classical approximation was surprisingly accurate.

6. Early universe

6.1. Magnetic moment constraint in a plasma

The depolarization of the spin in the fluctuating electromagnetic field of the early universe affects primordial nucleosynthesis. Although there is a growing awareness that

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7 In Eq. (A.4) of Ref. [15] there is a factor $q^2/\omega^2$ missing in the term below the light cone.
systematic errors are large in the determination of the nucleosynthesis limit on the effective number of neutrino species, it still seems reasonable to assume that an extra neutrino degree of freedom is not allowed [16]. Such an assumption then yields a cosmological limit on the neutrino magnetic moment.

In order to avoid populating the right-handed component of our Dirac neutrinos before BBN we need to require that $\Gamma_{\text{tot}}$ is less than the Hubble rate at all times between the muon annihilation and neutrino freeze-out epochs:

$$\Gamma_{\text{tot}} = 0.0132\mu^2T^3 < H = \frac{T^2}{m_{\text{pl}}} \left( \frac{4\pi^2g_s}{45} \right)^{1/2}. \quad (6.1)$$

Evidently, the most stringent bound comes from imposing this constraint at as high a temperature as possible. We take $g_s \approx 10.75$ for the effective number of thermal degrees of freedom which contribute to the energy density and thus account for the electrons, 3 left-handed neutrino species, and the photons. With $T = 100$ MeV this leads to

$$\mu < 6.2 \times 10^{-11}\mu_B, \quad (6.2)$$

where $\mu_B = e/2m_e$ is the Bohr magneton. This result puts previous estimates [5], where the infrared singularity in the cross section of neutrino–electron scattering mediated by a $t$-channel photon was estimated by a momentum cut-off at the Debye mass, on a solid basis. Here we used the full resummed photon propagator to take into account the screening effects correctly.

To obtain a more precise limit we should examine the relevant Boltzmann equations. This is however not really warranted since the most stringent limit on neutrino magnetic moments is still obtained from energy loss considerations of helium burning globular cluster stars. Plasmon decay would cool these stars too fast unless $\mu \lesssim 3 \times 10^{-12}\mu_B$ [17,18]. Of course, this limit is valid only for neutrinos with mass less than a few keV. There are additional bounds from SN 1987A [2,18] which are valid up to the experimental $\nu_\tau$ mass limit of about 24 MeV and which are also more restrictive than the cosmological limit, even though their exact numerical values differ significantly between the works of different authors and are based on rather sparse data. Still, unless these astrophysical limits are plagued with implausibly large systematic errors we arrive at the conclusion that neutrino dipole moments can affect nucleosynthesis only in connection with a large-scale primordial magnetic field.

6.2. Large-scale magnetic field

Another source for spin depolarization would be neutrino interactions with a background magnetic field. Let us assume for a moment that such a hypothetical (constant) field exists and compare the neutrino spin-flip rate due to a background field to spin flip rate induced by the fluctuations of thermal photons. This mechanism has been considered previously by several authors [7,8,19] and it is of interest to compare it with the depolarization in a stochastic electromagnetic field. In the classical picture the depolarization in a random field is a consequence of the random walk that the polarization vector
$P$ performs on the unit sphere, while in a constant background field the polarization is attenuated by the helicity-measuring scattering that projects the coherently rotating polarization vector onto the third axis, corresponding to a damping of the off-diagonal elements in the density matrix.

Though it has been derived several times before it easy enough to extract this damping directly from Eq. (2.3). We can simply calculate the eigenvalues of $M$ and see that the damping of the third component, in the limit of small $D$, is

$$
\Gamma_b = \frac{4\mu^2 B^2}{\omega_{\text{ref}}^2 + 4\mu^2 B^2_\perp}.
$$

(6.3)

At temperatures $1 \text{ MeV} \leq T \leq 100 \text{ MeV}$ one finds for effectively massless neutrinos [20] that $\omega_{\text{ref}} = \xi\langle p_0 \rangle$ with

$$
\xi = \frac{7\sqrt{2\pi^2 G_F T^4}}{45 m^2} \left( 1 + C_v \frac{2m^2}{m^2} \right),
$$

(6.4)

where $C_v = 1$ for electron neutrinos so that $\xi \simeq 1.1 \times 10^{-20} (T/\text{MeV})^4$, whereas $C_v = 0$ for muon and tau neutrinos. The main contribution to $D$ comes from neutrino elastic and inelastic scattering with leptons and equals half the total collision frequency [21,22]. Including scattering with electrons and neutrinos as well as annihilation one finds at temperature around $T = 1 \text{ MeV}$

$$
D = 2.04 G^2 T^5,
$$

(6.5)

for electron neutrinos. Since from Eqs. (6.4) and (6.5) we have that $\xi\langle p_0 \rangle / D \simeq 100$ the small $D$ approximation in Eq. (6.3) is good and would remain so even if we include damping due to collisions with muons in $D$.

Because the electrical conductivity of the universe is large [23], a background magnetic field is imprinted on the plasma and comoves with the expansion of the universe. Thus flux conservation implies that the mean r.m.s. field scales with temperature like

$$
\frac{3}{8} \langle B^2 \rangle^{1/2} = \langle B^2 \rangle^{1/2} = B_0 (T/T_0)^2.
$$

We should however point out that in the early universe a large conductivity implies a large Reynolds number, and hence there is a possibility for turbulence which can redistribute magnetic energy to various length scales. This was verified in [24], where full magnetohydrodynamics was simulated by using a simple MHD generalization of the cascade model much used in studying hydrodynamical turbulence. In the magnetic case it features a transport of magnetic energy from small length scales to large length scales. In a realistic situation the issue is then: has the coherence length of the background field grown large enough so it can be treated as a constant mean field on a given scale? Here we shall just assume that such a mean field exists. The limit from a large scale background field concerns the combination $\mu B_0$. Requiring that $\Gamma_b < H$ and using Eq. (6.3) we find that

$$
\mu B_0 < \frac{T_0^2}{T^2} \left( \frac{3}{8} \right)^{1/2} \xi\langle p_0 \rangle \left( \frac{H}{D - H} \right)^{1/2}.
$$

(6.6)
This constraint should be imposed at the temperature where the right-hand side is minimized (and $D > H$). With the damping rate in Eq. (6.5) and $\langle p_0 \rangle = 3T$ it happens at $T = 1.5$ MeV which is above the kinetic freeze-out temperature ($2D \gtrsim H$ gives $T \gtrsim 1$ MeV), and we obtain

$$\left( \frac{\mu}{\mu_B} \right) \left( \frac{B_0}{\text{MeV}^2} \right) < 2.3 \times 10^{-19} \frac{T_0^2}{\text{MeV}^2}. \quad (6.7)$$

Here we adopted $\alpha = 10.75$ in the Hubble rate. We notice that this mechanism of depolarization in a large scale field is most efficient at a rather low temperature, in contradistinction to the case of a random field which gave a stricter constraint at high temperature. The rate $\Gamma_b$ has been computed previously [7,8] but was not applied correctly at the neutrino freeze-out in [8] and an approximate formula for $\Gamma_b$ was used in [7] leading to a somewhat too stringent bound. Thus, the bound from a large scale field is better than the one from small scale fluctuations if $B_0 > 3.7 \times 10^{-9}$ (MeV)$^2$ at $T_0 = 1$ MeV, which is a rather weak field compared to other typical scales. The bound on a large scale magnetic field from BBN is [25] $B_0 < 4 \times 10^{-5}$ MeV$^2$ at $T_0 = 0.01$ MeV. (If the field is not homogeneous, then the limit is less stringent by an order of magnitude.) There is thus room for the large scale field to have been large enough to be the dominating depolarization mechanism without contradicting BBN. The existence or non-existence of such a primordial field should become less speculative with the forthcoming experiments trying to measure the strength of the intergalactic magnetic field [26].

7. Summary

We have performed a detailed comparison of the classical and quantum descriptions of neutrino depolarization by magnetic moment interactions in the presence of a stochastic electromagnetic field. Our main result is a general expression of the depolarization rate in terms of the electric and magnetic field correlation functions. By virtue of the fluctuation–dissipation theorem they are essentially equivalent to the (imaginary parts) of the medium’s dielectric response functions. Our analysis is exact up to second order in the magnetic moment $\mu$ and to the order that the electromagnetic field correlation functions are calculated.

We have evaluated the correlation functions and depolarization rate explicitly for the case of a relativistic QED plasma. The neutrino depolarization rate is dominated by spin-flip scattering on relativistic electrons and positrons. Even though the cross section of this process peaks in the forward direction so that the distribution of momentum transfers favors values of order $eT$, an average momentum transfer is of order the neutrino momentum. Therefore, the depolarization rate is not approximated well by the classical description which is based on the assumption that a typical momentum transfer is small relative to the neutrino momentum. A numerically reliable result requires our full quantum treatment.
Our precise calculation of the depolarization rate puts a previous estimate [5] on a firm basis. Imposing the constraint that a right-handed neutrino must not have been in equilibrium at nucleosynthesis we derive the limit $\mu < 6.2 \times 10^{-11}\mu_B$ on the neutrino Dirac magnetic moment. Other astrophysical limits are more restrictive by about an order of magnitude, revealing that neutrino magnetic moments can affect big-bang nucleosynthesis only in connection with large-scale primordial magnetic fields which would provide for an additional mechanism for left–right transitions.

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References

For a recent review, see S. Sarkar, Big bang nucleosynthesis and physics beyond the standard model, Rep. Prog. Phys. 59 (1996) 1493.