Solutions for Assignment of Week 09
Introduction to Astroparticle Physics

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Assignment of 12 January 2010

1 Energy transfer in WIMP-nucleus collision

A WIMP with mass m_χ and nonrelativistic velocity v strikes a nucleus at rest with mass m_A.

(i) Show that the energy transfer to the nucleus is

$$\Delta E = \frac{m_A\, m_\chi^2}{(m_A + m_\chi)^2}\, v^2\, (1 - \cos\theta)$$

where θ is the scattering angle in the CM frame.

(ii) For which scattering angle and WIMP mass is the transfer maximal? Interpretation?

(iii) Assume the galactic WIMP velocity distribution follows an isothermal halo model with

$$\frac{\mathrm{d}n}{\mathrm{d}v} = n_0\, \frac{4\, v^2}{\sqrt{\pi}\, \sigma^3}\, \mathrm{e}^{-v^2/\sigma^2}$$

where σ is the velocity dispersion. Assume further an isotropic scattering cross section that is independent of velocity. What is the average energy transfer per collision, assuming the Earth is at rest relative to an isotropic WIMP velocity distribution?

(iv) Give numerical values for the average energy transfer, assuming $\sigma = 220$ km s^{-1} and taking WIMP masses of 50, 100, and 200 GeV and the target nuclei oxygen ($A = 16$), silicon ($A = 28$), calcium ($A = 40$), germanium ($A = 74$), xenon ($A = 132$), or tungsten ($A = 184$).

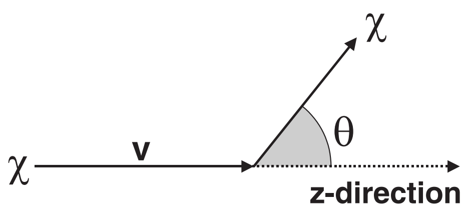
Solution

(i) Energy transfer

Take the direction of the WIMP velocity to be the z–direction. The CM system is defined by the WIMP and nucleus momenta to be opposite and equal. Because the kinematics is nonrelativistic and because the nucleus is at rest in the laboratory system, the transformation to the CM system is enabled by $p_{\rm CM} = m_\chi(v - v_{\rm CM}) = m_A v_{\rm CM}$, implying

$$v_{\rm CM} = \frac{m_\chi}{m_A + m_\chi}\, v \qquad \text{and} \qquad p_{\rm CM} = m_A v_{\rm CM} = \frac{m_A\, m_\chi}{m_A + m_\chi}\, v$$

The quantity $m_A m_\chi/(m_A + m_\chi)$ is the reduced mass of the two-body system.



After the collision with angle θ in the CM system, the nucleus momentum parallel and transverse to the z–direction is in the CM system

$$p_{{\rm CM},\|} = -p_{\rm CM}\cos\theta \qquad \text{and} \qquad p_{{\rm CM},\perp} = p_{\rm CM}\sin\theta$$

and therefore in the original laboratory system

$$p_{A,\|} = p_{\rm CM} - p_{\rm CM}\cos\theta \qquad \text{and} \qquad p_{A,\perp} = p_{\rm CM}\sin\theta$$

The energy transfer (measured in the lab system) is the same as the total final nucleus energy because it was originally at rest,

$$\Delta E = \frac{p_A^2}{2m_A} = \frac{p_{A,\|}^2 + p_{A,\perp}^2}{2m_A} = \frac{p_{\rm CM}^2}{2m_A}\left[\sin^2\theta + (1-\cos\theta)^2\right] = \frac{p_{\rm CM}^2}{2m_A}\, 2\,(1-\cos\theta)$$

which is the result we are looking for.

(ii) Maximum energy transfer

The maximum of ΔE arises for back-scattering with $\theta = \pi$ and therefore $\cos\theta = -1$. If the WIMP mass is equal to the nuclear mass, the energy transfer is

$$\Delta E = \frac{m_\chi^3}{(m_\chi + m_\chi)^2}\, v^2 \times 2 = \frac{m_\chi v^2}{2}$$

and thus identical to the initial kinetic energy: The energy transfer is complete, the WIMP remains at rest. Experiments are most sensitive to WIMPs with a mass matched to the mass of the target nucleus.

(iii) Average energy transfer for an isothermal halo model

The scattering probability scales with $\sigma_0 v$. If σ_0 is velocity independent, the probability for a scattering event scales with v times the distribution function. The energy transfer in a given collision scales with v^2. Therefore, the average of this per collision is

$$\left\langle v^2 \right\rangle_{\rm coll} = \frac{\int_0^\infty {\rm d}v\, v^3 f(v)}{\int_0^\infty {\rm d}v\, v\, f(v)} = 2\,\sigma^2$$

where here σ is the velocity dispersion, not the cross section. The cross section is assumed isotropic in the CM frame, so $\langle \cos\theta \rangle = 0$. Therefore

$$\langle \Delta E \rangle = 2\,\sigma^2 \,\frac{m_A\, m_\chi^2}{(m_A + m_\chi)^2}$$

(iv) Average energy transfer for different mass combinations

The average energy transfer in our case is

$$\langle \Delta E \rangle = 10.8 \text{ keV}\, \frac{m_\chi}{10 \text{ GeV}}\, \frac{m_A\, m_\chi}{(m_A + m_\chi)^2}$$

For the stated examples one finds the energy transfers (in keV) given here.

Element:	O	S	Ca	Ge	Xe	W
A	16	28	40	74	132	184
$m_\chi = 50$ GeV	9.5	12.1	13.2	13.1	11.1	9.4
100 GeV	12.1	17.6	21.3	26.0	26.6	25.0
200 GeV	13.9	21.9	28.5	41.0	50.7	53.5

2 Annual modulation of WIMP signal

Consider the same isothermal halo model of the previous exercise. The Earth moves in the halo with a velocity $\mathbf{v}_\mathrm{E}$. The halo itself is assumed to be non-rotating, so the WIMP velocity distribution is isotropic in a non-rotating frame.

(i) What is the velocity distribution in the laboratory frame, relevant for a WIMP detection experiment?

(ii) Assuming $v_\mathrm{E} = \sigma$, how much larger is the detection rate compared to $v_\mathrm{E} = 0$? How much larger is the average energy transfer per collision?

(iii) On its orbit around the Sun, the Earth velocity relative to the halo varies in a range ± 15 km s^{-1} relative to the average, taken to be $\langle v_\mathrm{E}\rangle = \sigma = 220$ km s^{-1}. How large is the annual variation of the detection rate?

Solution

(i) Laboratory velocity distribution

The isothermal velocity distribution corresponds to a three-dimensional distribution function

$$\mathrm{d}^3\mathbf{v}\, f(\mathbf{v}) \propto \mathrm{d}^3\mathbf{v}\, \mathrm{e}^{-\mathbf{v}^2/\sigma^2}$$

where we do not worry about the overall normalization that will be fixed in the end. The velocity in the laboratory frame is $\mathbf{u} = \mathbf{v} - \mathbf{v}_\mathrm{E}$. The differential is the same: $\mathrm{d}^3\mathbf{u} = \mathrm{d}^3\mathbf{v}$. Therefore, the laboratory distribution function is given by

$$\mathrm{d}^3\mathbf{u}\, g(\mathbf{u}) = \mathrm{d}^3\mathbf{u}\, f(\mathbf{u} + \mathbf{v}_\mathrm{E}) \propto \mathrm{d}^3\mathbf{u}\, \mathrm{e}^{-(\mathbf{v}+\mathbf{v}_\mathrm{E})^2/\sigma^2}$$

Since we do not have directional sensitivity, we are only interested in the scalar distribution function

$$\begin{aligned}
\mathrm{d}u\, g(u) &= \mathrm{d}u\, u^2 \int \mathrm{d}\phi\, \mathrm{d}\cos\theta\, \mathrm{e}^{-(\mathbf{u}+\mathbf{v}_\mathrm{E})^2/\sigma^2} \\
&= \mathrm{d}u\, u^2\, 2\pi \int_{-1}^{+1} \mathrm{d}\cos\theta\, \mathrm{e}^{-(u^2+v_\mathrm{E}^2-2uv_\mathrm{E}\cos\theta)/\sigma^2} \\
&= \mathrm{d}u\, u^2\, 2\pi\, \frac{\sigma^2}{2uv_\mathrm{E}} \left[\mathrm{e}^{-(u-v_\mathrm{E})^2/\sigma^2} - \mathrm{e}^{-(u+v_\mathrm{E})^2/\sigma^2}\right]
\end{aligned}$$

In normalized form it is

$$g(u) = \frac{u}{\sqrt{\pi}\, v_\mathrm{E}\sigma} \left[\mathrm{e}^{-(u-v_\mathrm{E})^2/\sigma^2} - \mathrm{e}^{-(u+v_\mathrm{E})^2/\sigma^2}\right]$$

Note that for $v_\mathrm{E} \to 0$ this agrees with the original isotropic distribution as can be shown by expanding in powers of v_E.

(ii) Modified detection rate and recoil energy

Assuming the cross section is velocity independent, the detection rate is proportional to $\langle u \rangle$. With $v_{\mathrm{E}} = 0$ the average laboratory velocity is $\langle u \rangle_0 = 2\sigma/\sqrt{\pi}$. With $v_{\mathrm{E}} = \sigma$ it is

$$\langle u \rangle_\sigma = \left(\frac{1}{\mathrm{e}\sqrt{\pi}} + \frac{3\mathrm{erf}(1)}{2} \right) \sigma = \frac{2 + 3\mathrm{e}\sqrt{\pi}\,\mathrm{erf}(1)}{4\mathrm{e}} \langle u \rangle_0 = 1.304 \langle u \rangle_0$$

As discussed in the previous exercise, the average recoil energy per collision we have for $v_{\mathrm{E}} = \sigma$

$$\langle \Delta E \rangle \propto \frac{\int_0^\infty \mathrm{d}u\, u^3 g(u)}{\int_0^\infty \mathrm{d}u\, u\, g(u)} = \frac{\sigma^2}{6} \left(19 + \frac{4}{2 + 3\mathrm{e}\sqrt{\pi}\mathrm{erf}(1)} \right)$$

to be compared with $2\sigma^2$ for $v_{\mathrm{E}} = 0$. So it is a factor 1.61 larger.

(iii) Annual Modulation

Now let us set $v_{\mathrm{E}} = \sigma + w$ with w some small velocity. To lowest order in w one finds for the average velocity (proportional to the average detection rate)

$$\langle u \rangle_{\sigma+w} = \left(\frac{1}{\mathrm{e}\sqrt{\pi}} + \frac{3\mathrm{erf}(1)}{2} \right) \sigma + \left(\frac{1}{\mathrm{e}\sqrt{\pi}} + \frac{\mathrm{erf}(1)}{2} \right) w + \mathcal{O}(w^2)$$

The fractional modification is

$$\frac{\langle u \rangle_{\sigma+w}}{\langle u \rangle_\sigma} = 0.427\, \frac{w}{\sigma}$$

If $w = \pm 15$ km s^{-1} and $\sigma = 220$ km s^{-1} the fractional variation of the detection rate is $\pm 1.2\%$ relative to the average.