Solutions for Assignment of Week 07
Introduction to Astroparticle Physics

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1 Maximum weak interaction cross section and transparency of the universe to high-energy neutrinos

Consider a very high-energy neutrino emitted by a source of cosmic rays. Is the universe transparent to neutrinos of any energy or is there a cutoff as we have found for high-energy photons? (i) To solve this problem, first recall that for $s \ll m_{Z,W}^2$ the cross section varies roughly as $\sigma \sim \alpha^2 s/m_{Z,W}^4$, for very high energies with $s \gg m_{Z,W}^2$ as $\sigma \sim \alpha^2/s$ where s is the square of the CM energy. So roughly what is the largest possible cross section? (ii) Considering that the density of cosmic background neutrinos is of order 300 cm^{-3}, what is the minimum of the mean free path? How does this compare with the Hubble distance?

Solution

(i) Maximum cross section

The cross section scales as s for small energies and as $1/s$ for large energies, so the maximum is roughly at the turnover $s \sim m_{Z,W}^2 \sim (100\ \text{GeV})^2$. The maximum cross section therefore is roughly

$$\sigma_{\text{max}} \sim \frac{\alpha^2}{m_{W,Z}^2} \sim \frac{\left(10^{-2}\right)^2}{(100\ \text{GeV})^2} = 4 \times 10^{-36}\ \text{cm}^2$$

(ii) Minimum mean free path

Taking the density of cosmic background neutrinos to be $n_\nu \sim 300\ \text{cm}^{-3}$, the minimum mean free path against neutrino-neutrino scattering is roughly

$$\lambda_{\text{min}} \sim (\sigma_{\text{max}} n_\nu)^{-1} \sim 1 \times 10^{33}\ \text{cm}$$

Compare this with the Hubble distance $H_0^{-1} \sim 10^{28}$ cm. The neutrino mfp is always much larger than the largest possible distances in our visible universe, so in principle we can see high-energy neutrinos from everywhere.

2 Is the Earth opaque for high-energy neutrinos?

High-energy neutrino telescopes, such as IceCube at the Southpole or ANTARES in the Mediterranean search for high-energy cosmic-ray neutrinos. In particular, they look for those coming through the Earth (upward going neutrinos) because from the other direction there is a huge signal from ordinary cosmic rays coming down from the atmosphere. What are the highest energy neutrinos that one can see through the Earth? (i) To solve this problem, first estimate the scattering cross section of a high-energy neutrino on a nucleon: The interaction is with the constituent quarks which have energies corresponding to their momentum uncertainty (they are localized in the nucleon). (ii) Then estimate the mean free path in the Earth and compare with its diameter of 12700 km.

Solution

(i) Cross section estimate

For approximately massless particles (the quarks in the nucleon, the high-E neutrino), the interaction cross section is roughly

$$\sigma \sim G_{\rm F}^2\, s$$

where $s \sim 2E_q E_\nu$. Quarks are localized in the nucleons with a diameter of roughly 2 fm. By Heisenberg's uncertainty relation they have a momentum of roughly

$$p_q \sim \frac{1}{2\ \text{fm}} = 0.5 \times 10^{13}\ \text{cm}^{-1} = 0.1\ \text{GeV}$$

With the Fermi constant $G_{\rm F} = 1.166 \times 10^{-5}\ \text{GeV}^{-2}$ one finds

$$\sigma \sim \left(1.166 \times 10^{-5}\ \text{GeV}^{-2}\right)^2 \times 2 \times 0.1\ \text{GeV} \times E_\nu = 1 \times 10^{-38}\ \text{cm}^2\, \frac{E_\nu}{\text{GeV}}$$

This result corresponds almost exactly to what is measured (see figure next page).

(ii) Mean-free path in the Earth

A typical density of the Earth's material is 5.5 g cm^{-3}. Dividing by the atomic mass unit of 1.661×10^{-24} g leads to a nucleon density of 3.3×10^{24} cm^{-3}. Therefore, the mean free path is approximately

$$\lambda = (\sigma n)^{-1} \sim \left(1 \times 10^{-38}\ \text{cm}^2\, \frac{E_\nu}{\text{GeV}} \times 3.3 \times 10^{24}\ \text{cm}^{-3}\right)^{-1} = 3 \times 10^8\ \text{km} \times \frac{\text{GeV}}{E_\nu}$$

This distance becomes smaller than the Earth's diameter for $E_\nu \gtrsim 30$ TeV.

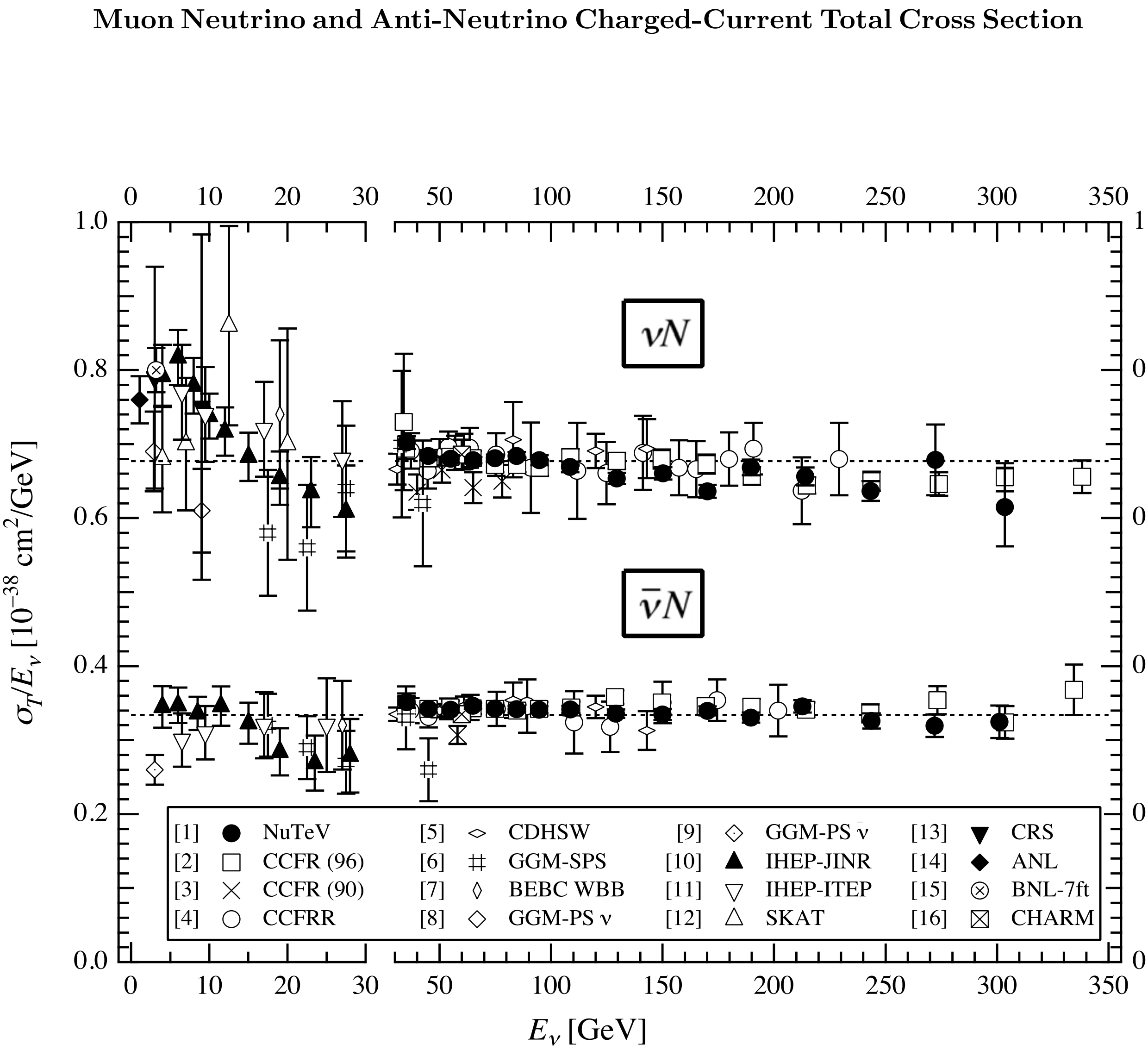

Figure 40.9: σ_T/E_ν for the muon neutrino and anti-neutrino charged-current total cross section as a function of neutrino energy. The error bars include both statistical and systematic errors. The straight lines are the isoscalar-corrected total cross-section values averaged over 30-200 *GeV* as measured by the experiments in Refs. [3–5]: $\sigma^{\nu\, Iso}/E_\nu = (0.677 \pm 0.014) \times 10^{-38}\text{cm}^2/\text{GeV}$; $\sigma^{\bar{\nu}\, Iso}/E_{\bar{\nu}} = (0.334 \pm 0.008) \times 10^{-38}\text{cm}^2/\text{GeV}$. The average ratio of the anti-neutrino to neutrino cross section in the energy range 30-200 GeV is $\sigma^{\bar{\nu}\, Iso}/\sigma^{\nu\, Iso} = 0.504 \pm 0.003$ as measured by Refs. [1–5]. Note the change in the energy scale at 30 GeV. (Courtesy W. Seligman and M.H. Shaevitz, Columbia University, 2007)

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3 Freeze-out of Dirac neutrinos

If neutrinos are Dirac particles, each family has four states, similar to electrons. However, the r.h. Dirac states have no gauge interactions and are in this sense sterile. They only couple with the mass to other particles. Their interaction rate, relative to active neutrinos, is suppressed by an approximate factor $(m/E)^2$. Which approximate value of the Dirac mass would be necessary to equilibrate these states after the QCD phase transition which happens at $T_{\rm QCD} \sim 170$ MeV?

Solution

The weak interaction rate of active neutrinos is of order

$$\Gamma_{\rm weak} \sim G_{\rm F}^2\, T^5$$

For r.h. Dirac neutrinos include a factor $(m_{\rm D}/T)^2$ so that

$$\Gamma_{\rm D} \sim G_{\rm F}^2\, m_{\rm D}^2 T^3$$

The cosmic expansion rate in the radiation dominated phase is roughly

$$H \sim T^2/m_{\rm Pl}$$

and the requirement $\Gamma > H$ at the QCD transition translates into

$$m_{\rm D} \gtrsim \left(G_{\rm F}^2 T_{\rm QCD} m_{\rm Pl}\right)^{-1/2} \sim 60 \text{ keV}$$

Since neutrino masses are today known to be in the sub-eV range, the r.h. Dirac states, if they exist, would not have been in equilibrium after the QCD transition. Therefore, a possible earlier population would have been diluted and plays no role today.

4 Matter-radiation equality including neutrinos

(i) Determine the redshift of matter-radiation equality, assuming neutrinos are nearly massless. What is the photon temperature at that time and what the neutrino temperature? (ii) How large should neutrino masses be to act as "matter" at this epoch?

Solution

(i) Matter-radiation equality with massless neutrinos

The energy density in photons plus neutrinos today is according to the discussion in the lectures

$$\rho_{\rm rad} = \left[1 + \frac{21}{8}\left(\frac{4}{11}\right)^{4/3}\right]\rho_\gamma \qquad \text{where} \qquad \rho_\gamma = \frac{\pi^2}{15}\, T_\gamma^4 = 0.261 \text{ eV cm}^{-3}$$

where $T_\gamma = 2.725$ K was used. Dividing with the critical density of $\rho_{\rm crit} = 5.80$ keV cm^{-3} we find

$$\Omega_{\rm rad} = \frac{\rho_{\nu\bar\nu} + \rho_\gamma}{\rho_{\rm crit}} = 7.56 \times 10^{-5}$$

whereas $\Omega_{\rm M} = 0.27$. Since matter scales with $(z+1)^3$ and radiation with $(z+1)^4$, matter and radiation are equal for

$$z_{\rm eq} + 1 = \frac{\Omega_{\rm M}}{\Omega_{\rm rad}} = 3571$$

The photon temperature at that epoch is

$$T_{\rm eq} = (z_{\rm eq}+1)\, T_\gamma^{\rm today} = 3571 \times 2.725~{\rm K} = 9731~{\rm K} = 0.839~{\rm eV}$$

The neutrino temperature is a factor $(3/11)^{1/3}$ smaller and thus

$$T_{\nu,\,{\rm eq}} = 6310~{\rm K} = 0.544~{\rm eV}$$

(ii) When do neutrinos act as matter at that epoch?

To understand how the energy density of massive neutrinos differs from that of massless ones note that at the relevant epoch they have long since decoupled. Therefore, the cosmic expansion simply redshifts their distribution. With $y = m_\nu/T$, the relative energy density of massive neutrinos over massless ones is, therefore,

$$g(y) = \int_0^\infty {\rm d}x\, \frac{x^2\sqrt{y^2+x^2}}{{\rm e}^x+1} \Bigg/ \int_0^\infty {\rm d}x\, \frac{x^3}{{\rm e}^x+1} = \frac{120}{7\pi^4} \int_0^\infty {\rm d}x\, \frac{x^2\sqrt{y^2+x^2}}{{\rm e}^x+1}$$

which is 1 for $y \to 0$. Numerically the integral is found to be $g(1) = 1.06$ and $g(2) = 1.22$. So for m_ν/T up to 2–3 it is probably justified to treat neutrinos as radiation, i.e. for masses up to about 1 eV.

While experimental limits imply that $m_\nu < 2.2$ eV, limits from precision cosmology imply $m_\nu \lesssim 0.2$ eV and so neutrinos should be treated as radiation at the epoch of matter-radiation equality.