Solutions for Assignment of Week 06
Introduction to Astroparticle Physics

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1 Glauber states

A natural basis for possible states of the radiation field are the number states $|n\rangle_{\mathbf{k},\boldsymbol{\epsilon}}$ for each mode $\mathbf{k}$ and polarization $\boldsymbol{\epsilon}$. Let us consider a single mode and drop the indices $\mathbf{k}$ and $\boldsymbol{\epsilon}$. Macroscopic fields, such as a radio wave or a laser beam, are highly occupied and have nonvanishing expectation values of the electric or magnetic field. They are represented by Glauber states, a coherent superposition of number states. With a complex number α, a Glauber state is

$$|\alpha\rangle = \mathrm{e}^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle = \mathrm{e}^{-|\alpha|^2/2}\, \mathrm{e}^{\alpha a^\dagger} |0\rangle$$

(i) Show that $|\alpha\rangle$ is normalized: $\langle\alpha|\alpha\rangle = 1$. (ii) Show that $|\alpha\rangle$ is an eigenstate of the destruction operator with eigenvalue α. (iii) What is the average occupation number $\langle\alpha|a^\dagger a|\alpha\rangle$. (iv) What is the root mean square (rms) variation of the occupation number? (v) Can you imagine why a laser beam comes out as a Glauber state and not as a number eigenstate?

Solution

(i) Normalization

From the definition of the Glauber state one finds immediately

$$\langle\alpha|\alpha\rangle = \left(\mathrm{e}^{-|\alpha|^2/2}\right)^2 \sum_{n,m=0}^{\infty} \frac{(\alpha^*)^m}{\sqrt{m!}} \frac{\alpha^n}{\sqrt{n!}} \underbrace{\langle m|n\rangle}_{\delta_{mn}} = \mathrm{e}^{-|\alpha|^2} \underbrace{\sum_{n=0}^{\infty} \frac{(\alpha^*\alpha)^n}{n!}}_{\mathrm{e}^{|\alpha|^2}} = 1$$

(ii) Effect of destruction operator

Again use the definition of the Glauber state to calculate the effect of the destruction operator

$$a|\alpha\rangle = \mathrm{e}^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \underbrace{a|n\rangle}_{\sqrt{n}\,|n-1\rangle} = \alpha\,\mathrm{e}^{-|\alpha|^2/2} \sum_{n=1}^{\infty} \frac{\alpha^{n-1}}{\sqrt{(n-1)!}}\,|n-1\rangle$$

The term $n = 0$ in the sum has disappeared because $a|0\rangle = 0$. Now substitute $n - 1 \to m$ and recover the expression for $|\alpha\rangle$. Therefore

$$a|\alpha\rangle = \alpha|\alpha\rangle$$

(iii) Average occupation number

Sandwich number operator between states and use result from (ii)

$$\langle n\rangle = \langle\alpha|a^\dagger a|\alpha\rangle = \langle\alpha|\alpha^*\alpha|\alpha\rangle = \alpha^*\alpha\langle\alpha|\alpha\rangle = |\alpha|^2$$

(iv) rms fluctuation of occupation number

First determine the expectation value of the squared occupation number

$$\langle n^2\rangle = \langle\alpha|a^\dagger a\, a^\dagger a|\alpha\rangle = \langle\alpha|\alpha^* \underbrace{a\, a^\dagger}_{a^\dagger a + 1 \text{ from commutation relation}} \alpha|\alpha\rangle = |\alpha|^2 \left(|\alpha|^2 + 1\right) = |\alpha|^4 + |\alpha|^2$$

Then variance

$$\langle n^2\rangle - \langle n\rangle^2 = |\alpha|^4 + |\alpha|^2 - (|\alpha|^2)^2 = |\alpha|^2 = \langle n\rangle$$

And therefore rms fluctuation

$$\sqrt{\langle n^2\rangle - \langle n\rangle^2} = \sqrt{\langle n\rangle}$$

Therefore, if $\langle n\rangle \gg 1$ the relative fluctuations become ever smaller and $|\alpha\rangle$ becomes ever closer to a number eigenstate with $n = |\alpha|^2$.

(v) Laser beam

The interaction Hamiltonian between photons and electrons or atoms is linear in A and thus linear in the creation and annihilation operator. The Glauber states thus are essentially eigenstates of the interaction operator.

2 Electron-positron annihilation

Consider the process $e^+e^- \to 2\gamma$. An exact QED calculation yields the cross section

$$\sigma_{\rm ann} = \frac{\pi\alpha^2}{m_e^2}\,\frac{1-v^2}{2v}\left[\frac{3-v^4}{2v}\ln\left(\frac{1+v}{1-v}\right)-2+v^2\right]$$

where v is the velocity of the e^- or e^+ in the CM frame. (i) Express v in terms of the CM squared energy s. (ii) Derive the cross section in the non-relativistic limit ($v \ll 1$) and ultrarelativistic limit ($s \gg m_e^2$). (iii) The nonrelativistic cross section is found to diverge with v^{-1}. Show that in an electron-positron plasma the annihilation rate remains finite. (Consider the inverse mean free path of a positron in a gas of electrons with density n_e.)

Solution

(i) Velocity in terms of CM energy

In the CM frame e^- and e^+ by definition have opposite equal momenta and their energy-momentum four vectors are $(E, \mathbf{p})$ and $(E, -\mathbf{p})$. Therefore, the sum of the two four vectors is $(2E, \mathbf{0})$. Squaring it yields $s = 4E^2$. With $E = m_e/\sqrt{1-v^2}$ one finds easily

$$v = \sqrt{1-\frac{4m_e^2}{s}}$$

(ii) Limiting expressions

In the nonrelativistic limit one finds that the expression in square brackets expands as

$$[\ldots] = 1 + 2v^2 + \mathcal{O}(v^4)$$

and so the cross section is to lowest order

$$\sigma_{\rm ann} = \frac{\pi\alpha^2}{m_e^2}\,\frac{1}{2v}$$

For high-energy limit insert the expression for v in the expression for cross section and expand with m_e^2 the small expansion parameter. To zeroth order (except for the logarithmic term) one finds by straightforward calculation

$$\sigma_{\rm ann} = \frac{2\pi\alpha^2}{s}\left[\ln\left(\frac{s}{m_e^2}\right)-1\right]$$

This is the usual result in any such case: The scale of the cross section is given by $\pi\alpha^2/s$ times numerical factors, including logarithmic terms.

(iii) Nonrelativistic annihilation rate

The relation between the interaction rate (or inverse mean free path) for one particle (here e.g. the positron) against annihilation in an electron gas of number density n_e is

$$\Gamma_{\rm ann} = n_e\left\langle\sigma_{\rm ann} v_{\rm rel}\right\rangle$$

where the average is taken over all relative velocities $v_{\rm rel}$ in the gas. In the nonrelativistic limit the relative velocity between two particles is simply $v_{\rm rel} = 2v$ with v the velocity of one of them in the CM frame. Therefore, one finds trivially

$$\Gamma_{\rm ann} = n_e \frac{\pi\alpha^2}{m_e^2} \left\langle \frac{1}{2v}\, v_{\rm rel} \right\rangle = n_e \frac{\pi\alpha^2}{m_e^2}$$

The annihilation rate here therefore does not depend on the detailed velocity distribution but only on the density of targets.

This result is quite general: If two particles with mass annihilate into radiation (massless stuff), the cross section always diverges with v^{-1}, assuming it is an s-wave process with no relative angular momentum. The annihilation rate remains finite.

3 Pair creation

Consider now the reverse process $2\gamma \to e^+e^-$ where the cross section is found to be

$$\sigma_{\rm pair} = 2v^2\,\sigma_{\rm ann}$$

where v is the velocity of the final-state e^- or e^+ in the CM frame. Of course, the initial-state photons must have at least the threshold energy $\omega \geq m_e$ to be able to produce a pair. (i) Derive the cross section in the non-relativistic limit ($v \ll 1$) and ultrarelativistic limit ($s \gg m_e^2$). (ii) Consider a high-energy photon ($E \gg m_e$) propagating in a bath of low-energy ones ($\omega \ll m_e$). Which energy E is required to be able to produce an electron-positron pair? (iii) Assuming the low-energy photon is a typical photon from the cosmic microwave background ($T = 2.725$ K), what is E? (iv) How far will photons with energies around this value travel in the universe? (v) The MAGIC high-energy gamma-ray telescope has recently observed the quasar 3C 279 at a redshift of $z = 0.536$ in energies 100–300 GeV.[1] Which distance does this correspond to? The energy range corresponds to which typical background photon range, assuming we are around the maximum of the pair-creation cross section? Which approximate upper limit on the exta-galactic background light (EBL) can be inferred?

Solution

(i) Limiting expressions

In the nonrelativistic limit simply multiply the pair annihilation cross section with $2v^2$

$$\sigma_{\rm pair} = \frac{\pi\alpha^2}{m_e^2}\,v$$

It vanishes at threshold as expected.[2]

In the ultra-relativistic case $v = 1$ and therefore the cross section is simply a factor 2 times the pair-annihilation cross section. Therefore to zeroth order

$$\sigma_{\rm pair} = \frac{4\pi\alpha^2}{s}\left[\ln\left(\frac{s}{m_e^2}\right) - 1\right]$$

(ii) Pair production threshold

The energy-momentum four vector of the high-energy photon is $P = (E, \mathbf{p})$, of the low-energy one $K = (\omega, \mathbf{k})$. The CM energy squared is

$$s = (P + K)^2 = P^2 + K^2 + 2PK = 0 + 0 + 2(E\omega - \mathbf{p}\cdot\mathbf{k})$$

If the photon collision is head-on, the two momenta are in opposite directions so that

$$s = 2(E\omega + pk) = 4E\omega$$

The pair-production threshold requires that in the CM system we have at least the energy $2m_e$ available, i.e. $s > (2m_e)^2$, implying the minimal required energy

$$E > m_e^2/\omega$$

[1] J. Albert et al. (MAGIC Collaboration), "Very-high-energy gamma rays from a distant quasar: How transparent is the universe?", Science 320, 1752 (2008) and http://arXiv.org/abs/0807.2822

[2] The corresponding Eq. (6.36) in the accompanying textbook by Bergström and Goobar misses the factor v.

(iii) Cosmic microwave background as target

With a temperature of $T = 2.725$ K a typical energy is $3T \sim 0.7$ meV. The treshold energy is therefore

$$E \gtrsim m_e^2/\omega \sim 400 \text{ TeV}$$

where 1 TeV $= 10^{12}$ eV. Of course, in detail one has to worry about the energy distribution of the background photons.

(iv) Mean free path of high-energy photons in the universe

The density of CMB photons is 411 cm^{-3} as discussed in the lectures. The maximum pair production threshold, slightly above threshold, is

$$\sigma_{\text{pair}}^{\text{max}} \sim \frac{\pi\alpha^2}{m_e^2} \sim 2.5 \times 10^{-25} \text{ cm}^2$$

Therefore the mean free path

$$\lambda_{\text{mfp}} \sim (n_\gamma \sigma)^{-1} \sim 1 \times 10^{22} \text{ cm} \sim 3 \text{ kpc}$$

Therefore, one could not even look as far as the galactic center (distance 8.5 kpc) with photon energies in the few hundred TeV range.

For lower-energy photons, the dominant absorption target is the cosmic infra-red background and at yet lower energies the density of extra-galactic background light (EBL).

(v) Limit on extra-galactic background light from transparency in 100 GeV range

From Hubble's law in linear approximation for cosmic expansion

$$z = H_0 D$$

we find

$$D = z/H_0 = 0.536 \times (3 \times 10^5 \text{ km s}^{-1})/74 \text{ km s}^{-1} \text{ Mpc}^{-1} = 2.2 \text{ Gpc}$$

Comparing with the mfp of 3 kpc above for a density of 411 photons/cm^3 one infers an upper limit of the photon density of

$$n_\gamma \lesssim 411 \text{ cm}^{-3} \times 3 \text{ kpc}/2 \text{ Gpc} \sim 0.6 \times 10^{-3} \text{ cm}^{-3}$$

The energy of 300 GeV is roughly 1000 times smaller than the 400 TeV above, so the typical background photon energies are 1000 times larger and thus around 0.7 eV (roughly visible light). In other words, the density of visible photons must be roughly six orders of magnitude below the density of CMB photons.

4 Transparency of the universe at early times

Returning to the early universe, at which redshift would it have become transparent to thermal photons, assuming all electrons are free? In reality it becomes transparent earlier at redshift $z \sim 1100$. Can you imagine what happened at that redshift?

Solution

Photons decouple fairly late when electrons are nonrelativistic and all positrons have disappeared. The Thomson scattering cross section is

$$\sigma = \frac{8\pi\alpha^2}{3m_e^2} = 6.65 \times 10^{-25}\ \mathrm{cm}^2$$

The electron density is

$$n_e \sim \eta_\mathrm{B}\, n_\gamma = \eta_\mathrm{B} n_\gamma^0\, (1+z)^3 = 2.88 \times 10^{-7}\ \mathrm{cm}^{-3}\ (1+z)^3$$

where η_B is the baryon/photon ratio, not exactly equal to the electron/photon ratio because some baryons are neutrons. The present-day CMB density is 411 cm^{-3}. The interaction rate is therefore

$$\lambda_\mathrm{mfp}^{-1} = \sigma n_e \sim 0.6 \times 10^{-6}\ \mathrm{Mpc}^{-1}\,(1+z)^3$$

To compare this with the Hubble expansion rate we assume that the decoupling happens in the matter-dominated epoch where roughly

$$H = H_0\,\sqrt{\Omega_\mathrm{M}}\,(1+z)^{3/2} = 1.3 \times 10^{-4}\ \mathrm{Mpc}^{-1}\,(1+z)^{3/2}$$

where we have used $\Omega_\mathrm{M} = 0.27$ and $H_0 = 74\ \mathrm{km\ s}^{-1}\ \mathrm{Mpc}^{-1}$. These numbers imply

$$H \lesssim \lambda_\mathrm{mfp}^{-1} \qquad \Rightarrow \qquad z \lesssim 35$$

In reality electrons, protons and α particles combine to form hydrogen and helium. The Rayleigh scattering cross section is much smaller than the Thomson scattering cross section because atoms are electrically neutral. Therefore, the universe becomes transparent at the epoch of recombination that happens roughly at $z = 1100$.

The universe today is ionized again, i.e., most of the intergalactic gas that has not condensed to form stars or galaxies is ionized. The re-ionization happened by the UV emission from the first stars. With some assumptions the redshift of re-ionization is around[3] $z = 11$, late enough that the universe would not become opaque again.

[3] E. Komatsu et al. (WMAP Collaboration), "Five-year Wilkinson Microwave Anisotropy Probe (WMAP) observations: Cosmological interpretation," Astrophys. J. Suppl. **180**, 330 (2009). See also http://arXiv.org/abs/0803.0547