Solutions for Assignment of Week 05
Introduction to Astroparticle Physics

Georg G. Raffelt
Max-Planck-Institut für Physik (Werner-Heisenberg-Institut)
Föhringer Ring 6, 80805 München
Email: raffelt(at)mppmu.mpg.de

`http://wwwth.mppmu.mpg.de/members/raffelt` → `Teaching`

Assignment of 24 November 2009

1 Entropy conservation in the expanding universe

Show that in a comoving volume of the universe the total entropy is conserved,

$$\frac{\mathrm{d}}{\mathrm{d}t}(a^3 s) = 0 \qquad \text{where} \qquad s = \frac{\rho + p}{T}\,.$$

Use the results

$$\frac{\mathrm{d}p}{\mathrm{d}T} = \frac{\rho + p}{T} \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(a^3\rho\right) = -p\,\frac{\mathrm{d}}{\mathrm{d}t}\left(a^3\right)$$

where the former follows from thermodynamic reasoning and the latter was discussed in the lectures for deriving the second Friedmann Eqn.

Solution

From $\mathrm{d}(p\,a^3)/\mathrm{d}t = a^3\mathrm{d}p/\mathrm{d}t + p\,\mathrm{d}(a^3)/\mathrm{d}t$ we infer $a^3\mathrm{d}p/\mathrm{d}t = \mathrm{d}(p\,a^3)/\mathrm{d}t - p\,\mathrm{d}(a^3)/\mathrm{d}t$. Inserting this on the r.h.s. of the second eqn yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(a^3\rho\right) = -\frac{\mathrm{d}}{\mathrm{d}t}\left(a^3 p\right) + a^3\frac{\mathrm{d}p}{\mathrm{d}t} \qquad \Rightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left[a^3(\rho+p)\right] = a^3\,\frac{\mathrm{d}p}{\mathrm{d}t}$$

Now consider the time derivative of the entropy

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{a^3(\rho+p)}{T}\right) = \frac{1}{T}\,\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\left[a^3(\rho+p)\right]}_{a^3\,\frac{\mathrm{d}p}{\mathrm{d}t}} - \frac{a^3}{T}\,\underbrace{\frac{\rho+p}{T}}_{\frac{\mathrm{d}p}{\mathrm{d}T}}\,\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{a^3}{T}\Bigg(\frac{\mathrm{d}p}{\mathrm{d}t} - \underbrace{\frac{\mathrm{d}p}{\mathrm{d}T}\,\frac{\mathrm{d}T}{\mathrm{d}t}}_{\frac{\mathrm{d}p}{\mathrm{d}t}}\Bigg) = 0$$

which thus is conserved.

2 Chemical potential

(i) Consider a relativistic species (e.g. electrons/positrons) in the early universe and assume a small asymmetry between particles and antiparticles. Express the asymmetry parameter

$$\eta = \frac{n_{e^-} - n_{e^+}}{n_\gamma}$$

in terms of the relativistic chemical potential μ in the approximation $\mu \ll T$.

(ii) Consider the opposite limit of a degenerate Fermi gas in the $T \to 0$ limit, e.g. the neutrons in a neutron star. Express the number density in terms of the Fermi momentum.

(iii) How large is the Fermi momentum for nuclear density, $\rho_{\rm nuc} = 3 \times 10^{14}$ g cm^{-3}, assuming there are only neutrons or assuming there are equal densities of protons and neutrons ("symmetric nuclear matter")? How large is the nonrelativistic Fermi energy in symmetric nuclear matter? How large is the typical distance between nucleons if we use the length L of a cube that is on average available to one nucleon?

(iv) Compare the typical distance between nucleons with a typical distance over which pions as exchange particles provide a significant interaction potential.

Solution

(i) Small asymmetry

The net electron density is given by the phase-space integral

$$\begin{aligned}
n_{e^-} - n_{e^+} &= 2 \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \left[\frac{1}{\mathrm{e}^{(E-\mu)/T} + 1} - \frac{1}{\mathrm{e}^{(E+\mu)/T} + 1} \right] \\
&= \frac{T^3}{\pi^2} \int_0^\infty \mathrm{d}x\, x^2 \left[\frac{1}{\mathrm{e}^{x-\xi} + 1} - \frac{1}{\mathrm{e}^{x+\xi} + 1} \right] \\
&= \frac{T^3}{\pi^2} \int_0^\infty \mathrm{d}x\, x^2 \frac{\mathrm{e}^x(\mathrm{e}^\xi - \mathrm{e}^{-\xi})}{(\mathrm{e}^{x-\xi} + 1)(\mathrm{e}^{x+\xi} + 1)}
\end{aligned}$$

where $x = E/T$ and $\xi = \mu/T$. In principle one can simply solve this integral, but the result involves functions that are not very illuminating. Expand to lowest order in $\xi \ll 1$ with

$$\mathrm{e}^{x\pm\xi} = \mathrm{e}^x\, \mathrm{e}^{\pm\xi} = \mathrm{e}^x[1 \pm \xi + \mathcal{O}(\xi^2)]$$

Therefore in integral expression

$$\begin{aligned}
&\text{Numerator} \to 2\xi \mathrm{e}^x \\
&\text{Denominator} \to (\mathrm{e}^x + 1)^2 + \xi^2 \mathrm{e}^{2x} = (\mathrm{e}^x + 1)^2 \quad \text{to lowest order} \\
&\frac{\mathrm{e}^x}{(\mathrm{e}^x + 1)^2} = \frac{1}{(\mathrm{e}^{x/2} + \mathrm{e}^{-x/2})^2}
\end{aligned}$$

Asymmetry therefore, after the substitution $y = x/2$,

$$\eta = \frac{n_{e^-} - n_{e^+}}{n_\gamma} = \left(\frac{2\zeta_3}{\pi^2} T^3\right)^{-1} \frac{16\mu T^2}{\pi^2} \underbrace{\int_0^\infty \mathrm{d}y \left(\frac{y}{\mathrm{e}^y + \mathrm{e}^{-y}}\right)^2}_{\pi^2/48}$$

$$= \frac{\pi^2}{6\zeta_3} \frac{\mu}{T} = 1.368 \frac{\mu}{T}$$

Therefore, η is roughly the same as μ/T and very small for an asymmetry of order 10^{-9}.

(ii) High degeneracy

In the integral of (i), the anti-particle contribution is completely suppressed for $T \to 0$, so only particles survive with the number density

$$n = 2 \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{1}{\mathrm{e}^{(E-\mu)/T} + 1}$$

The occupation number is 1 for energies below the Fermi surface, and zero above, so

$$n = \frac{1}{\pi^2} \int_0^{p_\mathrm{F}} \mathrm{d}p\, p^2 = \frac{p_\mathrm{F}^3}{3\pi^2}$$

(iii) Nuclear density

With $m_N = 938$ MeV one finds for the nucleon number density at nuclear density

$$n_\mathrm{nuc} = \frac{\rho_\mathrm{nuc}}{1.67 \times 10^{-24}\ \mathrm{g}} = 1.79 \times 10^{38}\ \mathrm{cm}^{-3}$$

and therefore the Fermi momentum for neutron matter

$$p_\mathrm{F} = 1.75 \times 10^{13}\ \mathrm{cm}^{-1} = 344\ \mathrm{MeV}$$

Symmetric nuclear matter: equal densities of p and n, so divide by $2^{1/3}$ and find 273 MeV.

The Fermi energy in the symmetric case is

$$\mu - m_N = \frac{p_\mathrm{F}^2}{2m_N} = \frac{(273\ \mathrm{MeV})^2}{2 \times 938\ \mathrm{MeV}} = 40\ \mathrm{MeV}$$

The typical distance between nucleons is derived from $L^3 = 1/n_\mathrm{nuc}$

$$L = (3\pi^2)^{1/3}\, p_\mathrm{F}^{-1} = 1.77 \times 10^{-13}\ \mathrm{cm} = 1.77\ \mathrm{fm} \qquad \text{where} \qquad 1\ \mathrm{fm} = 10^{-15}\ \mathrm{m}$$

Assuming nucleons are densely packed, this gives us the typical size of a nucleon.

(iv) Pion exchange

The Yukawa potential has the form $\mathrm{e}^{-m_\pi r}/r$, so take as a typical range of the potential

$$m_\pi^{-1} = \frac{1}{135\ \mathrm{MeV}} = 1.46\ \mathrm{fm}$$

Nucleons in nuclear matter are sufficiently dense that their interaction is strong.

3 Conditions at the deconfinement transition

The transition from the quark-gluon plasma to hadrons occurs in the early universe at $T_{\mathrm{QCD}} \sim 170$ MeV. (i) Taking pions to be relativistic, what is a typical distance between them immediately after the transition? (ii) What is the density of protons and neutrons, assuming a vanishing chemical potential? Is this approximation good, considering that today $\eta_B \sim 10^{-9}$?

Solution

(i) Pion density

The number density of one species of pions, e.g. π^0, is in the relativistic approximation

$$n_{\pi^0} = \frac{\zeta_3}{\pi^2} T^3 = \frac{\zeta_3}{\pi^2} (170 \text{ MeV})^3 = 0.78 \times 10^{38} \text{ cm}^{-3}$$

Taking this once more as the inverse of the volume of a cube of length L one finds

$$L = n_{\pi^0}^{-1/3} = 2.34 \text{ fm}$$

(ii) Nucleon density

In the lectures we have derived the energy density of a fermion species with mass m in the low-temparature limit $T \ll m$

$$\rho = \mathrm{e}^{-m/T} \frac{m}{2} \left(\frac{2mT}{\pi}\right)^{3/2}$$

In this limit the energy of every particle is essentially its mass, so the number density of protons plus anti-protons, for example, is

$$n_{p\bar{p}} = \mathrm{e}^{-m_N/T} \frac{1}{2} \left(\frac{2m_N T}{\pi}\right)^{3/2} = 0.85 \times 10^{37} \text{ cm}^{-3} = 0.11\, n_{\pi^0}$$

using $m_N = 938$ MeV and $T = 170$ MeV. The Boltzmann suppression is not yet very large, so $n_{p\bar{p}}$ is still roughly 10^8 times the asymmtry density. It is an excellent approximation to neglect the chemical potential.

4 Some particle properties

To answer the following simple questions, visit the homepage of the particle data group at `http://pdg.lbl.gov` and look up most of the answers. (i) What is the mass difference between protons and neutrons? (ii) Between π^0 and $\pi^\pm$? What are the dominant decay channels of neutral and charged pions, and what are the mean lifetimes for them? Can you imagine why the decay $\pi^0 \to \nu\bar{\nu}$ does not seem to occur? And why is the charged-pion decay $\pi^+ \to e^+ + \nu_e$ so much slower than the muon decay channel? Hint: Think of the left-handedness of weak interactions and of angular momentum conservation! (iii) What is the spin and mass of the Δ resonance? (iv) Baryons containing a strange quark are called hyperons. Which are the lightest hyperons and what is their quark content? Can you imagine what is the difference between the Λ and the Σ^0?

Solution

(i) Proton/neutron mass difference

The proton mass is 938.27 MeV, the neutron mass 939.57 MeV and their mass difference is

$$Q = m_n - m_p = 1.2933 \text{ MeV}$$

So there is enough energy for the decay $n \to p + e^- + \bar{\nu}_e$, but not for $n \to p + \mu^- + \bar{\nu}_\mu$.

(ii) Pions

The pion masses and mass differences are

$$m_{\pi^0} = 134.98 \text{ MeV} \qquad m_{\pi^\pm} = 139.57 \text{ MeV} \qquad m_{\pi^\pm} - m_{\pi^0} = 4.59 \text{ MeV}$$

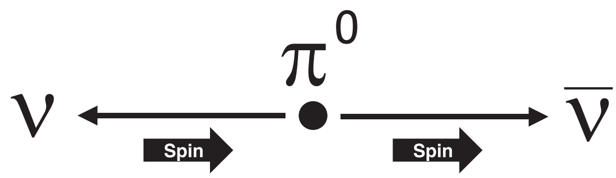
So here the charged particle is the heavier one.

The decay channels and mean lifetimes are

$$\begin{aligned} &\pi^0 \to \gamma + \gamma && \tau_{\pi^0} = (8.4 \pm 0.6) \times 10^{-17} \text{ s} \\ &\pi^+ = \mu^+ + \nu_\mu && \tau_{\pi^\pm} = 2.60 \times 10^{-8} \text{ s} \end{aligned}$$

So the charged pion lives very much longer than the neutral one.

The charged pion decay must occur via the weak interaction because neutrinos are produced. Neutrinos are left-handed, meaning that ν has negative helicity, $\bar{\nu}$ has positive helicity. If a π^0 decays at rest, the daughter neutrinos must emerge in opposite directions, so their required handedness means that their spins point in the same direction and thus add up to 1 unit of angular momentum. However, the pion has spin 0, so angular-momentum conservation prevents the two-neutrino decay.



The same reasoning applies to the weak decays of charged pions. However, the muon mass is comparable to the pion mass, so here handedness is almost meaningless because the muon is produced nonrelativistically for a pion decaying at rest. But the electron mass is much smaller, so one expects a suppression roughly by $(m_e/m_\mu)^2 \sim 0.3 \times 10^{-4}$, not too far off from the true suppression of about 10^{-4}.

(iii) Delta resonance

Lowest-lying non-strange baryon. Spin 3/2, mass 1232 MeV.

(iv) Hyperons

Lowest-lying hyperons are the Λ and Σ^0 with quark content uds and spin 1/2. Their masses are $m_\Lambda = 1115.7$ MeV and $m_\Sigma = 1192.6$ MeV. Its main decay channel is $\Sigma^0 \to \Lambda + \gamma$.

One can represent the u/d quark content by a property called "isospin." Think of u as "isospin up" and d as "isospin down." Combine two spins 1/2 to a spin 1 and a spin 0

$$\frac{\mathbf{1}}{\mathbf{2}} \otimes \frac{\mathbf{1}}{\mathbf{2}} = \mathbf{1} \oplus \mathbf{0}$$

So the Λ is the isospin-singlet, the Σ^0 a member of the triplet. The ud content of the two neutral hyperons is therefore $(ud + du)/\sqrt{2}$ and $(ud - du)/\sqrt{2}$, respectively.

5 Cosmic-ray muons

Primary cosmic rays consist of protons and heavier nuclei, hitting the atmosphere with a steeply falling energy spectrum. The interactions with nuclei of the air leads to particle showers, containing large amounts of pions. Charged pions decay according to $\pi^+ \to \mu^+ + \nu_\mu$ and $\pi^- \to \mu^- + \bar{\nu}_\mu$. At sea level the muon flux is $\sim 1\ \mathrm{cm}^{-2}\ \mathrm{min}^{-1}$. The production is at an altitude of ~ 15 km. (i) For an energy of 3 GeV, which fraction of muons reaches sea level before decaying? (ii) On the other hand, how far do $\pi^\pm$ of the same energy get after production before decaying? Can you imagine what would happen if the pion lifetime were much longer?

Solution

(i) The muon mass is 105 MeV, so for 3 GeV energy the relativistic Lorentz factor is $\gamma = E/m = 28.6$. The muon lifetime of 2.197 μs corresponds to a distance (after multiplying with the speed of light) of 0.66 km. Multiplying with the Lorentz factor provides the effective average distance before decaying of 18.8 km. So the atmosphere is well matched to the muon decay properties!

(ii) For charged pions the corresponding distance (in the laboratory frame) is about 170 m. If pions lived much longer, they would be reabsorbed by further interactions with the air: the muon flux would be much smaller.