Solutions for Assignment of Week 03
Introduction to Astroparticle Physics

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Assignment of 09 November 2009

1 Second Friedmann Equation

The Friedmann equation and continuum equation are

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi}{3}\,G_{\rm N}\rho - \frac{k}{a^2} \qquad \text{and} \qquad \dot\rho = -3(\rho+p)\,\frac{\dot a}{a}\,.$$

Differentiating the Friedmann equation with respect to time and using the continuum equation, derive the second Friedmann equation

$$\frac{\ddot a}{a} = -\frac{4\pi}{3}\,G_{\rm N}(\rho+3p)\,.$$

Solution

Differentiating the 1st Friedmann Eqn with respect to time and inserting for $\dot\rho$ the continuum eqn yields

$$2\frac{\dot a}{a}\left(\frac{\ddot a}{a} - \frac{\dot a^2}{a^2}\right) = \frac{8\pi}{3}\,G_{\rm N}\dot\rho + 2k\frac{\dot a}{a^3} = -8\pi\,G_{\rm N}(\rho+p)\,\frac{\dot a}{a} + 2\frac{k}{a^2}\,\frac{\dot a}{a}$$

Now use once more the 1st Friedmann Eqn to replace in the r.h.s.

$$\frac{k}{a^2} = \frac{8\pi}{3}\,G_{\rm N}\rho - \left(\frac{\dot a}{a}\right)^2$$

It is now straightforward to solve for $\ddot a/a$ and obtain the 2nd Friedmann Eqn.

2 Measures of distance

Assume that a cosmological model is perfectly characterized by Ω_{M} and Ω_Λ, i.e. ignore the contribution of radiation. An observer receives light from a distant source (emitter E) and measures that the spectral lines have suffered the redshift z_{E}. (i) Write down general integral expressions for the lookback time (how long ago was this signal emitted) and for the coordinate, luminosity and angle distances of the source. These quantities will depend on Ω_{M}, Ω_Λ and z_{E}. (ii) Solve them explicitly for a flat, matter-dominated universe ($\Omega_{\mathrm{M}} = 1$ and $\Omega_\Lambda = 0$). (iii) For this model, find the redshift where the angular size of an object is smallest. (iv) What is this redshift in a realistic flat model ($\Omega_{\mathrm{M}} = 0.27$ and $\Omega_\Lambda = 0.73$)? (v) For this realistic model, plot the different distance measures as a function of redshift.

Solution

(i) General integral expressions

Following what was shown in the lectures, we write the Friedmann Eqn in the form

$$\dot{y}^2 = \Omega_{\mathrm{M}}\, y^{-1} + \Omega_{\mathrm{K}} + \Omega_\Lambda\, y^2$$

where

$$\Omega_{\mathrm{K}} = 1 - \Omega_{\mathrm{M}} - \Omega_\Lambda, \qquad y = \frac{a}{a_0} = \frac{1}{1+z}\,, \qquad \tau = H_0 t, \qquad \dot{y} = \frac{\mathrm{d}y}{\mathrm{d}\tau}$$

and so time is always expressed in units of H_0^{-1}. The Friedmann Eqn is equivalent to

$$\mathrm{d}\tau = \frac{\mathrm{d}y}{\sqrt{\Omega_{\mathrm{M}}\, y^{-1} + \Omega_{\mathrm{K}} + \Omega_\Lambda\, y^2}}\,.$$

Lookback time

Simple integration provides the lookback time $\Delta t = \Delta\tau H_0$

$$\Delta\tau = \int_{\tau_{\mathrm{E}}}^{1} \mathrm{d}\tau = \int_{y_{\mathrm{E}}}^{1} \frac{\mathrm{d}y}{\sqrt{\Omega_{\mathrm{M}}\, y^{-1} + \Omega_{\mathrm{K}} + \Omega_\Lambda\, y^2}}$$

If expressed in terms of redshift $z = y^{-1} - 1$ these results are equivalent to

$$\mathrm{d}\tau = -\frac{\mathrm{d}z}{(1+z)\sqrt{(1+z)^2(1+\Omega_{\mathrm{M}}z) - z(2+z)\Omega_\Lambda}}$$

$$\Delta\tau = \int_0^{z_{\mathrm{E}}} \frac{\mathrm{d}z}{(1+z)\sqrt{(1+z)^2(1+\Omega_{\mathrm{M}}z) - z(2+z)\Omega_\Lambda}}$$

Coordinate distance

Beginning with the Robertson-Walker metric

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}\Omega^2\right)$$

we consider a light ray ($\mathrm{d}s^2 = 0$) propagating along the radial direction ($\mathrm{d}\Omega = 0$) and so

$$\frac{\mathrm{d}r}{\sqrt{1-kr^2}} = -\frac{\mathrm{d}t}{a(t)}$$

The minus sign was chosen to have a light signal coming to us at $r = 0$ from a distant source, so $\mathrm{d}r$ is negative with increasing time. Multiplying both sides with $H_0 a_0$ and noting $\tau = H_0 t$ and $y(t) = a(t)/a_0$ leads to

$$H_0 a_0 \frac{\mathrm{d}r}{\sqrt{1-kr^2}} = -\frac{\mathrm{d}\tau}{y}$$

The l.h.s. is the differential of the coordinate distance (in units of H_0^{-1}). On the r.h.s. we may insert the differential from the Friedmann Eqn $\mathrm{d}\tau = \mathrm{d}y/\sqrt{\cdots}$ and so

$$\begin{aligned} H_0 a_0 \frac{\mathrm{d}r}{\sqrt{1-kr^2}} &= -\frac{\mathrm{d}y}{y\sqrt{\Omega_\mathrm{M}\, y^{-1} + \Omega_\mathrm{K} + \Omega_\Lambda\, y^2}} \\ &= \frac{\mathrm{d}z}{\sqrt{(1+z)^2(1+\Omega_\mathrm{M} z) - z(2+z)\Omega_\Lambda}} \end{aligned}$$

In integrated form we thus find for the coordinate distance in units of H_0

$$\begin{aligned} H_0 D_\mathrm{c} = H_0 a_0 \int_0^{r_\mathrm{E}} \frac{\mathrm{d}r}{\sqrt{1-kr^2}} &= \int_{y_\mathrm{E}}^1 \frac{\mathrm{d}y}{\sqrt{\Omega_\mathrm{M}\, y + \Omega_\mathrm{K}\, y^2 + \Omega_\Lambda\, y^4}} \\ &= \int_0^{z_\mathrm{E}} \frac{\mathrm{d}z}{\sqrt{(1+z)^2(1+\Omega_\mathrm{M} z) - z(2+z)\Omega_\Lambda}} \end{aligned}$$

Particle horizon

As an additional remark let us note that the coordinate distance corresponding to infinite redshift is called the "particle horizon." It is the coordinate distance that a light signal or gravitational wave signal has travelled since the big bang, or conversely, the largest coordinate distance of objects that we can see in the sky. Objects further away can not be seen because their light signals could not yet have reached us since the big bang.

So in our Ω_M-Ω_Λ models the particle horizon (in units of H_0^{-1}) is

$$\begin{aligned} H_0 D_\mathrm{hor} &= \int_0^1 \frac{\mathrm{d}y}{\sqrt{\Omega_\mathrm{M}\, y + \Omega_\mathrm{K}\, y^2 + \Omega_\Lambda\, y^4}} \\ &= \int_0^\infty \frac{\mathrm{d}z}{\sqrt{(1+z)^2(1+\Omega_\mathrm{M} z) - z(2+z)\Omega_\Lambda}} \end{aligned}$$

Luminosity and angle distance

In the lectures we have derived the luminosity and angle distance. After multiplying with H_0 they are

$$H_0 D_{\mathrm{L}} = (1+z_{\mathrm{E}})\, H_0 a_0 r_{\mathrm{E}} = \frac{H_0 a_0 r_{\mathrm{E}}}{y_{\mathrm{E}}}$$

$$H_0 D_{\mathrm{A}} = \frac{H_0 a_0\, r_{\mathrm{E}}}{(1+z_{\mathrm{E}})} = H_0 a_0\, r_{\mathrm{E}}\, y_{\mathrm{E}}$$

To evaluate these expressions we need an explicit expression for $H_0 a_0 r_{\mathrm{E}}$. We recall from the lectures that

$$\chi_{\mathrm{E}} \equiv \int_0^{r_{\mathrm{E}}} \frac{\mathrm{d}r}{\sqrt{1-kr^2}} = \begin{cases} \mathrm{asin}(r_{\mathrm{E}}) & \text{for } k=+1 \\ r_{\mathrm{E}} & \text{for } k=0 \\ \mathrm{Asinh}(r_{\mathrm{E}}) & \text{for } k=-1 \end{cases}$$

or turning this around

$$r_{\mathrm{E}} = S_k(\chi_{\mathrm{E}}) \equiv \begin{cases} \sin(\chi_{\mathrm{E}}) & \text{for } k=+1 \\ \chi_{\mathrm{E}} & \text{for } k=0 \\ \sinh(\chi_{\mathrm{E}}) & \text{for } k=-1 \end{cases}$$

From the above integral expressions we note that $H_0 D_{\mathrm{c}} = H_0 a_0 \chi_{\mathrm{E}}$. Therefore,

$$H_0 a_0 r_{\mathrm{E}} = H_0 a_0\, S_k\left(\frac{1}{H_0 a_0} \int_{y_{\mathrm{E}}}^1 \frac{\mathrm{d}y}{\sqrt{\Omega_{\mathrm{M}}\, y + \Omega_{\mathrm{K}}\, y^2 + \Omega_\Lambda\, y^4}} \right)$$

Finally recall from the definition of the Ω parameters that

$$\Omega_{\mathrm{tot}} = \Omega_{\mathrm{M}} + \Omega_\Lambda = 1 + \frac{k}{(H_0 a_0)^2} \qquad \Rightarrow \qquad \Omega_{\mathrm{K}} = 1 - \Omega_{\mathrm{M}} - \Omega_\Lambda = -\frac{k}{(H_0 a_0)^2}$$

For $k=0$ we may choose the arbitrary scale factor such that $H_0 a_0 = 1$ and anyway, because here $S_k(\chi) = \chi$ for $k=0$, the factor $H_0 a_0$ drops out. For $k=\pm 1$ we find

$$H_0 a_0 = \frac{1}{\sqrt{|\Omega_{\mathrm{K}}|}} = \frac{1}{\sqrt{|1-\Omega_{\mathrm{M}}-\Omega_\Lambda|}}$$

Therefore, we now have explicit expressions for the different distance measures in terms of integral expressions. Of course, everything can be expressed in terms of z–integrals.

(ii) Flat, matter dominated case

We now specialize to the simple case $\Omega_{\rm M} = 1$, $\Omega_\Lambda = 0$ and therefore $\Omega_{\rm K} = 0$ and $k = 0$. The lookback time is now found by direct integration

$$\Delta\tau = \int_{y_{\rm E}}^{1} {\rm d}y\,\sqrt{y} = \frac{2}{3}\left(1 - y_{\rm E}^{3/2}\right)$$

and we may use $y_{\rm E} = 1/(1 + z_{\rm E})$. The age of the universe is found for $y_{\rm E} = 0$ and thus the familiar $t_0 = \frac{2}{3} H_0^{-1}$.

The coordinate distance is

$$H_0 D_{\rm c} = \int_{y_{\rm E}}^{1} \frac{{\rm d}y}{\sqrt{y}} = 2\left(1 - y_{\rm E}^{1/2}\right)$$

Therefore, a light signal emitted shortly after the big bang has traversed the coordinate distance $2H_0^{-1}$ or, comparing with age of the universe, the distance $3t_0$. In other words, we receive light from 3 times further away than in the Newtonian picture. When the light began travelling, all distances were smaller and so it got further than it would have in a static Newtonian space.

We note that in the flat case $H_0 D_{\rm c} = H_0 a_0 r_{\rm E}$ and therefore

$$H_0 a_0 r_{\rm E} = 2\,(1 - y_{\rm E}^{1/2})$$

The coordinate and luminosity distances are then

$$H_0 D_{\rm L} = 2\,\frac{1 - y_{\rm E}^{1/2}}{y_{\rm E}}$$

$$H_0 D_{\rm A} = 2\,y_{\rm E}(1 - y_{\rm E}^{1/2})$$

(iii) Redshift of smallest angular size in the flat, matter-dominated case

The function $H_0 D_{\rm A}$ has a maximum. Differentiate the function with respect to $y_{\rm E}$, set the result to zero, and find

$$y_{\rm E}^{\rm min} = \frac{4}{9} \qquad \text{and} \qquad z_{\rm E}^{\rm min} = \frac{1}{y_{\rm E}^{\rm min}} - 1 = \frac{5}{4}$$

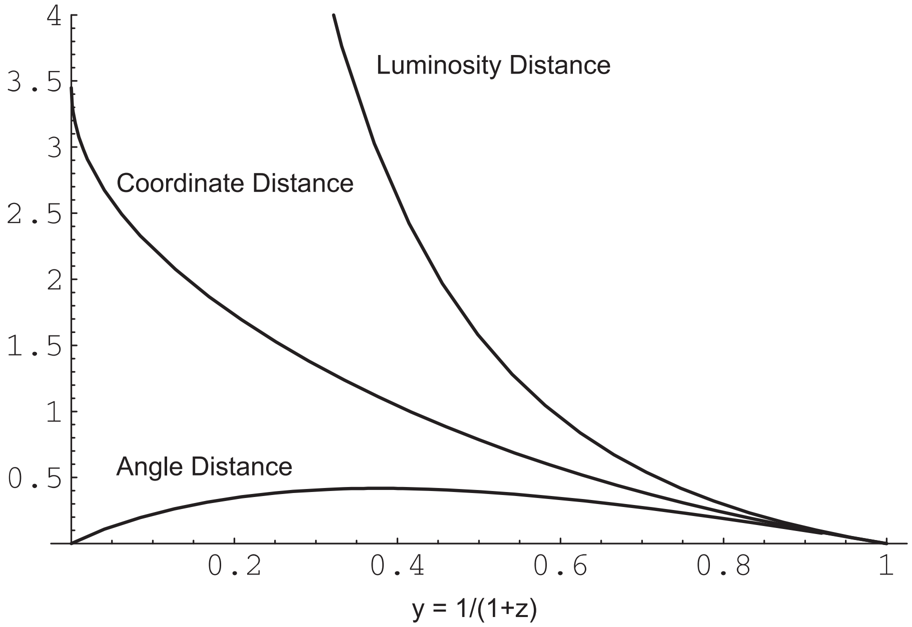
(iv and v) Realistic flat model

Now we have $\Omega_{\rm K} = 0$, $k = 0$ and $\Omega_{\rm M} = 1 - \Omega_\Lambda$. The coordinate distance is then

$$H_0 D_{\rm c} = \int_{y_{\rm E}}^{1} \frac{{\rm d}y}{\sqrt{(1 - \Omega_\Lambda)\,y + \Omega_\Lambda\,y^4}}$$

and the luminosity and angle distance are once more obtained by dividing or multiplying with $y_{\rm E}$.

The integral can not be expressed in a simple analytic form, although it can be expressed as an incomplete beta function. The coordinate, luminosity and angle distance as a function of redshift can be numerically plotted as shown here.



Numerically one finds for the redshift of smallest angular size (largest angle distance)

$$z_{\rm E}^{\rm min} = 1.64$$

which is larger than in the matter-only flat case.

While this was not asked in the assignment, consider also the particle horizon, i.e. the coordinate distance of the big bang. For the flat matter-dominated case we found $D_{\rm hor} = 2H_0^{-1}$ whereas the numerical evaluation for the realistic flat case with $\Omega_{\rm M} \sim 0.27$ yields $D_{\rm hor} \sim 3.5\, H_0^{-1}$. With the Hubble distance $H_0^{-1} = 4.0$ Gpc the particle horizon is $D_{\rm hor} = 14$ Gpc. This is "the size of our visible universe."

Actually the furthest light signal we can receive is the cosmic microwave background. At earlier times the universe was not transparent to light. The CMB decoupled at a redshift of approximately 1100, so $y_{\rm E} \sim 1/1100$. The integral for the horizon is practically the same for the lower limit of integration being zero or $1/1100$, so the coordinate distance to the surface of last scattering is practically the same as the distance to the big bang. At the time of decoupling, then, the coordinate distance to the surface of last scattering was $1/1100$ of what it is today, i.e. 14 Gpc$/1100 = 13$ Mpc. Today this is about the distance to the next galaxy cluster.

3 FLRW models models of the universe with matter and vacuum energy

Assume that a cosmological model in the post-radiation epoch is perfectly characterized by $0 \leq \Omega_\mathrm{M} < \infty$ and $-\infty < \Omega_\Lambda < +\infty$. In the plane defined by these two parameters, identify the regions where (i) the universe is flat, positively curved and negatively curved, (ii) is accelerating, coasting, or decelerating, (iii) the locus where a static Einstein universe is possible, and (iv) the universe expands forever or eventually recollapses.

Solution

Following what was shown in the lectures, we write the Friedmann Eqn in the form

$$\dot{y}^2 = \Omega_\mathrm{M}\, y^{-1} + \Omega_\mathrm{K} + \Omega_\Lambda\, y^2$$

where

$$\Omega_\mathrm{K} = 1 - \Omega_\mathrm{M} - \Omega_\Lambda, \qquad y = \frac{a}{a_0} = \frac{1}{1+z}, \qquad \tau = H_0 t, \qquad \dot{y} = \frac{\mathrm{d}y}{\mathrm{d}\tau}$$

(i) Curvature

Recalling that Ω_K is proportional to k it is clear that the universe is flat for $\Omega_\mathrm{K} = 0$ and thus for $\Omega_\mathrm{M} + \Omega_\Lambda = 1$, shown as a diagonal dashed line in the figure below. Space is negatively curved below this line and positively curved above it.

(ii) Acceleration

Differentiating the Friedmann Eqn with respect to τ provides

$$2\dot{y}\ddot{y} = -\Omega_\mathrm{M}\, y^{-2}\dot{y} + 2\Omega_\Lambda\, y\dot{y} \qquad \Rightarrow \qquad \ddot{y} = -\tfrac{1}{2}\,\Omega_\mathrm{M}\, y^{-2} + \Omega_\Lambda\, y$$

The universe is accelerating for $\ddot{y} > 0$. The present epoch is given by $y = 1$, so today the universe is accelerating if

$$\Omega_\Lambda > \tfrac{1}{2}\,\Omega_\mathrm{M}$$

This line is not shown in the figure.

(iii) Einstein universe

To identify the different types of solutions, we write the Friedmann Eqn in the form "kinetic energy + potential energy = constant"

$$\dot{y}^2 + V(y) = \Omega_K = 1 - \Omega_M - \Omega_\Lambda \qquad \text{where} \qquad V(y) = -\Omega_M\, y^{-1} - \Omega_\Lambda\, y^2$$

An Einstein solution is only possible when the system sits at the maximum or minimum of the potential. There is never a minimum (no stable static solution). A maximum exists only for $\Omega_\Lambda > 0$. The condition is $\mathrm{d}V/\mathrm{d}y = 0$ yields

$$\Omega_M\, y^{-2} - 2\Omega_\Lambda\, y = 0 \qquad \Rightarrow \qquad y_{\max} = \left(\frac{\Omega_M}{2\Omega_\Lambda}\right)^{1/3}$$

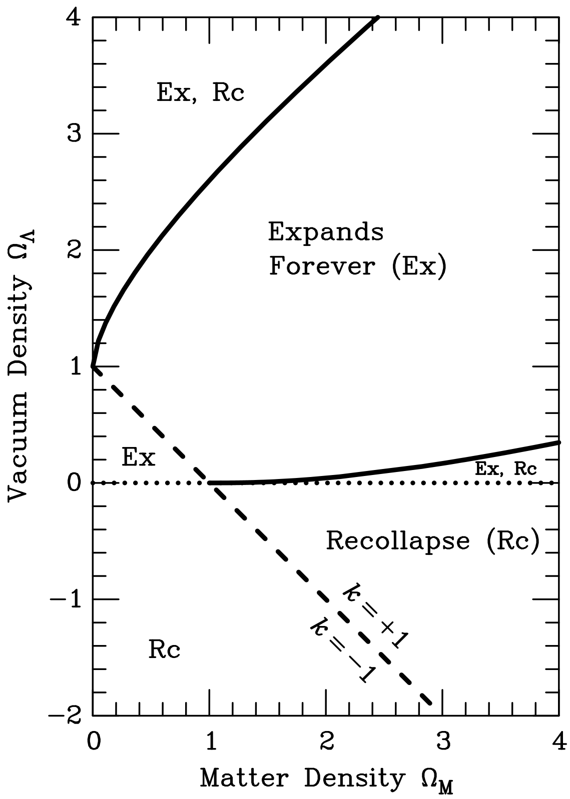
Now demanding that $\dot{y} = 0$ we insert this value into the Friedmann Eqn and find the condition

$$0 = \dot{y}^2\Big|_{y=y_{\max}} \qquad \Rightarrow \qquad \left(\frac{\Omega_M}{2\Omega_\Lambda}\right)^{2/3} = \frac{1 - \Omega_M - \Omega_\Lambda}{3\Omega_\Lambda}$$

One can now solve this equation to find the locus $\Omega_\Lambda(\Omega_M)$ that fulfills this condition. One way to express the solution is in parametric form, using $\mu \equiv \Omega_M/2\Omega_\Lambda$

$$\Omega_\Lambda = \frac{1}{1 + 2\mu - 3\mu^{2/3}} \qquad \text{and} \qquad \Omega_M = \frac{2\mu}{1 + 2\mu - 3\mu^{2/3}}$$

The denominator diverges for $\mu \to 1$, so there are two branches for $0 \le \mu < 1$ and one for $1 < \mu \le \infty$, beginning at the points $(\Omega_M, \Omega_\Lambda) = (0, 1)$ and $(1, 0)$ respectively (solid lines).



(iv) Forever expanding or re-collapsing

For negative vacuum energy the potential increases for large y, so the system will reach a largest y–value, then return, i.e. the universe reverses to a contracting phase and ends in a big crunch.

For vanishing vacuum energy we have three possibilities: $\Omega_{\mathrm{M}} < 1$ (expands forever, negative curvature), $\Omega_{\mathrm{M}} = 1$ (expands forever, flat), and $\Omega_{\mathrm{M}} > 1$ (re-collapses, positive curvature).

For positive vacuum energy $\Omega_{\Lambda} > 0$ there are three generic cases, depending on whether or not Ω_{K} is below, at or above the Einstein value. First we note that now $V(y) < 0$, so for $\Omega_{\mathrm{K}} < 0$ we have $\dot{y}^2 > 0$ for any y. In other words, for negative curvature the universe expands forever, irrespective of the other parameters.

Between the "Einstein lines" in the figure, Ω_{K} is above the maximum of the potential and we have a single, forever expanding solution. Above and below these lines, there are two solutions. The expanding solution in these regions would not have had a singlurity in the past (no big bang).