Solutions for Assignment of Week 01
Introduction to Astroparticle Physics

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Assignment of 27 October 2009

## 1 Cosmic expansion without center

According to Hubble's law all galaxies show on average a recession velocity that increases linearly with distance: $v_{\rm rec} = H_0 D$. Use the linearity of this relation to show that an observer on any other galaxy will likewise see himself in the apparent center of the expansion. The Hubble flow looks the same to all observers.

### Solution

According to Hubble's law an observer $A$ measures the recession velocity of a galaxy $G$, located at $\mathbf{r}_G$ with coordinate origin at $A$, as $\mathbf{v}_G = H_0\mathbf{r}_G$. Another observer $B$ on a different galaxy recedes from $A$ with $\mathbf{v}_B = H_0\mathbf{r}_B$. Subtracting the two equations leads to $\mathbf{v}_G - \mathbf{v}_B = H_0(\mathbf{r}_G - \mathbf{r}_B)$. In our nonrelativistic scenario the recession velocity of $G$ from $B$ is $\mathbf{v}'_G = \mathbf{v}_G - \mathbf{v}_B$ and its location relative to $B$ is $\mathbf{r}'_G = \mathbf{r}_G - \mathbf{r}_B$. Therefore, the subtraction of the two original equations implies $\mathbf{v}'_G = H_0\mathbf{r}'_G$, i.e. once more the Hubble law, but this time with $B$ at the coordinate origin. The linearity of Hubble's law implies that any observer on any galaxy has the impression of being at the center of an isotropic recession of all other galaxies.

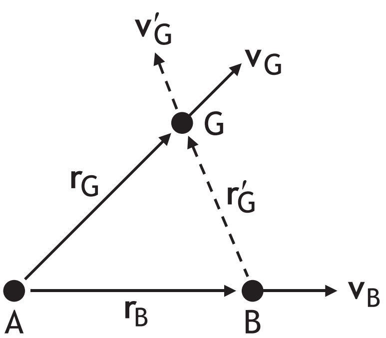


Figure 1: Observers $A$ and $B$ measure the recession velocity of a galaxy $G$.

## 2 Doppler effect (kinematical redshift)

Consider light of wavelength $\lambda$ emitted by a source that moves relative to an observer. Which wavelength or frequency is measured by the observer? Consider in particular the limiting cases of (i) parallel motion (the source moves along the line of sight) and (ii) transverse motion. Interpretation of the results?

### Solution

Consider a light wave with frequency $\omega$ and wave vector $\mathbf{k}$ where $|\mathbf{k}| = 2\pi/\lambda$. The photon's four momentum is therefore $K = (\omega, \mathbf{k})$. We use natural units with $c = \hbar = 1$ and do not distinguish between energy $E$ and frequency $\omega$ and not between momentum $\mathbf{p}$ and wave vector $\mathbf{k}$.

Let an observer (absorber) $A$ have $\mathbf{v}_A$ in the same coordinate system and hence the four velocity $V_A = (1, \mathbf{v}_A)/\sqrt{1 - v_A^2}$. The quantity $\omega_A = V_{A\mu} K^\mu$ is a relativistic scalar and thus the same in all frames. What is its interpretation? In the rest frame of $A$ we have $\mathbf{v}_A = \mathbf{0}$, so in that frame $\omega_A$ is simply the photon frequency. In other words, the quantity

$$\omega_A = \omega \, \frac{1 - \mathbf{v}_A \cdot \hat{\mathbf{k}}}{\sqrt{1 - v_A^2}} \tag{1}$$

is the frequency of the light wave that $A$ measures in his rest frame. We have used $\hat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}| = \mathbf{k}/\omega$ as a unit vector in the direction of photon propagation.

Next consider a source (emitter) $E$ and observer (absorber) $A$ with velocities $\mathbf{v}_E$ und $\mathbf{v}_A$, respectively. The ratio of frequencies measured by these observers is

$$\frac{\omega_A}{\omega_E} = \sqrt{\frac{1 - v_E^2}{1 - v_A^2}} \, \frac{1 - \mathbf{v}_A \cdot \hat{\mathbf{k}}}{1 - \mathbf{v}_E \cdot \hat{\mathbf{k}}} \, . \tag{2}$$

In the rest frame of $A$ we have $\mathbf{v}_A = \mathbf{0}$ and the velocity of the emitter is its velocity $\mathbf{v}$ relative to the absorber. We conclude that

$$\frac{\omega_A}{\omega_E} = \frac{\sqrt{1 - v^2}}{1 - \mathbf{v} \cdot \hat{\mathbf{k}}} \, , \tag{3}$$

where $v = |\mathbf{v}|$.

For a source receding from the observer, $\hat{\mathbf{k}}$ and $\mathbf{v}$ have opposite signs, so $\mathbf{v} \cdot \hat{\mathbf{k}} = -v$ and

$$\frac{\omega_A}{\omega_E} = \frac{\sqrt{1 - v^2}}{1 + v} = \sqrt{\frac{1 - v}{1 + v}} = 1 - v + \mathcal{O}(v^2) \, . \tag{4}$$

Therefore, the observed frequency is smaller than the one emitted by the source. The usual redshift variable is

$$z = \frac{\lambda_A}{\lambda_E} - 1 = \frac{\omega_E}{\omega_A} - 1 = \sqrt{\frac{1 + v}{1 - v}} - 1 = v + \mathcal{O}(v^2) \, . \tag{5}$$

If the source moves toward the observer, $\mathbf{v}$ and $\mathbf{k}$ are colinear, amounting to $v \to -v$ and thus to a blueshift (negative $z$).

Let the source move in a transverse direction to the observer, implying $\mathbf{v} \cdot \hat{\mathbf{k}} = 0$ and

$$\frac{\omega_A}{\omega_E} = \sqrt{1 - v^2} = 1 - \frac{v^2}{2} + \mathcal{O}(v^4)\,. \tag{6}$$

For small speeds this "transverse Doppler effect" is much smaller than the longitudinal one because the small velocity appears quadratically to lowest order. The interpretation is that of a relativistic time dilation: "Moving clocks run slow."

# 3 Redshift by gravitation

(i) A photon moves along a gravitational field (acceleration $g$, approximately homogeneous). After overcoming a height difference $H$, what is the photons's redshift? [Hint: Use the equivalence between a homogeneous gravitational field and an accelerated system of reference ("Einstein elevator"). In the freely falling frame the absorber acquires a velocity during the time that passes between emission and absorption of the light wave.] Express the result as a difference between the gravitational potentials at the emission and absorption points.

(ii) How large is therefore the redshift of a spectral line emitted from a star of radius $R$ and mass $M$, observed at a large distance? The solar mass is $M_\odot = 2 \times 10^{30}$ kg and its radius $R_\odot = 6.96 \times 10^5$ km. How large is the redshift here? For a neutron star, typical values are $M_{\rm NS} = 1.4\, M_\odot$ und $R_{\rm NS} = 12$ km. Redshift here in the Newtonian approximation?

### Solution

(i) A photon is emitted at E at time $t = 0$ and absorbed at A at time $t = H$ (natural units with $c = 1$). We analyze the experiment from the perspective of a freely falling observer who is at rest at time $t = 0$. While the photon moves upward, the elevator accelerates downward and at the time of absorption has reached the speed $v = gH$. From this observer's perspective there is no gravitational field, but he sees $A$ accelerating, reaching $v = gH$ at the moment of

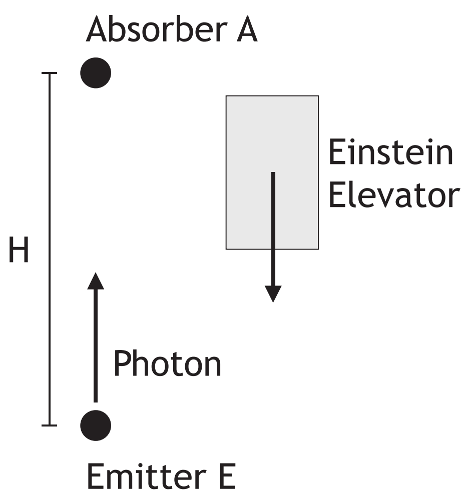


Figure 2: Redshift experiment in a gravitational field, analyzed by an observer in a freely falling "Einstein elevator."

absorption. Hence $A$ will measure the light wave with a kinematical Doppler effect of

$$z = gH\,. \tag{7}$$

All of this applies in linear approximation for a small height difference, so we may as well interpret the result in a differential sense as $\mathrm{d}z/\mathrm{d}R = g$. Noting that $g = -\mathrm{d}\Phi/\mathrm{d}R$ with $\Phi$ the gravitational potential we find

$$z = \Phi_A - \Phi_E\,. \tag{8}$$

Obviously one finds a corresponding blue shift if the photon runs from a higher to a lower gravitational potential. These results apply for weak gravitational fields, i.e. in the Newtonian limit.

(ii) In the Newtonian limit the gravitational potential outside of a spherical mass distribution $M$ is

$$\Phi = -\frac{G_\mathrm{N} M}{R} = -2.14\times 10^{-6}\,\frac{M}{M_\odot}\,\frac{R_\odot}{R}\,, \tag{9}$$

if we normalize the potential to zero at infinity. The numerical version of this result if found, for example, if we use natural units and recall that $G_\mathrm{N} = 1/m_\mathrm{Pl}^2$ with $m_\mathrm{Pl} = 1.22\times 10^{19}$ GeV. The mass 1 g corresponds to $0.561\times 10^{24}$ GeV and 1 $\mathrm{cm}^{-1}$ to $1.973\times 10^{-14}$ GeV. The redshift $z = -\Phi$ on the solar surface for a distant observer is therefore $2.14\times 10^{-6}$ and thus quite small, but has been precisely measured. (Of course, for a precise measurement by an Earth-based experiment one needs to take into account the gravitational potential on Earth.) Multiplying $z$ with the speed of light yields a Doppler effect corresponding to 640 m $\mathrm{s}^{-1}$. For a typical neutron star the mass is 1.4 times larger, the radius smaller by a factor of $1.72\times 10^{-5}$, implying $\Phi = -0.17$. The corresponding Doppler effect of $0.17\,c$ is small enough that for a crude estimate the Newtonian approximation is still o.k., but for an exact result one would have to use a fully relativistic treatment.

## 4 Friction by Hubble expansion

Consider a body (e.g. a galaxy) moving with a nonrelativistic speed $v$ relative to the Hubble flow ("peculiar velocity"). It will slow down relative to the Hubble flow by cosmic expansion. (i) How large is the deceleration as a function of $v$ and of the Hubble parameter $H_0$? (ii) Compare the result for the Earth on its orbit with the gravitational acceleration caused by the Sun? ($H_0 = 74$ km $\mathrm{s}^{-1}$ $\mathrm{Mpc}^{-1}$, $M_\odot = 2\times 10^{30}$ kg, average Earth-Sun distance 150 million km). (iii) How do these numbers compare in a hydrogen atom relative to the Coulomb acceleration, assuming a typical distance and velocity of the electron?

### Solution

(i) Cosmic redshift implies that the inverse momentum of a body evolves like the cosmic scale factor, $p^{-1}\propto a$, or equivalently $\dot p/p = -\dot a/a = -H$. Since for a nonrelativistic body we have $p = mv$ we conclude

$$\dot v = -H\,v\,. \tag{10}$$

(ii) The circumference of the Earth orbit around the Sun is $2\pi D$ where $D_\oplus = 150$ million km is the mean Sun–Earth distance. The Earth goes around the Sun once per year and a year is approximately $\pi\times 10^7$ s, so the Earth's orbital velocity is $v_\oplus = 30$ km $\mathrm{s}^{-1}$. The Hubble deceleration in the absence of other gravitational fields would be $g_{\rm H} = |\dot{v}_\oplus| = H_0 v_\oplus$. With $H_0 = 1/t_{\rm H}$ and $t_{\rm H} = 4.2 \times 10^{17}$ s we find $g_{\rm H} = 7 \times 10^{-14}$ m s$^{-2}$. The Sun's gravitational acceleration at the Earth orbit is $g_\odot = G_{\rm N} M_\odot / D_\oplus^2 = 6 \times 10^{-3}$ m s$^{-2}$ and thus much larger than $g_{\rm H}$. Evidently the Sun's gravitational field is far more important than cosmic expansion. The solar system does not expand under the influence of cosmic expansion.

(iii) To estimate the properties of the hydrogen atom, we note that the Coulomb potential at a distance $r$ is, in natural rationalized units, $e^2/4\pi r = \alpha/r$ where $\alpha = e^2/4\pi \approx 1/137$ is the fine structure constant. The virial theorem informs us that in a Coulomb potential the average potential energy should be twice the negative average kinetic energy, $\Phi = -2T$ with $T = p^2/2m_e$. Heisenberg's uncertainty relation reads $\Delta x\, \Delta p > 1$ (in natural units with $\hbar = 1$). Taking the momentum uncertainty to be equal to the momentum and the location uncertainty equal to the distance $r$ from the nucleus, the virial condition $\langle \alpha/r \rangle = \langle p^2 \rangle / m$ translates with $p \sim 1/r$ into the estimate $r \sim 1/(\alpha m_e)$, which actually is the exact expression for the Bohr radius (in natural units). Inserting this back into $mv = p = 1/r$ yields $v \sim \alpha$ for an estimate of a typical velocity, i.e. an electron in a hydrogen atom moves typically with 1/137 of the speed of light. Comparing the Coulomb acceleration at the Bohr radius of $g_{\rm B} \sim \alpha/r^2$ with the Hubble acceleration $g_{\rm H} \sim \alpha H_0$ yields for the ratio $g_{\rm B}/g_{\rm H} \sim \alpha\,(m_e/H_0)$. Recalling that $H_0$ in natural units is identical with the Hubble energy of about $1.55 \times 10^{-33}$ eV and the electron mass is $m_e = 0.511$ MeV one finds $g_{\rm B}/g_{\rm H} \sim 2 \times 10^{36}$.