9.8 Photon–axion oscillations (1 Feb 2010)

9.8.1 Linearizing the equation of motion

The above wave equation for the combined photon-axion system can be written in the form

$$(\omega^2 + \partial_z^2 - \mathsf{M}^2)\mathcal{A} = 0$$

where

$$\mathsf{M}^2 = \begin{pmatrix} m_{\gamma\perp}^2 & 0 & 0 \\ 0 & m_{\gamma,\parallel}^2 & G_{a\gamma\gamma}B\omega \\ 0 & G_{a\gamma\gamma}B\omega & m_a^2 \end{pmatrix} \qquad \text{and} \qquad \mathcal{A} = \begin{pmatrix} A_\perp \\ A_\parallel \\ a \end{pmatrix}$$

We have absorbed a relative phase between $A_\parallel$ and a in the definition of the axion field, so the M^2 matrix is now real.

In the ultra-relativistic limit when $|n-1| \ll 1$, always relevant in this context, we may linearize the wave equation.

To this end we observe that for a plane wave, and if the matrix were diagonal, $\partial_z \to \pm \mathrm{i}k$ and thus

$$\omega^2 + \partial_z^2 = \omega^2 - (\mathrm{i}\partial_z)^2 = (\omega + \mathrm{i}\partial_z)(\omega - \mathrm{i}\partial_z) = (\omega + k)(\omega - k) = \omega^2 - k^2 = m^2$$

where m would be the mass of the given diagonal component.

The trick to a linearized wave equation is to separate a common fast time variation of the plane waves from the slow difference of the different components, assuming the ultra-relativistic limit where for every component $|m^2| \ll \omega^2$. In this case $k \sim \omega$ and

$$\omega^2 + \partial_z^2 = (\omega + k)(\omega + \mathrm{i}\partial_z) \to 2\omega(\omega + \mathrm{i}\partial_z)$$

So we may linearize the Klein-Gordon Eqn in the form

$$\left(\omega + \mathrm{i}\partial_z - \frac{\mathsf{M}^2}{2\omega}\right)\mathcal{A} = 0 \qquad \text{or} \qquad \mathrm{i}\partial_z\mathcal{A} = \left(-\omega + \frac{\mathsf{M}^2}{2\omega}\right)\mathcal{A}$$

Ignoring the mass term (refractive term), the equation is

$$\partial_z\mathcal{A} = \mathrm{i}\omega\,\mathcal{A}$$

with the solution

$$\mathcal{A}(z) = \mathrm{e}^{\mathrm{i}\omega\, z}\,\mathcal{A}(0)$$

The sign in the exponential gives us the direction in which the wave is running (left-moving or right-moving). The sign chosen in the linearization picks out the direction of motion.

The $e^{i\omega z}$ term is just a common phase to all components of $\mathcal{A}$ and is irrelevant for probabilities (squared amplitudes), so for our purpose of seeking transitions between different components of $\mathcal{A}$ we may always drop a common phase.

Altogether the physically relevant equation of motion is

$$i\partial_z \mathcal{A} = \mathsf{H}\,\mathcal{A} \qquad \text{where} \qquad \mathsf{H} = \frac{\mathsf{M}^2}{2\omega}$$

This is the same for all relativistic mixing phenomena, notably neutrino flavor oscillations where M is the matrix of neutrino masses.

This equation looks like an ordinary Schrödinger equation with the Hamiltonian operator H, except that the "evolution" is along the z coordinate, not a time evolution.

The equation is solved by

$$\mathcal{A}(z) = \mathsf{U}(z)\,\mathcal{A}(0) \qquad \text{where} \qquad \mathsf{U}(z) = e^{-i\,Hz} = \exp\left(-i\,\frac{\mathsf{M}^2}{2\omega}\,z\right)$$

In general, the matrix M^2 depends on the spatial coordinate. This is the case when the medium (giving photons a refractive index) and/or the magnetic field are not homogeneous. So the general equation is

$$i\partial_z \mathcal{A}(z) = \mathsf{H}(z)\,\mathcal{A}(z)$$

with the boundary condition that $\mathcal{A}(0)$ is known. A formal solution is in this case

$$\mathsf{U}(z) = \mathcal{S}\,\exp\left[-i\int_0^z dz'\,H(z')\right]$$

Since the matrix $\mathsf{H}(z)$ does not necessarily commute with itself for different z, this is to be interpreted as a space-ordered expression, indicated by $\mathcal{S}$. (For an ordinary Schrödinger equation one would have a space-ordered exponential.)

One way of defining the space-ordered exponential is by the limiting expression

$$\mathcal{S}\,\exp\left[-i\int_0^z dz'\,H(z')\right] = \lim_{n\to\infty}[1-i\mathsf{H}(z_n)dz]\cdot[1-i\mathsf{H}(z_{n-1})dz]\cdots[1-i\mathsf{H}(z_2)dz]\cdot[1-i\mathsf{H}(z_1)dz]$$

where

$$z_j = j\,dz \qquad \text{and} \qquad dz = \frac{z}{n}$$

or somewhat circularly by the solution of the original differential equation. Depending on the space variation of $\mathsf{H}(z)$ there can be quite unexpected solutions.

9.8.2 Explicit solution for two-flavor case

Let us focus on the 2-component system consisting of axions and photons with $\parallel$ polarization to the external B field

$$\mathcal{A} = \begin{pmatrix} A \\ a \end{pmatrix} \qquad \text{where} \qquad A \equiv A_{\parallel}$$

and the Hamiltonian is a 2×2 matrix

$$\mathsf{H} = \frac{\mathsf{M}^2}{2\omega} = \frac{1}{2\omega}\begin{pmatrix} m_\gamma^2 & G_{a\gamma\gamma}B\omega \\ G_{a\gamma\gamma}B\omega & m_a^2 \end{pmatrix} = \frac{m_\gamma^2 + m_a^2}{4\omega} + \begin{pmatrix} q/2 & \Delta \\ \Delta & -q/2 \end{pmatrix}$$

where

$$q = \frac{m_\gamma^2 - m_a^2}{2\omega} \qquad \text{photon-axion momentum transfer}$$
$$\Delta = \frac{G_{a\gamma\gamma}B}{2} \qquad \text{mixing energy}$$

After ignoring a piece proportional to the unit matrix, leading once more to a common phase, one has

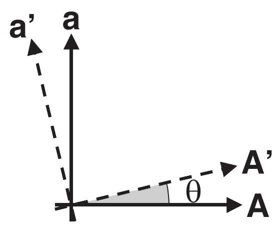
$$\mathsf{H} = \begin{pmatrix} q/2 & \Delta \\ \Delta & -q/2 \end{pmatrix}$$

Assume that all quantities are constant (independent of z, e.g. homogeneous medium and B field).

The equation of motion implies that the fields or particles A and a are not **propagation eigenstates**, they are the **interaction eigenstates**. For example, an electron radiates a pure photon (coupled by the usual gauge interactions), but the propagation eigenstate is a superposition of A and a.

The mass matrix can be diagonalized by a unitary transformation in the form of a rotation

$$\begin{pmatrix} A' \\ a' \end{pmatrix} = \mathsf{R}_\theta \begin{pmatrix} A \\ a \end{pmatrix} \qquad \text{where} \qquad \mathsf{R}_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$



Note that the inverse matrix is

$$\mathsf{R}_\theta^{-1} = \mathsf{R}_\theta^\dagger = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \mathsf{R}_{-\theta}$$

The mixing angle θ is chosen such that the "mass matrix" becomes diagonal

$$\begin{pmatrix} m_{\gamma'}^2 & 0 \\ 0 & m_{a'}^2 \end{pmatrix} = \mathsf{U}\,\mathsf{M}^2\mathsf{U}^\dagger = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} m_\gamma^2 & 2\omega\Delta \\ 2\omega\Delta & m_a^2 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

So if we begin with a pure axion (e.g. produced in the Sun) propagating in a B field (e.g. the dipole magnet of a helioscope), then proceed as follows:

- Rotate initial state into the propagation basis
- Accrue separate phases for the two components over the propagation distance z
- Rotate back into the interaction basis to find amplitude of produced photon wave

Therefore, the amplitudes at a location z are immediately found to be

$$\begin{pmatrix} A_z \\ a_z \end{pmatrix} = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\phi_{\gamma'}} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\phi_{a'}} \end{pmatrix} \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} A_0 \\ a_0 \end{pmatrix}$$

where $s = \sin\theta$, $c = \cos\theta$, and

$$\phi_{\gamma'} = \frac{m_{\gamma'}^2}{2\omega}\, z \qquad \text{and} \qquad \phi_{a'} = \frac{m_{a'}^2}{2\omega}\, z$$

Assuming we begin with a pure axion beam so that

$$\begin{pmatrix} A_0 \\ a_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

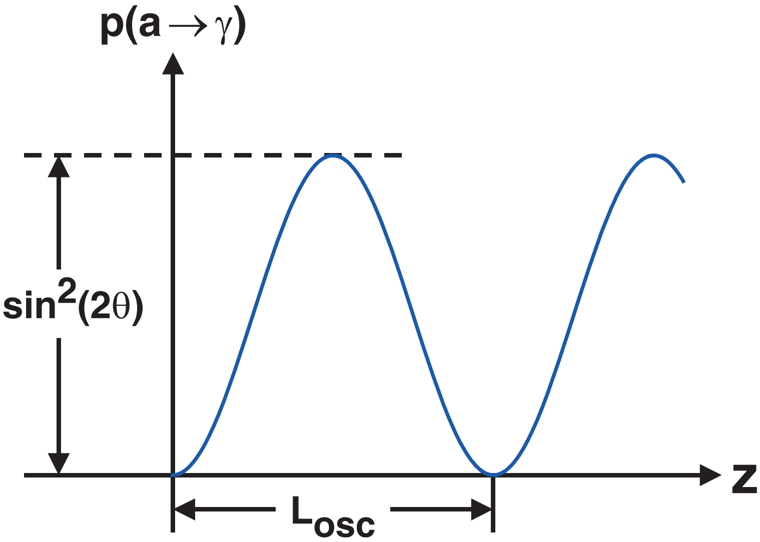
the axion to photon transition probability is after simple evaluation and transformations

$$p(a \to \gamma) = |A_z|^2 = c^2 s^2 \left| \mathrm{e}^{-\mathrm{i}\phi_{\gamma'}} - \mathrm{e}^{-\mathrm{i}\phi_{a'}} \right|^2 = \sin^2(2\theta)\, \sin^2(\pi z / L_{\mathrm{osc}})$$

where the oscillation length is

$$L_{\mathrm{osc}} = \frac{4\pi\omega}{m_{\gamma'}^2 - m_{a'}^2}$$

Therefore, in a transverse B field, axions will convert to photons and back periodically. Completely analogous to rotation of polarization of light in a birefringent medium.



Finally determine mixing parameters for axion–photon system. First of all, the mixing angle is determined by the requirement that the off-diagonal element of the following matrix vanishes

$$\begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} q/2 & \Delta \\ \Delta & -q/2 \end{pmatrix} \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$$

In other words

$$\frac{2\Delta}{q} = \frac{2cs}{c^2 - s^2} = \tan(2\theta) \qquad \text{or} \qquad \sin^2(2\theta) = \frac{\Delta^2}{\Delta^2 + (q/2)^2}$$

The propagation eigenvalues of the quantities $m^2/2\omega$ are best determined from the eigenvalue equation of the propagation matrix

$$\begin{pmatrix} q/2 & \Delta \\ \Delta & -q/2 \end{pmatrix} \begin{pmatrix} A \\ a \end{pmatrix} = \lambda \begin{pmatrix} A \\ a \end{pmatrix}$$

which has a nontrivial solution for

$$\det \begin{pmatrix} q/2 - \lambda & \Delta \\ \Delta & -q/2 - \lambda \end{pmatrix} = 0 \qquad \Rightarrow \qquad \lambda^2 = (q/2)^2 + \Delta^2$$

Therefore

$$\frac{m_{\gamma'}^2 - m_{a'}^2}{2\omega} = \lambda_+ - \lambda_- = 2\sqrt{(q/2)^2 + \Delta^2}$$

and oscillation length therefore

$$L_{\rm osc} = \frac{\pi}{\sqrt{(q/2)^2 + \Delta^2}}$$

The conversion probability is therefore found to be

$$p(a \to \gamma) = \sin^2(2\theta)\,\sin^2(\pi z/L_{\rm osc}) = \frac{\Delta^2}{(q/2)^2 + \Delta^2}\,\sin^2\left(\sqrt{(q/2)^2 + \Delta^2}\,z\right)$$

If the travelling distance L is much shorter than the oscillation length, the sine function can be expanded and the conversion probability is

$$p(a \to \gamma) = (\Delta L)^2 = \left(\frac{G_{a\gamma\gamma} B L}{2}\right)^2$$

independent of the axion and photon masses.

In the opposite limit we consider the average over a broad range of energies so that the oscillatory term averages to 1/2. Moreover, assume the mixing energy Δ is very small and assume vacuum so that the photon mass vanishes,

$$\langle p(a \to \gamma)\rangle = 2\left(\frac{G_{a\gamma\gamma} B\omega}{m_a^2}\right)^2$$

9.9 CAST solar axion search

Axions or axion-like particles with a two-photon vertex are produced in the Sun by the Primakoff process and can be measured in the laboratory with a long and strong dipole magnet.

Numerical integration over a standard solar model yields an axion luminosity of

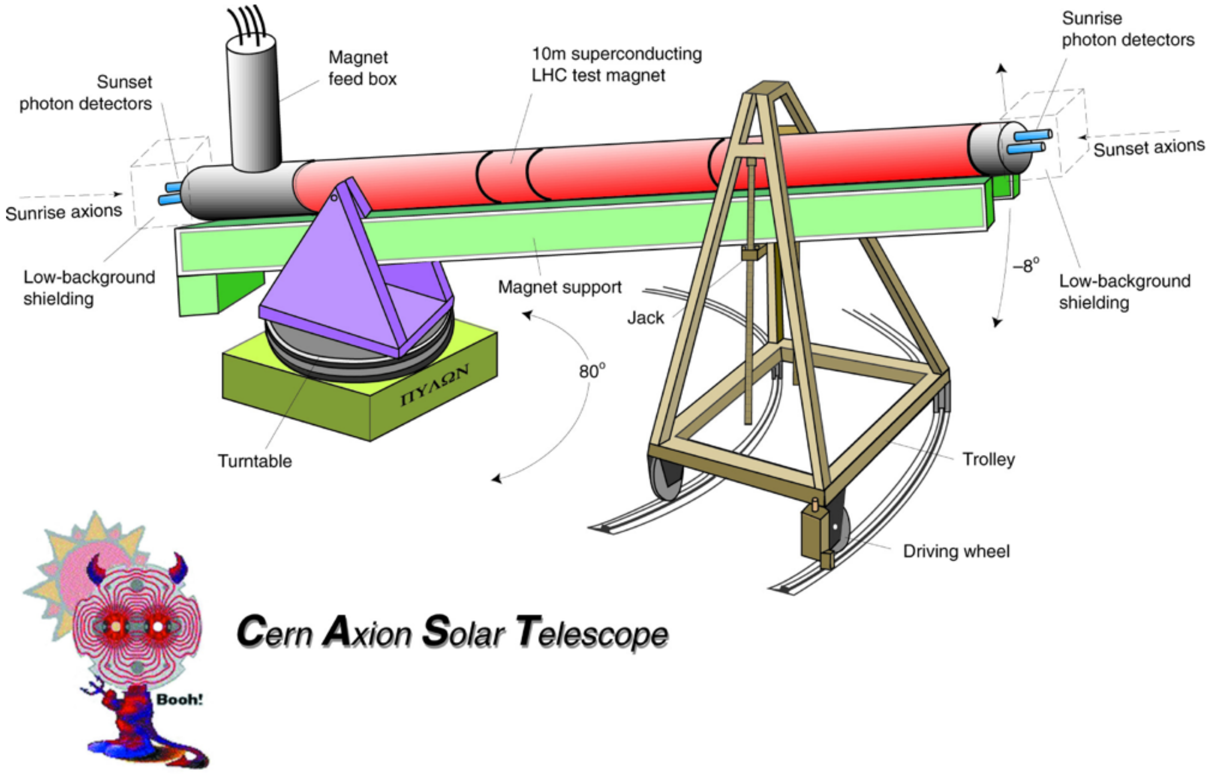
$$L_a = G_{10}^2 \; 1.85 \times 10^{-3} \, L_\odot \qquad \text{where} \qquad G_{10} = \frac{G_{a\gamma\gamma}}{10^{-10}\ \mathrm{GeV}^{-1}}$$

A simple estimate was performed in a homework problem.

The solar interior temperature is roughly 1 keV. The expected axion spectrum peaks at 3 keV and has an average energy of 4.2 keV. The number flux at Earth would be

$$F_a = G_{10}^2 \; 3.75 \times 10^{11}\ \mathrm{cm}^{-2}\ \mathrm{s}^{-1}$$

Useful dipole magnets from accelerators. The largest helioscope experiment is the CERN Axion Solar Telescope (CAST), using a prototype LHC magnet mounted to follow the Sun.



The approximate magnet properties are $L = 9.26$ m, $B = 9.0$ T, aperture of one pipe $A = 14.5\ \mathrm{cm}^2$.

The conversion probability within the magnet is therefore, assuming a very small axion mass such that the oscillation length exceeds the magnet length,

$$p(a \to \gamma) = \left(\frac{G_{a\gamma\gamma}BL}{2}\right)^2 = G_{10}^2\, 1.70 \times 10^{-17}$$

Therefore, at the far end of the CAST magnet one expects the x-ray flux

$$F_\gamma = p(a \to \gamma) F_a = G_{10}^4\ 6.4 \times 10^{-6}\ \mathrm{cm}^{-2}\ \mathrm{s}^{-1}$$

Since the cross-sectional area of one pipe is 14 cm^2 and one day has 86400 s, so one expects from one pipe

$$\mathrm{Rate}_\gamma = 7.7\ G_{10}^4\ \text{events day}^{-1}$$

The true sensitivity depends on the intrinsic detector background, not just having enough signal events.

The CAST experiment did not detect any solar axions, but in the end found a limit

$$G_{10} \lesssim 0.88 \quad (95\%\ \mathrm{CL}) \qquad \text{for} \quad m_a \lesssim 0.02\ \mathrm{eV}$$

For larger masses, the conversion efficiency is suppressed according to the discussion above: The oscillation length becomes shorter than the field region.

Way out: provide the photons with a mass themselves, i.e. fill the magnet pipes with a gas at variable pressure. At a given pressure, the electron density provides a plasma mass $\omega_{\rm plas}$ and makes photons and axions degenerate. The momentum transfer q vanishes exactly on resonance: Maximal conversion rate.

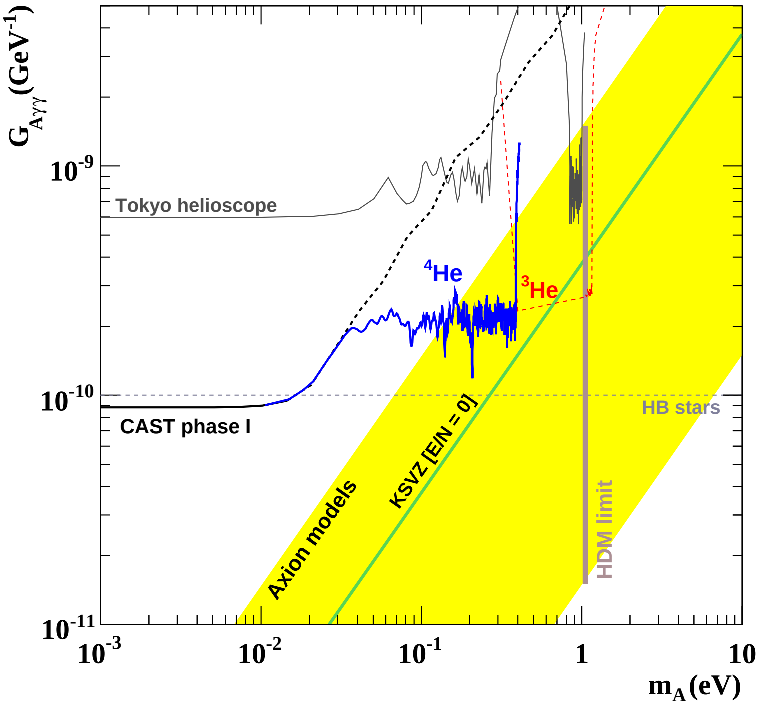
Disadvantage: Have to step through many pressure steps, but one can reach the mass-interaction-strength condition relevant for true axions, not just axion-like particles.

As filling gas one can only take hydrogen or helium, otherwise binding energy too large and the electrons are not "quasi free" to act like a plasma.

At the liquid helium temperature of the CAST magnet, filling with helium one can only go up to a pressure corresponding roughly to $m_a = 0.4$ eV, corresponding to the vapor pressure of helium at roughly 4 K.

This phase was completed and again limits found that are shown in the plot below, taken from Arik et al. (CAST Collaboration), "Probing eV-scale axions with CAST", arXiv:0810.4482.

To connect to the hot dark matter limit of about 1 eV one can fill the pipe with 3-He that has a higher vapor pressure. This final phase of CAST is currently taking place.



9.9.1 Photon regeneration experiments

One can perform similar experiments purely in the laboratory by sending a laser beam through a magnet, blocking it half way. In the first part, axions would be produced and converted back to photons on the other side ("shining light through a wall").

However sensitivity very much smaller than for CAST. Still, several such experiments have been performed—just in case the Sun does not emit axions after all.

9.9.2 Conversion in astrophysical magnetic field

The universe is filled with magnetic fields. For example, typical field strength in the galaxy of order μG, in intergalactic space limits are below nG.

But large distances available. Therefore BL could become large.

In principle, photon-axion-photon conversion can be important for transparency of the universe at high energies. Some speculations that such effects may have been observed.

However, always requires very low-mass particles, not honest QCD axions.