6.1.5 Quantized electron field (30 Nov. 2009)

For fermions (electrons) one postulates the existence of a field

$$\Psi(\mathbf{x}, t)$$

For particles with mass m the field will obey the Klein-Gordon Eqn

$$(\Box + m^2)\, \Psi(\mathbf{x}, t) = 0$$

The Fourier modes are assumed to be quantized in an analogous way to bosons, except that the commutation relations become anti-commutation relations:

$$\begin{aligned}
&\{a, a^\dagger\} = a\, a^\dagger + a^\dagger\, a = 1 \\
&\{a, a\} = \{a^\dagger, a^\dagger\} = 0
\end{aligned}$$

From the second line follows, in particular,

$$a^2 = (a^\dagger)^2 = 0$$

For the occupation number states therefore

$$\begin{aligned}
&a^\dagger|0\rangle = |1\rangle \\
&a^\dagger|1\rangle = (a^\dagger)^2|0\rangle = |0\rangle
\end{aligned}$$

Only occupation numbers 0 and 1 possible (Pauli exclusion principle).

For "Dirac particles" we have antiparticles (here positrons), obeying the same relations. Usually denoted by b and $b^\dagger$.

Four possible "polarization states" unified to a single "Dirac field" with four components:

$$\Psi(\mathbf{x}, t) = \begin{pmatrix} \psi_1(\mathbf{x}, t) \\ \psi_2(\mathbf{x}, t) \\ \psi_3(\mathbf{x}, t) \\ \psi_4(\mathbf{x}, t) \end{pmatrix}$$

The eigenvectors in this space are "Dirac spinors." For nonrelativistic particles, the upper two components are the two spin states of the electron, the lower of the positron.

The quantized Dirac field is found, in analogy to the photon field, as

$$\Psi(\mathbf{x}, t) = \sum_{s,\mathbf{p}} \frac{1}{\sqrt{2EV}} \left[u_{s,\,\mathbf{p}}\, \mathrm{e}^{-\mathrm{i}(Et - \mathbf{p}\cdot\mathbf{x})} a_{s,\mathbf{p}} + v_{s,\,-\mathbf{p}}\, \mathrm{e}^{\mathrm{i}(Et - \mathbf{p}\cdot\mathbf{x})} b^\dagger_{s,\mathbf{p}} \right]$$

where s denotes the spin and $E = \sqrt{\mathbf{p}^2 + m^2}$.

The "spinors" u are for electrons whereas v for positrons.

The four components of the Dirac field are not independent. Besides each component fulfilling the Klein-Gordon Eqn, the Dirac field obeys a linear wave Eqn, the Dirac Eqn

$$(\mathrm{i}\gamma_\mu \partial^\mu - m)\Psi(\mathbf{x}, t) = 0$$

where the γ^μ are 4×4 "Dirac matrices" for which different representations exist.

The field Ψ contains the destruction operator for particles and the creation operator of antiparticles, the conjugate field $\Psi^\dagger$ the creation of particles and destruction of antiparticles.

The Dirac field ψ, as opposed to the Maxwell field A, is not a Hermitean operator and therefore not a quantum observable. The Dirac field has no classical counterpart. (Pauli exclusion!)

Operators representing observables are always field bilinears of the form $\Psi^\dagger \ldots \Psi$. For example, $\Psi^\dagger\Psi$ is an operator representing the particle density. It contains operators of the form $a^\dagger a$ and $bb^\dagger$. By anticommutation: $bb^\dagger = -b^\dagger b + 1$, so apart from overall constants $\Psi^\dagger\Psi$ is the number operator for particles minus antiparticles, i.e. the "net particle density" or charge density

$$n_{e^-} - n_{e^+} = \langle \mathrm{state} | \Psi^\dagger \Psi | \mathrm{state} \rangle$$

6.2 Electrodynamics as a gauge field theory

Observable quantities of electron field always field bilinears of the form

$$\Psi^\dagger \Psi$$

Such bilinears are invariant under "gauge transformations" of the first kind

$$\Psi(x) \to \mathrm{e}^{\mathrm{i}\alpha}\, \Psi(x) \qquad \text{and} \qquad \Psi^\dagger(x) \to \mathrm{e}^{-\mathrm{i}\alpha}\, \Psi^\dagger(x)$$

where α is a real number (a phase).

Postulate that total Hamiltonian is also invariant under gauge transformations of the second kind (local phase transformations)

$$\Psi(x) \to \mathrm{e}^{\mathrm{i}\alpha(x)}\, \Psi(x) \qquad \text{and} \qquad \Psi^\dagger(x) \to \mathrm{e}^{-\mathrm{i}\alpha(x)}\, \Psi^\dagger(x)$$

where $\alpha(x)$ is a real scalar function.

In this case bilinears $\Psi^\dagger\Psi$ are still invariant, but there are also derivatives of Ψ in the Hamiltonian (or in the Lagrangian). They transform as

$$\partial_\mu \Psi(x) \to \mathrm{e}^{\mathrm{i}\alpha(x)}\, \partial_\mu \Psi(x) + [\mathrm{i}\partial^\mu \alpha(x)]\, \mathrm{e}^{\mathrm{i}\alpha(x)}\, \Psi(x)$$

Invariance is restored if everywhere in the original Hamiltonian (or Lagrangian) we use the "covariant derivative"

$$D^\mu = \partial^\mu - \mathrm{i}eA^\mu(x)$$

and under a gauge transformation of the electron fields transform the vector potential as

$$A^\mu(x) \to A^\mu(x) + \frac{1}{e}\,\partial^\mu \alpha(x)$$

Covariant derivative then transforms as

$$D^\mu \Psi(x) \to \mathrm{e}^{\mathrm{i}\alpha(x)}\, D^\mu \Psi(x)$$

Bilinears constructed from Ψ and $D^\mu\Psi$ are invariant.

Gauge invariance of the free EM fields ensures that Hamiltonian or Lagrangian is also invariant.

Assumption of gauge invariance leads to unique interaction structure between electrons and photons. As in classical electrodynamics, canonical momentum for electrons

$$P^\mu \to P^\mu - eA^\mu \qquad \text{where} \qquad P^\mu \to \frac{1}{\mathrm{i}}\,\partial^\mu$$

Relativistic covariance best seen in Lagrangian formulation because Lagrangian is a relativistic scalar, whereas Hamiltonian is 00 component of the energy-momentum tensor.

QED Lagrangian is then

$$\mathcal{L}_{\text{QED}} = \bar{\Psi}(\mathrm{i}\gamma^{\mu} D_{\mu} - m)\Psi - \frac{1}{4} F_{\mu\nu}F^{\mu\nu}$$

Note that $\bar{\Psi} = \Psi^{\dagger}\gamma^{0}$ and that

$$-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2) \qquad \text{whereas} \qquad \mathcal{H} = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2)$$

Note that the Hamiltonian density is the starting point for the canonical quantization procedure that is simple, but not intrinsically covariant. The Langrangian density is the starting point for the path-integral quantization procedure.

The interaction term between the EM field and the electron field is

$$\mathcal{L}_{\text{int}} = eA_{\mu}\bar{\Psi}\gamma^{\mu}\Psi$$

The crucial point is that the interaction between electrons and photons has the structure

$$eA\,\Psi^{\dagger}\Psi$$

Therefore, the interaction can, for example, destroy a photon and at the same time create an electron and a positron etc.

The gauge coupling constant, here e, is a dimensionless number. In the electromagnetic case

$$e = \sqrt{4\pi\,\alpha} = 0.30$$

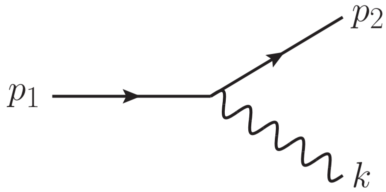
where $\alpha \approx 1/137$ is the fine-structure constant.

6.3 Feynman graphs and some processes

A formal perturbative treatment of processes can be represented by Feynman graphs, depicting the interaction of electrons, positrons and photons. The expansion parameter is the fine-structure constant $\alpha = 1/137$.

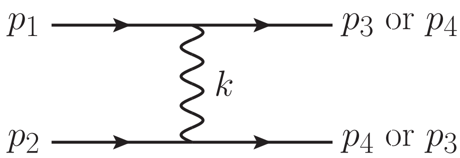
6.3.1 Radiation of a single photon

The process $e \to e + \gamma$ is not possible in vacuum because of energy-momentum conservation, but allowed in a medium when the photon refractive index is larger than 1 and so effectively $\omega^2 - \mathbf{k}^2 < 0$ ("space-like photons"): Cherenkov effect



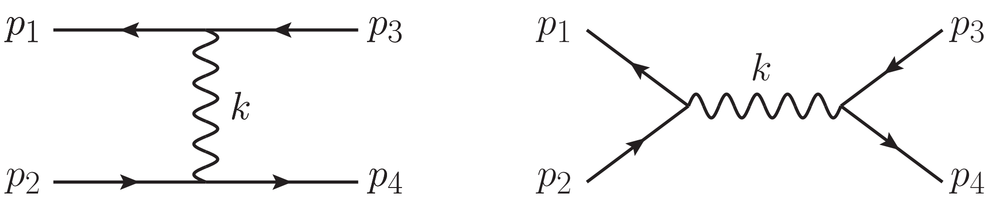
6.3.2 Coulomb scattering between two electrons (Møller scattering)

The process $e + e \to e + e$ is among the simplest second-order processes. One electron emits a photon that is absorbed (before or afterwards) by the other.



6.3.3 Bhabha scattering

The process $e^- + e^+ \to e^- + e^+$ is similar to Coulomb scattering, but involves the additional annihilation graph. Note also that the arrows follow the flow of charge, not the flow of momenum. (For an antiparticle the flow of momentum is opposite to the flow of charge.)

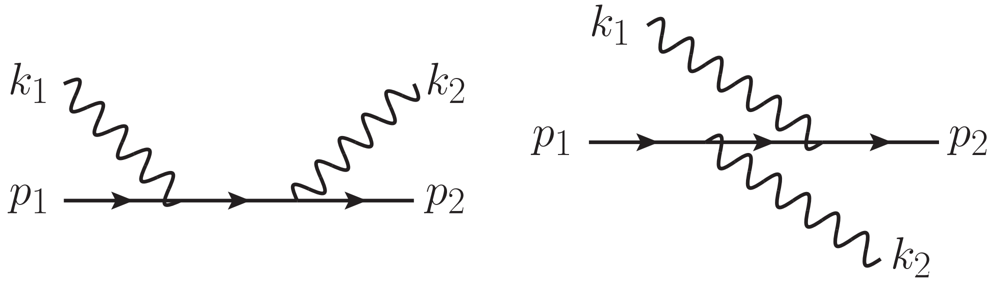


For the "Coulomb graph", the intermediate photon is space like, for the annihilation graph it is "time-like"

$$k^2 = (p_4 - p_2)^2 < 0 \qquad \text{Coulomb graph}$$
$$k^2 = (p_1 + p_2)^2 > 0 \qquad \text{Annihilation graph}$$

6.3.4 Compton scattering

This basic process can occur by two amplitudes, the final-state photon can be emitted before or after the first.



As an example, we give in this case the explicit result for the total scattering cross section:

$$\sigma = \frac{\pi\alpha^2}{m_e^2}\left[\frac{16}{(\hat{s}-1)^2} + \frac{\hat{s}+1}{\hat{s}^2} + \frac{2(\hat{s}^2 - 6\hat{s} - 3)}{(\hat{s}-1)^3}\ln(\hat{s})\right] \qquad \text{where} \qquad \hat{s} = \frac{s}{m_e^2}$$

For an electron with four momentum $P = (E, \mathbf{p})$ and a photon with $K = (\omega, \mathbf{k})$ the CM energy is defined by

$$s = (P+K)^2 = \begin{cases} (E+\omega)^2 & \text{CM frame where } \mathbf{p} = -\mathbf{k} \\ 2\omega m_e + m_e^2 & \text{Laboratory frame where } \mathbf{p} = 0 \end{cases}$$

In the nonrelativistic limit $\omega \to 0$ this is the Thomson cross section

$$\sigma = \frac{8\pi\alpha^2}{3m_e^2}$$

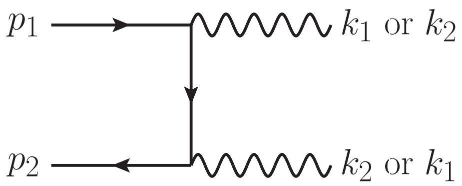
whereas in the ultrarelativistic limit $m_e \to 0$ relative to other energy scales it is

$$\sigma = \frac{\pi\alpha^2}{s}\left[2\ln\left(\frac{s}{m_e^2}\right) + 1\right]$$

Apart from overall factors and logarithmic corrections, such results can be guessed from an "educated dimensional analysis". Each photon vertex contributes a factor α. The cross section has the dimension of an area or inverse energy squared. In the relativistic limit (relevant for the early universe), the only natural energy scale is the CM energy.

6.3.5 Pair annihilation or pair creation

The Compton graph "turned sideways" gives us the pair annihilation or creation graph



The pair creation cross section ($\gamma\gamma \to e^- e^+$) is found to be

$$\sigma = \frac{\pi\alpha^2}{m_e^2}\,\frac{1-v^2}{2}\left[(3-v^4)\ln\left(\frac{1+v}{1-v}\right) + 2v(v^2-2)\right]$$

where v is the velocity of the produced e^- or e^+ in the CM frame

$$v = \sqrt{1 - \frac{4m_e^2}{s}}$$

Near threshold (for small v) the cross section expands as

$$\sigma = \frac{\pi\alpha^2}{m_e^2}\,v$$

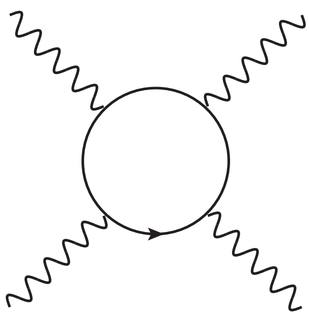
In the other extreme of $v \to 1$, i.e. $s \gg m_e^2$, one finds

$$\sigma = \frac{4\pi\alpha^2}{s}\left[\ln\left(\frac{s}{m_e^2}\right) - 1\right]$$

Pair annihilation ($e^- e^+ \to \gamma\gamma$) has the same high-energy limit except for a factor 1/2. This factor comes from the "statistics" of the final-state photons: They are not distinguishable, so the two-body phase space is reduced by this factor.

6.3.6 Photon-photon scattering

Completely new processes include photon-photon scattering by a loop of virtual electrons/positrons.



6.4 Equilibrium of gauge interactions in the early universe

Now estimate when in the early universe photons are in thermal equilibrium, as an example for all particles/processes governed by gauge interactions.

Consider specifically the photon annihilation rate into fermion pairs. In thermal equilibrium this is the same as the reverse rate.

The reaction rate (or inverse mean free path) of a typical photon is on average

$$\Gamma = n_\gamma \langle \sigma v_{\rm rel} \rangle$$

with the cross section

$$\sigma = \frac{4\pi\alpha^2}{s}\left[\ln\left(\frac{s}{m_e^2}\right) - 1\right]$$

In thermal equilibrium the number density of photons is

$$n_\gamma = \frac{2\zeta_3}{\pi^2} T^3$$

The relative velocity between photons 1 and 2 is given by

$$\mathbf{v}_{\rm rel} = \mathbf{v}_1 - \mathbf{v}_2 \qquad \Rightarrow \qquad v_{\rm rel}^2 = v_1^2 + v_2^2 - 2v_1 v_2 \cos\theta = 2(1-\cos\theta)$$

The CM energy is given by

$$s = (K_1 + K_2)^2 = K_1^2 + K_2^2 + 2K_1K_2 = 0 + 0 + 2(\omega_1\omega_2 - k_1k_2\cos\theta) = \omega_1\omega_2 2(1-\cos\theta)$$

Therefore the dominant term is

$$\frac{v_{\rm rel}}{s} = \frac{1}{\omega_1\omega_2}\,\frac{1}{\sqrt{2(1-\cos\theta)}}$$

So we should average the second factor over relative angles and find

$$\left\langle \frac{1}{\sqrt{2(1-\cos\theta)}} \right\rangle = \int_{-1}^{+1} \frac{1}{\sqrt{2(1-\cos\theta)}}\,\mathrm{d}\cos\theta \Big/ \int_{-1}^{+1} \mathrm{d}\cos\theta = 1$$

Averaging the former factors we note that for a Bose-Einstein distribution

$$\langle \omega^{-1} \rangle = \frac{\pi^2}{12\,\zeta_3} T^{-1} \approx 0.684\, T^{-1}$$

and therefore

$$\left\langle \frac{1}{\omega_1\omega_2} \right\rangle = \left(\frac{\pi^2}{12\,\zeta_3}\right)^2 T^{-2} \approx 0.468\, T^{-2} \sim \frac{1}{2T^2}$$

The logarithmic term in the cross section is taken to be a constant. At high T the electron will have a thermal mass of order eT, so roughly $s/m_e^2 \sim 2(3T)(3T)/(0.3T)^2 = 200$ and $\ln(s/m_e^2) - 1 \sim 4$.

Altogether for the interaction rate then

$$\Gamma \sim \frac{2\zeta_3}{\pi^2} T^3 4\pi\alpha^2 \frac{4}{2T^2} = \frac{16\zeta_3}{\pi} \alpha^2 T \sim 6\alpha^2 T$$

Something we could have guessed immediately on dimensional grounds except for the numerical coefficient.

Possible reaction channels include all charged fermions. At sufficiently high T, muons and tau leptons will play the same role as neutrinos, so

3 channels from charged leptons

Quarks have charges 2/3 and 1/3, respectively. The rate is proportional to $\alpha^2 = e^4/(4\pi)^2$, so the effective channels are

$$\left(\frac{2}{3}\right)^4 \times 3 \text{ colors} \times 3 \text{ flavors} = \frac{16}{9} \qquad \text{from up-type quarks}$$

$$\left(\frac{1}{3}\right)^4 \times 3 \text{ colors} \times 3 \text{ flavors} = \frac{1}{9} \qquad \text{from down-type quarks}$$

So roughly 2 effective channels from quarks and thus a total of about 5 channels. Altogether

$$\Gamma \sim 30\, \alpha^2 T$$

Compare with Hubble expansion rate, assuming standard-model particles with $g^* = 106.75$

$$H = \frac{T^2}{m_{\rm Pl}} 1.66\sqrt{g^*} \sim 17 \frac{T^2}{m_{\rm Pl}}$$

The equlibrium condition $H \lesssim \Gamma$ then is

$$17 \frac{T^2}{m_{\rm Pl}} \lesssim 30\, \alpha^2 T \qquad \Rightarrow \qquad T \lesssim 2\alpha^2\, m_{\rm Pl}$$

With $\alpha = 1/137$ this is approximately $T \lesssim 1 \times 10^{15}$ GeV. However, the effective coupling constants depend on energy (see later) and at this scale we roughly have $\alpha \sim 1/25$. Equilibrium condition then

$$T \lesssim 3 \times 10^{16} \text{ GeV}$$

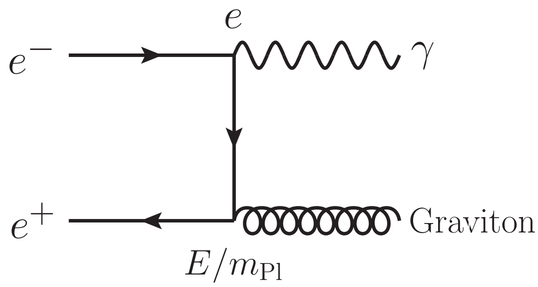
Roughly identical with the "grand unification scale".

Gauge interactions enter thermal equilibrium roughly at the GUT scale, assuming the universe ever reheated to such large T after cosmic inflation. Otherwise they are in equilibrium directly after reheating.

6.5 Will gravitons thermalize ever? (1 Dec. 2009)

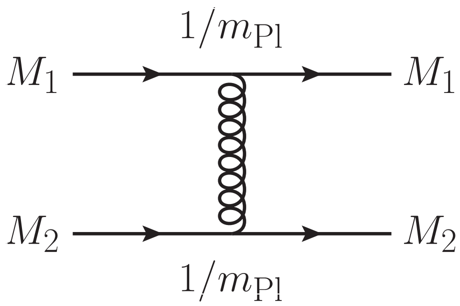
We can ask if the hypothetical quanta of gravitational radiation, gravitons, will ever thermalize.

They would interact by all sorts of processes, like the one shown here, where a graviton substitutes, for example, for a photon.



The "charge" substituting for e must be an energy scale E divided by the Plank mass.

Compare Newton's and Coulomb potential between two bodies. The potential involves Newton's constant and therefore $1/m_{\rm Pl}^2$, one factor being associated with one vertex each.



Therefore, on dimensional grounds the rate will be something like

$$\Gamma \sim \alpha (T/m_{\rm Pl})^2\, T = \alpha T^3/m_{\rm Pl}^2$$

and for equilibrium should be larger than

$$H \sim T^2/m_{\rm Pl}$$

and thus

$$T \gtrsim m_{\rm Pl}/\alpha$$

This would be before the Planck epoch where we have no idea about the relevant physics.

Gravitons never reached thermal equilibrium in the universe as we understand it.