2.6 Horizons and the future of the universe (16 Nov. 2009)

2.6.1 Horizons

One last important distance in the universe is that of a "horizon." One means a surface (or the distance to a surface) beyond which nothing can be observed. Well known is the Schwarzschild radius of a black hole. Light signals from inside can not reach us—the Schwarzschild radius is the surface nearest to the black hole from which we can still receive signals.

In cosmology, there are two concepts of horizon.

- **Particle horizon D_{P}**
  The largest coordinate distance in the past from which we receive information, i.e. the "radius of the visible universe." Events outside of this region can not have causal influence on us today.

- **Event horizon D_{E}**
  The largest coordinate distance in the future that we can causally influence. Or conversely, all objects or events outside of this horizon will never become visible to us in future.

Depending on the dynamics of the universe, either of these horizons can be finite or infinite. If it is infinite one can also say there is no such horizon.

2.6.2 General expression for coordinate distance

We recall that the present-day coordinate distance between two observers A and B is

$$D_{\mathrm{c}}^{AB} = a_0 \int_A^B \frac{\mathrm{d}r}{\sqrt{1-kr^2}}$$

Assuming more specifically that a light-signal is emitted by A at some time t_A and received at B at some time t_B we use $\mathrm{d}s^2 = 0$ and the Robertson-Walker metric to conclude

$$D_{\mathrm{c}}^{AB} = a_0 \int_{r_A}^{r_B} \frac{\mathrm{d}r}{\sqrt{1-kr^2}} = a_0 \int_{t_A}^{t_B} \frac{\mathrm{d}t}{a(t)} = \int_{t_A}^{t_B} \frac{\mathrm{d}t}{y(t)}$$

where as usual $y = a/a_0$. The evolution $y(t)$ is determined by the Friedmann Eqn which we can write in the form

$$\frac{\dot{y}^2}{y^2} = H_0^2 \sum_j \Omega_j y^{n_j}$$

assuming we have different barotropic fluid components. For our most relevant case, a flat matter-Lambda model, this is explicitly

$$H_0 \mathrm{d}t = \frac{\mathrm{d}y}{\sqrt{\Omega_{\mathrm{M}} y^{-1} + \Omega_\Lambda y^2}} \qquad \text{where} \qquad \Omega_{\mathrm{M}} + \Omega_\Lambda = 1$$

Inserting this above provides

$$H_0 D_{\rm c}^{AB} = \int_{y_A}^{y_B} \frac{\mathrm{d}y}{\sqrt{\Omega_{\rm M} y + \Omega_\Lambda y^4}}$$

and we recall that the Ω parameters by definition refer to the present epoch.

2.6.3 Particle and event horizons for simple flat models

For flat models characterized by the two components $\Omega_{\rm M}$ and Ω_Λ the explicit expressions for the particle and event horizons, for a present-day observer, are

$$H_0 D_{\rm P} = \int_0^1 \frac{\mathrm{d}y}{\sqrt{\Omega_{\rm M} y + \Omega_\Lambda y^4}}$$

$$H_0 D_{\rm E} = \int_1^\infty \frac{\mathrm{d}y}{\sqrt{\Omega_{\rm M} y + \Omega_\Lambda y^4}}$$

Matter dominated

For a purely matter-dominated case ($\Omega_\Lambda = 0$ and $\Omega_{\rm M} = 1$) the integrals are easily performed explicitly

$$H_0 D_{\rm P} = 2$$

$$H_0 D_{\rm E} \to \infty$$

The universe visible to us today is limited, but we can see ever larger regions and in the infinite future can see an infinite region of space.

Vacuum dominated

In the opposite case $\Omega_\Lambda = 1$ and $\Omega_{\rm M} = 0$ we find

$$H_0 D_{\rm P} \to \infty$$

$$H_0 D_{\rm E} = 1$$

An infinite region of space from the past is visible. On the other hand, objects that today are further away than H_0^{-1} will never become visible: The light emitted by them can not catch up with the cosmic expansion ("superluminal expansion").

Since the expansion rate $H = H_0$ is fixed in this case, the event horizon stays the same forever. However, since all galaxies recede from us, they will eventually pass through this horizon and disappear from view.

In this picture we see fewer and fewer objects until we are alone within our horizon.

Realistic case

In our universe with $\Omega_\mathrm{M} = 0.27$ and $\Omega_\Lambda = 0.73$ the present-day horizons are numerically

$$H_0 D_\mathrm{P} = 3.45$$
$$H_0 D_\mathrm{E} = 1.12$$

Therefore, some regions of space that were visible in the past have already disappeared from view.

2.6.4 Future of our universe

The universe today is already vacuum dominated, so eventually all galaxies will disappear from view and only gravitationally bound systems will survive this dilution effect.

However, even our galaxy can not last forever. It emits light, evaporates matter and dark matter particles and gravitational waves. Even burnt-out stars will eventually collide and collapse to form black holes. But even black holes do not last forever, they evaporate.

So in the end everything will get infinitely diluted, every elementary particle, or perhaps atoms or molecules, isolated in their own horizon.

If protons are unstable on some huge time scale, as is often assumed, then in the end only individual electrons, positrons, neutrinos, photons and gravitational waves will sit in isolation, eventually being redshifted until their wavelength exceeds the horizon.

An even more extreme scenario obtains if the vacuum energy is of the phantom type, leading to a "big rip." See next homework assignment for details.

3 The contents of the universe

In the section on the expanding universe we have repeatedly mentioned and used that the universe today seems to have several components: vacuum energy, matter (dark and visible) and radiation.

How do we know all of this? The answer today derives largely from "precision cosmology" and is based on the observed structures in the universe, particularly the statistical properties of the matter distribution and of the temperature fluctuations of the cosmic microwave background radiation.

These are rather advanced topics that we will probably not find time to develop in detail. A first overview is given in a powerpoint presentation—see the collection of slides "slides04.pdf".

One concludes that there are large amounts of dark matter in the universe in a weakly interacting form. This is also inferred from local evidence (galactic rotation curves, galaxy clusters, gravitational lensing)—see the slides.

All the radiation and matter in the universe today must have arisen in the hot early phase. Let us first take a brief glance at what happened at which epoch in the universe and then take it from there to study the early phases.

Redshift	Time	Temp.	Event
0	13.7 Gyr	2.725 K	Today
0.76	6.8 Gyr		Universe begins accelerating
1100	380,000 yr	0.26 eV	Recombination (neutral hydrogen & helium forms) Universe becomes transparent $\rightarrow$ Origin of CMB Baryons begin following structure formation
3600	60,000 yr		Matter-radiation equality Structures begin growing
3×10^8	3 min	80 keV	Nucleosynthesis of light elements begins
3×10^9	1 s	1 MeV	Neutrinos decouple Proton/neutron ratio freezes out
		150 MeV	QCD phase transition Free quarks group inseparably into hadrons
		250 GeV	Electroweak phase transition Fermions acquire mass from Higgs field W and Z bosons become massive Weak interaction becomes "weak"
		$\sim 10^{16}$ GeV	Grand unification between electroweak and strong interactions?
		10^{19} GeV	Planck epoch Need quantum gravity to understand

In addition, somewhere on this timeline the dark matter particles must have emerged or frozen out, but when depends on the relevant particle.

Likewise, at some large temperature cosmic inflation is assumed to have happened, and it is not clear to which temperature the universe was reheated afterwards. The GUT temperature likely was never reached.

From big bang nucleosynthesis we know with reasonable certainty that the universe was once hotter than about 1 MeV, the rest is hypothesis.

Most of the interesting particle-physics stuff happens in the early universe to which we now turn.

4 Radiation-dominated universe (17 Nov. 2009)

4.1 Friedmann Equation

Before matter-radiation equality, the universe was dominated by radiation. Henceforth we may usually negelect the contribution of matter, curvature and vacuum energy—they all become negligible.

Friedmann equation therefore

$$H^2 = \frac{8\pi}{3} G_\mathrm{N}\, \rho_\mathrm{rad} = \frac{8\pi}{3} \frac{\rho_\mathrm{rad}}{m_\mathrm{Pl}^2} \qquad \text{or} \qquad H = \frac{1}{m_\mathrm{Pl}} \sqrt{\frac{8\pi}{3}\,\rho_\mathrm{rad}}$$

with Planck mass as discussed earlier

$$m_\mathrm{Pl}^2 = 1.22 \times 10^{19}\ \mathrm{GeV}$$

4.2 Electromagnetic radiation

4.2.1 Energy density

Energy density of the electromagnetic radiation field is given by the phase-space integral

$$\rho_\gamma = 2 \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \frac{\omega}{\mathrm{e}^{\omega/T} - 1} = \frac{T^4}{\pi^2} \underbrace{\int_0^\infty \mathrm{d}x\, \frac{x^3}{\mathrm{e}^x - 1}}_{\pi^4/15} = \frac{\pi^2}{15} T^4$$

where for massless photons $\omega = |\mathbf{k}|$ is the energy of mode $\mathbf{k}$, the factor 2 is for two polarization degrees of freedom.

Note that the vacuum contribution (zero-point energy) of $\frac{1}{2}\omega$ per mode is not included here. It diverges and has no obvious interpretation as far as its gravitational effects are concerned. (Problem of vacuum energy or cosmological constant.)

As long as photons are the only form of radiation (no neutrinos etc.), the Friedmann Eqn becomes

$$H = \frac{1}{m_\mathrm{Pl}} \sqrt{\frac{8\pi}{3} \frac{\pi^2}{15} T^4} = \sqrt{\frac{8\pi^3}{45}} \frac{T^2}{m_\mathrm{Pl}}$$

This is the characteristic relationship between expansion rate and temperature in the early universe.

4.2.2 Number density

The photon number density is

$$n_\gamma = 2\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\frac{1}{\mathrm{e}^{\omega/T}-1} = \frac{T^3}{\pi^2}\underbrace{\int_0^\infty \mathrm{d}x\,\frac{x^2}{\mathrm{e}^x-1}}_{2\zeta_3} = 2\frac{\zeta_3}{\pi^2}T^3$$

where $\zeta_3 = 1.20206$ is the Riemann Zeta function at argument 3.

In the present-day universe with $T = 2.725$ K the number density is

$$n_\gamma = 411.1\ \mathrm{cm}^{-3}$$

4.2.3 Baryon-to-photon ratio

Compare the photon density with that of ordinary matter (baryons), i.e. protons or neutrons. The present-day baryon density is

$$n_\mathrm{B} = \Omega_\mathrm{B}\frac{\rho_\mathrm{crit}}{m_u} = 0.046\,\frac{1.04\times10^{-29}\ \mathrm{g\ cm}^{-3}}{1.66\times10^{-24}\ \mathrm{g}} = 2.88\times10^{-7}\ \mathrm{cm}^{-3}$$

where m_u is the atomic mass unit.

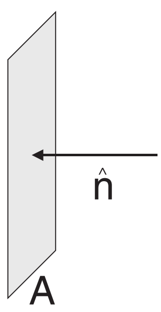
The baryon-to-photon ratio is then

$$\eta \equiv \frac{n_\mathrm{B}}{n_\gamma} = 0.70\times10^{-9}$$

This small number needs explaining as we will see. There are far more primordial photons in the universe than baryons.

4.2.4 Pressure

General gas with phase-space distribution $f_\mathbf{p}$. Pressure is equivalent to momentum flow per unit area through some surface A with normal vector $\hat{\mathbf{n}}$.



Pressure therefore proportional to velocity $\times$ momentum perpendicular to the surface, i.e.

$$\text{Pressure} \propto (\hat{\mathbf{n}} \cdot \mathbf{v})(\hat{\mathbf{n}} \cdot \mathbf{p}) = |\mathbf{v}|\,|\mathbf{p}|\,\cos^2\theta$$

If the fluid is isotropic, simply average over an isotropic distribution for all angles of modes moving toward the surface

$$\langle \cos^2\theta \rangle = \int_0^1 \cos^2\theta \,\mathrm{d}\cos\theta = \frac{1}{3}$$

Phase-space integral over all modes then

$$\text{Pressure} = \sum_{\text{spins}} \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\, f_{\mathbf{p}}\, |\mathbf{v}|\,|\mathbf{p}|\,\langle \cos^2\theta \rangle$$

For photons

$$\sum_{\text{spins}} \to 2, \qquad |\mathbf{v}| = 1, \qquad |\mathbf{p}| = \omega, \qquad f_{\mathbf{p}} = \frac{1}{\mathrm{e}^{\omega/T} - 1}$$

and so the phase-space integral is the same as for ρ_γ, except for the factor $\frac{1}{3}$. Therefore

$$p = \frac{\rho}{3}$$

4.2.5 Entropy

Based on general thermodynamic arguments one can show that the entropy density is

$$s(T) = \frac{\rho(T) + p(T)}{T}$$

One can actually show that in the expanding universe this implies that the entropy within a comoving volume is conserved (see next homework assignment)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(a^3 s\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(a^3\, \frac{\rho(T) + p(T)}{T}\right) = 0$$

Therefore, the evolution of the entropy density tracks the evolution of the cosmic scale factor a, or rather a^{-3}.

Viewed as a thermodynamic system, in the expanding universe it is comoving entropy that is conserved, not the energy.

For relativistic radiation, and notably electromagnetic radiation, where $p = \rho/3$ the entropy density is

$$s = \frac{\rho + p}{T} = \frac{4}{3}\frac{\rho}{T} \qquad \text{and} \qquad s_\gamma = \frac{4\pi^2}{45}T^3$$

comparable to the number density.

4.3 Electron-positron gas

4.3.1 Fermi-Dirac distribution

In thermal equilibrium all possible forms of "radiation" will be present. At sufficiently high temperature T this includes electrons and positrons, serving as an example for a thermal fermion gas.

We know from quantum electrodynamics that electrons and positrons can be produced in pairs, for example by the annihilation of two sufficiently high-energy photons

$$\gamma + \gamma \leftrightarrow e^- + e^+$$

The electron mass is $m_e = 0.511$ MeV, so if the reaction takes place in the CM frame, each photon must have at least this energy.

Given enough time, these pair processes will achieve thermal equilibrium. In this case the thermal occupation numbers of the momentum modes $\mathbf{p}$ of the electron (and positron) field is given by the Fermi-Dirac distribution

$$f_{\mathbf{p}} = \frac{1}{\mathrm{e}^{E_{\mathbf{p}}/T} + 1} < 1 \qquad \text{where} \qquad E_{\mathbf{p}} = \sqrt{m_e^2 + \mathbf{p}^2}$$

It reflects the Pauli principle: every mode can be occupied by at most 1 excitation and the thermal average is given by the Fermi-Dirac distribution.

Every momentum mode includes four different discrete possibilities. Each one can be occupied: two spin states (spin $\frac{1}{2}$ for fermions!) and the excitation can have positive or negative electric charge.

In other words, the phase space includes the continuum of momentum modes and four discrete possibilities.

4.3.2 Energy density

The energy density follows from the analogous phase-space integral as in the case of the electromagnetic field

$$\rho_{e^+e^-} = 4 \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{E}{\mathrm{e}^{E/T} + 1} = 4\,\frac{4\pi}{(2\pi)^3} \int_0^\infty \mathrm{d}p\, \frac{p^2 E}{\mathrm{e}^{E/T} + 1}$$

where $p = |\mathbf{p}|$ and

$$E = \sqrt{m_e^2 + p^2} \qquad \Rightarrow \qquad \mathrm{d}E = \frac{1}{2}\,\frac{2p\,\mathrm{d}p}{\sqrt{m_e^2 + p^2}} = \frac{p}{E}\,\mathrm{d}p$$

Phase-space integral therefore

$$\rho_{e^+e^-} = \frac{2}{\pi^2} \int_{m_e}^\infty \mathrm{d}E\, \frac{E^2\sqrt{E^2 - m_e^2}}{\mathrm{e}^{E/T} + 1}$$

“Massless” limit: $T \gg m_e$

In the limit $m_e \to 0$, the temperature is the only scale of the problem and the phase-space integration is very similar to bosons

$$\rho_{e^+e^-} = \frac{2}{\pi^2}\int_0^\infty \mathrm{d}E\,\frac{E^3}{\mathrm{e}^{E/T}+1} = \frac{2T^4}{\pi^2}\underbrace{\int_0^\infty \mathrm{d}x\,\frac{x^3}{\mathrm{e}^x+1}}_{\frac{7}{8}\frac{\pi^4}{15}} = \frac{7}{8}\,4\,\underbrace{\frac{\pi^2}{30}}_{\substack{\text{Same as for}\\ \text{1 boson}\\ \text{degree of freedom}}}T^4$$

Except for a factor of 4 (for the number of discrete degrees of freedom) and the factor 7/8 (Fermi rather than Bose statistics) the same as for photons.

At high T the electron-positron gas plays the same role as radiation.

Low-temperature limit: $T \ll m_e$

The energies E will be not much larger than the mass. Expand nonrelativistically

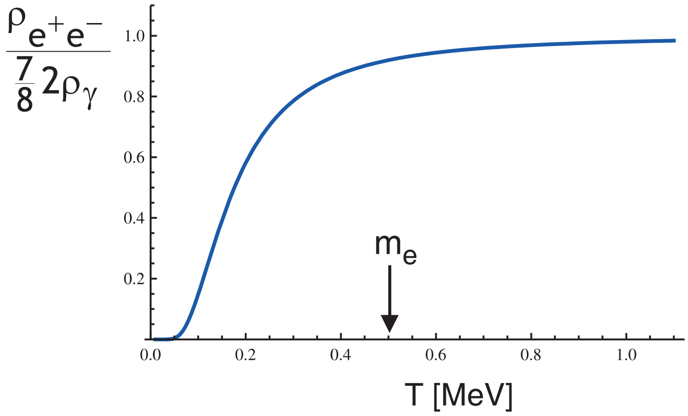
$$E = \sqrt{m_e^2+p^2} \approx m_e + \frac{p^2}{2m_e}$$

$$\mathrm{e}^{E/T}+1 \approx \underbrace{\mathrm{e}^{m_e/T}}_{\gg 1}\,\underbrace{\mathrm{e}^{p^2/2m_eT}}_{\mathcal{O}(1)} + 1 \approx \mathrm{e}^{m_e/T}\mathrm{e}^{p^2/2m_eT}$$

Energy density therefore

$$\begin{aligned}\rho_{e^+e^-} &= \frac{2}{\pi^2}\int_0^\infty \mathrm{d}p\,\frac{p^2E}{\mathrm{e}^{E/T}+1}\\ &\to \frac{2}{\pi^2}\int_0^\infty \mathrm{d}p\,p^2\,m_e\,\mathrm{e}^{-m_e/T}\,\mathrm{e}^{-p^2/2m_eT} = \frac{2\,m_e}{\pi^2}\underbrace{\mathrm{e}^{-m_e/T}}_{\substack{\text{Boltzmann}\\ \text{suppression}\\ \text{factor}}}\underbrace{\int_0^\infty \mathrm{d}p\,p^2\,\mathrm{e}^{-p^2/2m_eT}}_{\substack{\text{Maxwell-Boltzmann}\\ \text{distribution}}}\\ &= \frac{2\,m_e}{\pi^2}\,\mathrm{e}^{-m_e/T}\,(2m_eT)^{3/2}\underbrace{\int_0^\infty \mathrm{d}x\,x^2\,\mathrm{e}^{-x^2}}_{\sqrt{\pi}/4}\\ &= \mathrm{e}^{-m_e/T}\,\frac{m_e}{2}\left(\frac{2m_eT}{\pi}\right)^{3/2}\end{aligned}$$

For low T, the electron-positron population is suppressed by the mass threshold.



4.3.3 Chemical potential

If there is an initial asymmetry between electrons and positrons (true in the universe because today we have ordinary matter with electrons in it), the Fermi-Dirac distribution involves the chemical potential μ, which has dimensions of energy

$$f = \frac{1}{\mathrm{e}^{(E \mp \mu)/T} + 1}$$

where the upper sign refers to particles, the lower to antiparticles, i.e. μ is chosen positive if there is an excess of fermions (here: electrons).

The energy density for the electron-positron gas is in this case

$$\rho_{e^+e^-} = 2 \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \Bigg(\underbrace{\frac{E}{\mathrm{e}^{(E-\mu)/T} + 1}}_{\text{Electrons}} + \underbrace{\frac{E}{\mathrm{e}^{(E+\mu)/T} + 1}}_{\text{Positrons}} \Bigg)$$

In the early universe, the electron chemical potential is very small (see next homework assignment). On the other hand, in stars it is very large—the excess of electrons over positrons is huge.

At zero temperature, the chemical potential is also called the Fermi energy.

Often the chemical potential is introduced in its nonrelativistic form not including the particle mass,

$$\mu_{\text{nonrel}} = \mu - m$$

In this case the energy in the distribution functions must be the kinetic energy $E - m$.

However, in the relativistic context it is extremely confusing to measure the chemical potential relative to the rest mass. We always use the relativistic chemical potential.

In general, then, the thermal distributions are characterized by three energy scales: temperature T, chemical potential μ and particle mass m.

4.4 General expressions (relativistic limit)

In summary, the energy density for a thermal relativistic gas of electrons and positrons is

$$\rho = g\,\frac{\pi^2}{30}\,T^4 \times \begin{cases} 1 & \text{Bosons} \\ \frac{7}{8} & \text{Fermions} \end{cases}$$

where g is the number of discrete degrees of freedom per momentum mode, e.g. 2 polarization states for photons and 4 states for relativistic electrons/positrons.

The number density and entropy density are

$$n = g\,\frac{\zeta_3}{\pi^2}\,T^3 \times \begin{cases} 1 & \text{Bosons} \\ \frac{3}{4} & \text{Fermions} \end{cases}$$

$$s = \frac{\rho + p}{T} = \frac{4}{3}\,\frac{\rho}{T} = g\,\frac{4\pi^2}{90}\,T^3 \times \begin{cases} 1 & \text{Bosons} \\ \frac{7}{8} & \text{Fermions} \end{cases}$$

Cosmic energy density

$$\rho = \frac{\pi^2}{30}\,T^4\,\underbrace{\left(\sum_{\text{bosons}} g_{\text{B}} + \frac{7}{8}\sum_{\text{fermions}} g_{\text{F}}\right)}_{\substack{g^* \\ \text{Effective number of} \\ \text{thermal degrees of} \\ \text{freedom}}}$$

Friedmann Equation

$$H = \frac{T^2}{m_{\text{Pl}}}\,\sqrt{\frac{4\pi^3}{45}\,g^*} \approx \frac{T^2}{m_{\text{Pl}}}\,1.660\,\sqrt{g^*}$$

The cosmic expansion rate is therefore determined by the particle-physics degrees of freedom that are relativistic at a given temperature.

4.5 Expansion age

In our general discussion of solutions to FRWL models we have shown that a radiation dominated universe has the expansion age since the big bang of

$$t = \tfrac{1}{2}\, H^{-1}$$

With our result for H this is

$$t = \frac{1}{2}\sqrt{\frac{45}{4\pi^3}}\,\frac{1}{\sqrt{g^*}}\,\frac{m_{\rm Pl}}{T^2} = \frac{0.3012}{\sqrt{g^*}}\,\frac{m_{\rm Pl}}{T^2}$$

Of course, we still need to understand what happens when g^* changes, i.e. g^* is itself a function of time.

However, since $T \propto t^{1/2}$ for constant g^*, the temperature change is always slowest at the latest time. In other words, most cosmic time is accrued near the g^* of the last epoch. Therefore, the overall age is well approximated by using the final value of g^*.

To determine g^* we need to worry about two issues.

- Which thermal degrees of freedom (which particles) are available and what are their mass thresholds?
- Are they actually in thermal equilibrium at a given epoch? Need to know the interaction rate and compare with available time $t \sim H^{-1}$ to reach equilibrium.

For example, the cosmic microwave background today has a thermal distribution, but is not in thermal equilibrium: The CMB photons have propagated without interaction since they decoupled from the cosmic plasma at redshift $z = 1100$.

Electromagnetic radiation (e.g. starlight) produced after decoupling propagates without interaction and thus is not thermalized.

We begin with an overview of the particle-physics degrees of freedom and later turn to the question of their interactions to determine if and when they reach equilibrium.