2.4.6 Equation of state (09 Nov. 2009)

To solve Friedmann's equation we need to know how the gravitating mass–energy ρ evolves with time (or with cosmic scale factor).

For "dust" (non-relativistic bodies such as galaxies or dark-matter particles), dilution by cosmic expansion implies

$$\rho \propto a^{-3}$$

In general there is pressure p. Cosmic expansion does work against pressure

$$\mathrm{d}E = -p\,\mathrm{d}V \qquad \text{where} \qquad E \propto \rho a^3 \qquad \text{and} \qquad V \propto a^3$$

Therefore

$$\mathrm{d}(\rho a^3) = -p\,\mathrm{d}(a^3) \qquad \Rightarrow \qquad \dot\rho = -3(\rho + p)\,\frac{\dot a}{a}$$

Taking $\mathrm{d}/\mathrm{d}t$ of the Friedmann equation and using this result provides directly

$$\frac{\ddot a}{a} = -\frac{4\pi}{3}\,G_\mathrm{N}(\rho + 3p)$$

Altogether we have three equations available, but only two are independent

$$\begin{aligned}
\left(\frac{\dot a}{a}\right)^2 &= \frac{8\pi}{3}\,G_\mathrm{N}\rho - \frac{k}{a^2} && \text{1st Friedmann Eqn}\\
\frac{\ddot a}{a} &= -\frac{4\pi}{3}\,G_\mathrm{N}(\rho + 3p) && \text{2nd Friedmann Eqn}\\
\dot\rho &= -3(\rho + p)\,\frac{\dot a}{a} && \text{Continuum Eqn}
\end{aligned}$$

The first two Eqs together follow from the Einstein equation and are the "Friedmann equations" whereas the third represents energy-momentum conservation (continuum equation).

Need physical knowledge of EoS to solve.

2.4.7 Decelerated expansion

From the 2nd Friedmann Eqn one concludes that $\ddot a < 0$ (decelerated expansion) if

$$p > -\frac{1}{3}\,\rho$$

True for all "normal fluids" (have positive pressure, not "tension").

In this case $t_0 < H_0^{-1}$, cosmic age shorter than Hubble time.

2.4.8 Barotropic fluids

In practice one nearly always considers "barotropic fluids" with the EoS

$$p = w\rho \qquad \text{where} \qquad w = \text{const.}$$

Behavior under cosmic expansion

$$\begin{aligned}
&\mathrm{d}E = -p\,\mathrm{d}V \\
&\mathrm{d}(\rho a^3) = -p\,\mathrm{d}(a^3) = -w\rho\,\mathrm{d}(a^3) \\
&a^3\,\mathrm{d}\rho + 3a^2\rho\,\mathrm{d}a = -w\rho 3a^2\mathrm{d}a \\
&\frac{\mathrm{d}\rho}{\rho} = -3\,(1+w)\,\frac{\mathrm{d}a}{a} \qquad \text{integrate both sides} \\
&\log\rho = -3\,(1+w)\,\log a + \text{const.}
\end{aligned}$$

Therefore, for a barotropic fluid, the density evolves with scale factor as

$$\rho = \text{const.} \times a^{-3\,(1+w)}$$

Matter ("Dust")

$$p = 0, \qquad w = 0, \qquad \rho \propto a^{-3}$$

Simple dilution of galaxies (or dark matter particles) by cosmic expansion.

Radiation

$$p = \tfrac{1}{3}\rho, \qquad w = \tfrac{1}{3}, \qquad \rho \propto a^{-4}$$

Dilution of photons by volume factor a^{-3} and redshift factor a^{-1}.

Curvature

In Friedmann Eqn $H^2 = (8\pi/3)G_{\mathrm{N}}\rho - k/a^2$, so curvature acts like a density component where $\rho \propto a^{-2}$. From $\rho \propto a^{-3(1+w)}$ we conclude

$$p = -\tfrac{1}{3}\rho, \qquad w = -\tfrac{1}{3}, \qquad \rho \propto a^{-2}$$

Network of cosmic strings has a similar scaling. Also recall Milne universe (curvature driven): linear expansion, no acceleration, no deceleration.

Vacuum energy

$$p = -\rho, \qquad w = -1, \qquad \rho = \text{const.}$$

Vacuum energy (e.g. from quantum fluctuations) is invariant (scalar) under cosmic expansion, a property of vacuum. No dilution by cosmic expansion. Leads to accelerated expansion.

2.5 Simple Friedmann-Lemaître-Robertson-Walker (FLRW) models of the universe

2.5.1 Solutions for single-component EoS

Assume the Friedmann Eqn is dominated by a single-component barotropic fluid

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\,G_\mathrm{N}\,\rho = \frac{8\pi}{3}\,G_\mathrm{N}\,\rho_0\left(\frac{a_0}{a}\right)^{3\,(1+w)} = H_0^2\left(\frac{a_0}{a}\right)^{3(1+w)}$$

Use effectively redshift instead of scale factor

$$\begin{aligned} & y \equiv \frac{1}{1+z} = \frac{a}{a_0} \\ & y = 0 \qquad\qquad \text{big bang} \\ & y = 1 \qquad\qquad \text{today} \end{aligned}$$

Friedmann Eqn in these variables

$$\begin{aligned} \left(\frac{\dot{y}}{y}\right)^2 &= H_0^2\,y^{-3(1+w)} \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= H_0\,y^{-\frac{3(1+w)}{2}+1} = H_0\,y^{-\frac{3w+1}{2}} \\ H_0\mathrm{d}t &= y^{\frac{3w+1}{2}}\,\mathrm{d}y \end{aligned}$$

For $w > -1$ power-law behavior

$$H_0 t = \frac{2}{3\,(w+1)}\,y^{\frac{3(w+1)}{2}} \qquad \text{or} \qquad y = \frac{a}{a_0} = \left[\frac{3(w+1)}{2}\,H_0 t\right]^{\frac{2}{3(w+1)}}$$

Age obtained with $t \to t_0$ and $y \to 1$

$$t_0 = \frac{2}{3\,(w+1)}\,H_0^{-1}$$

Scaling of energy density with time: use $\rho \propto a^{-3(1+w)}$ and $t \propto a^{\frac{3(w+1)}{2}}$ and find

$$\rho \propto t^{-2}$$

for any barotropic fluid, independently of $w > -1$.

For $w = -1$ exponential expansion

$$\begin{aligned} & H_0\mathrm{d}t = \frac{\mathrm{d}y}{y} \qquad \Rightarrow \qquad H_0 t = \log y + \text{const.} \\ & y = \text{const.} \times \mathrm{e}^{H_0 t} \qquad \text{or} \qquad a(t) = a_0\mathrm{e}^{H_0 t} \end{aligned}$$

Case	EoS	Scaling of ρ	Evol. of $a(t)$	Age $[H_0^{-1}]$
Radiation	$p = \frac{1}{3}\rho$	a^{-4}	$t^{1/2}$	$\frac{1}{2}$
Matter	$p = 0$	a^{-3}	$t^{2/3}$	$\frac{2}{3}$
Curvature	$p = -\frac{1}{3}\rho$	a^{-2}	t	1
Vacuum	$p = -\rho$	const.	$e^{H_0 t}$	∞

If all of these components are present, radiation will dominate in the early universe, matter later, curvature later, and vacuum energy in the end.

In the absence of vacuum energy, curvature (if any) dominates in the end: The universe re-collapses (positive curvature) or develops toward the Milne solution (negative curvature).

2.5.2 Epoch of matter domination

The epoch of matter dominance is crucial for structures to develop. If radiation or vacuum energy dominates, the action of gravity can not lead to the growth of structures (see later in this course). Small density fluctuations can lead to stars, galaxies etc. only if there is a sufficiently long phase of matter domination.

Assuming the dynamics of the universe today is indeed dominated by 27% matter and 73% vacuum energy, at which redshift were the two contributions equal?

Matter density varies with $(1+z)^3$, so vacuum matter equality happens at

$$\frac{27\,(1+z)^3}{73} = 1 \qquad \Rightarrow \qquad z_{\Lambda\mathrm{M}} = 0.39$$

In the more distant past, radiation dominated. Let us ignore neutrinos for a first estimate and consider only photons. Present-day cosmic microwave background (CMB)

$$T_0 = 2.725\ \mathrm{K} = 0.235\ \mathrm{meV}$$

Energy density of thermal photon radiation

$$\rho_\gamma = 2\int \frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{\omega}{e^{\omega/T}-1} = \frac{2\,(4\pi)}{(2\pi)^3}\,T^4\int dx\,\frac{x^3}{e^x-1} = \frac{T^4}{\pi^2}\,\frac{\pi^4}{15} = \frac{\pi^2}{15}\,T^4$$

The coefficient is the radiation constant in natural units giving $\rho_\gamma = aT^4$.Usually given as

$$a = \frac{\pi^2}{15}\,\frac{k_\mathrm{B}^4}{\hbar^3 c^3} = 7.5657\times 10^{-15}\ \mathrm{erg\ cm^{-3}\ K^{-4}}$$

With the above value for T_0 we find

$$\rho_\mathrm{CMB} = 0.261\ \mathrm{eV\ cm^{-3}} \qquad \Rightarrow \qquad \Omega_\mathrm{CMB} = \frac{\rho_\mathrm{CMB}}{\rho_\mathrm{crit}} = 4.5\times 10^{-5} \ll 1$$

Since $\rho_\gamma \propto (z+1)^4$ and $\rho_{\rm M} \propto (z+1)^3$ and assuming today $\Omega_{\rm M} = 0.27$ the redshift of matter and radiation equality is

$$\frac{4.5 \times 10^{-5}\,(z+1)^4}{0.27\,(z+1)^3} = 1 \qquad \Rightarrow \qquad z_{\rm eq} + 1 = 6000$$

Including neutrinos increases the radiation density roughly by a factor of 2 and therefore delays the redshift of matter–radiation equality to (see later in these lectures)

$$z_{\rm eq} = 3580$$

This corresponds to a photon temperature at that time of $T_{\rm eq} = 0.84$ eV $= 9800$ K. (Not much hotter than the surface of the Sun with 5778 K.)

Derivation of the radiation content

The energy content of the electromagnetic radiation field is

$$\rho_\gamma = 2 \sum_{\text{modes }\mathbf{k}} f(\mathbf{k})\,\hbar\omega_{\mathbf{k}}$$

where the factor 2 comes from two polarization states, $\hbar\omega_{\mathbf{k}}$ with $\omega = c|\mathbf{k}|$ is the frequency, and $f(\mathbf{k})$ is the occupation number of mode $\mathbf{k}$. In a thermal medium we have the Bose-Einstein result $f = (\mathrm{e}^{\hbar\omega/k_{\rm B}T} - 1)^{-1}$. The summation is replaced by an integration as

$$\sum_{\text{modes }\mathbf{k}} \to \int \mathcal{D}(\mathbf{k})\,\mathrm{d}^3\mathbf{k} \qquad \text{where} \quad \mathcal{D}(\mathbf{k}) \quad \text{is the density of modes}$$

In a volume V with length L, the modes are proportional to $\exp(\pm 2\pi j/L)$ with $j = 1, 2, \ldots$ and so the momentum difference between neighboring modes is $\Delta k = 2\pi/L$. Hence the density of modes per unit volume V is

$$\mathcal{D}(\mathbf{k}) = \frac{1}{V}\,\frac{1}{(\Delta k)^3} = \frac{1}{(2\pi)^3}$$

leading to the above expression.

2.5.3 Realistic late-time models (matter plus vacuum energy)

Let us study late-time models of the universe, i.e. after the epoch of matter-radiation equality. The radiation density is then less and less important and the only relevant ingredients are matter and vacuum energy. Friedmann Eqn then

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G_\mathrm{N} \left(\rho_\mathrm{M}\frac{a_0^3}{a^3} + \rho_\Lambda\right) - \frac{k}{a^2}$$

where ρ_M is the present-day matter density and ρ_Λ that of vacuum energy.

Express present-day densities in terms of critical density and present-day Ω parameters

$$\rho^0_{\mathrm{M},\Lambda} = \Omega_{\mathrm{M},\Lambda}\, \rho^0_\mathrm{crit} = \Omega_{\mathrm{M},\Lambda} \frac{H_0^2}{8\pi\, G_\mathrm{N}/3}$$

Today: $\dot{a}/a = H_0$ and $a = a_0$

$$H_0^2 = H_0^2\,(\Omega_\mathrm{M} + \Omega_\Lambda) - \frac{k}{a_0^2} \qquad \Rightarrow \qquad -\frac{k}{a_0^2} = H_0^2\,(1 - \Omega_\mathrm{M} - \Omega_\Lambda)$$

Introduce curvature term

$$\Omega_\mathrm{K} \equiv 1 - \Omega_\mathrm{M} - \Omega_\Lambda$$

Friedmann Eqn then

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2 \left[\Omega_\mathrm{M}\left(\frac{a_0}{a}\right)^3 + \Omega_\mathrm{K}\left(\frac{a_0}{a}\right)^2 + \Omega_\Lambda\right]$$

Curvature term plays the role of some type of fluid. Note that by definition

$$\Omega_\mathrm{M} + \Omega_\mathrm{K} + \Omega_\Lambda = 1$$

Use relative coordinate (effectively redshift) and re-scaled time coordinate

$$y \equiv \frac{a}{a_0} = \frac{1}{1+z} \qquad \text{and} \qquad \tau \equiv H_0 t \qquad \text{and} \qquad \dot{y} \equiv \frac{\mathrm{d}y}{\mathrm{d}\tau}$$

Friedmann Eqn then

$$\dot{y}^2 = \frac{\Omega_\mathrm{M}}{y} + \Omega_\mathrm{K} + \Omega_\Lambda y^2 \qquad \Rightarrow \qquad \dot{y}^2 - \frac{\Omega_\mathrm{M}}{y} - \Omega_\Lambda y^2 = 1 - \Omega_\mathrm{M} - \Omega_\Lambda = \Omega_\mathrm{K}$$

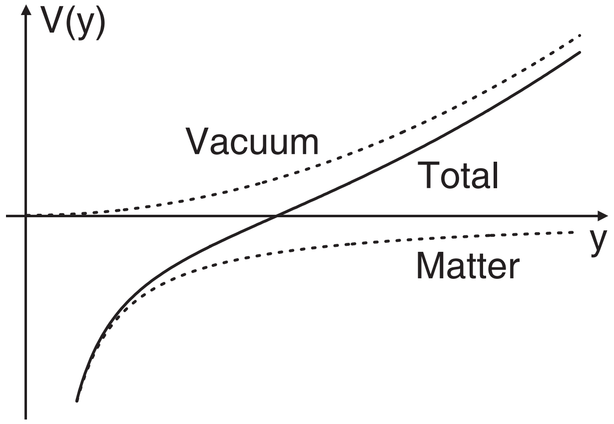
This is of the form

kinetic energy + potential energy = const.

Friedmann Eqn can be written like a mass point moving in a potential. (No surprise considering our Newtonian "derivation".)

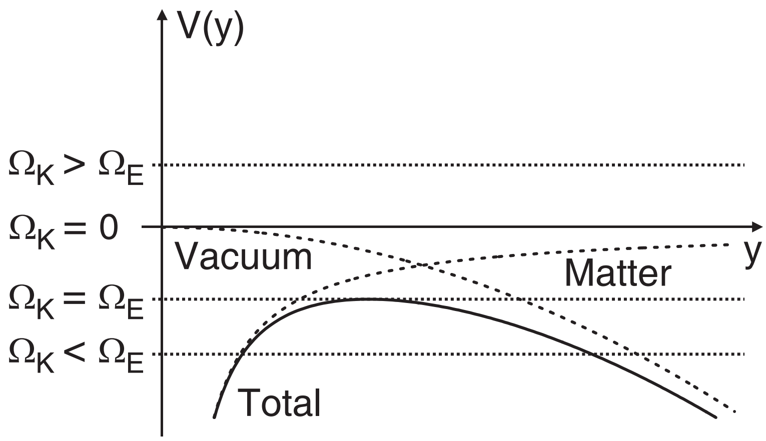
Negative vacuum energy: $\Omega_\Lambda < 0$

A particle in the potential $V(y) = -\Omega_{\rm M}\, y^{-1} - \Omega_\Lambda\, y^2$ eventually turns around, the universe returns to $y = 0$. The universe recollapses and ends in a "big crunch."



Positive vacuum energy: $\Omega_\Lambda > 0$

Potential has a maximum, so several generic solutions are possible.



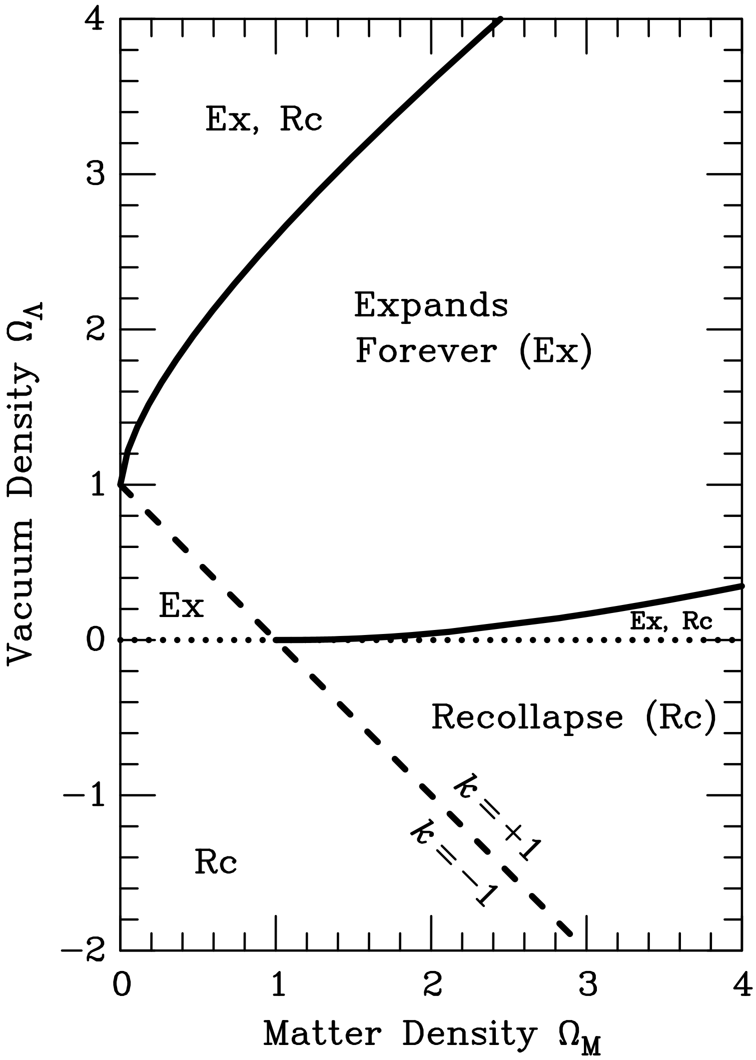
- **$\Omega_{\rm K}$ above maximum of potential**
  Universe expands forever, at some point accelerated expansion.

- **$\Omega_{\rm K}$ has Einstein value**
  Three solutions possible. One bound state that asymptotically reaches the maximum. One static solution (Einstein). One accelerated expansion solution.

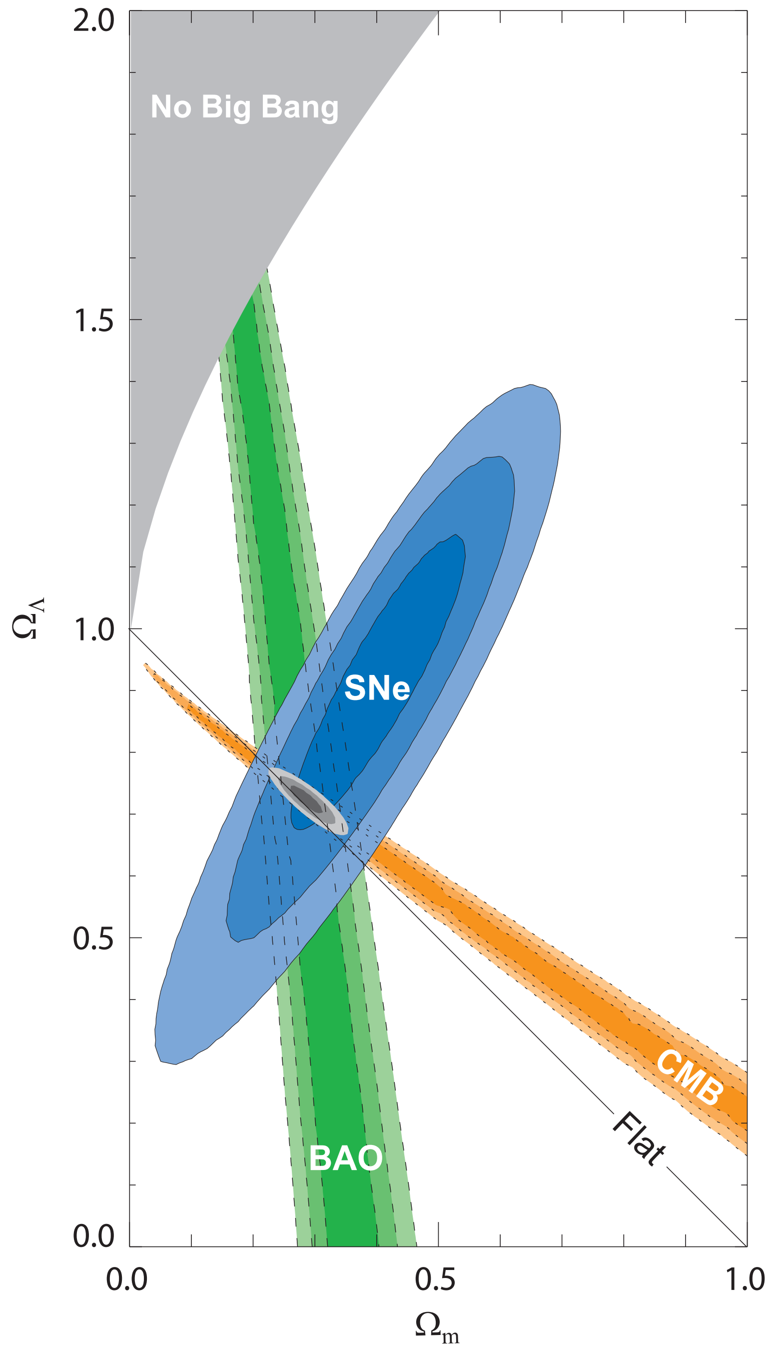
- **$\Omega_{\rm K}$ below Einstein value**
  One bound state that recollapses. One unbound state with accelerated expansion and no big bang in the past.

Second homework assignment: Identify the different solutions, depending on $0 \leq \Omega_{\rm M} < \infty$ and $-\infty < \Omega_\Lambda < +\infty$.

The solution is shown here. The thick solid lines mark the static Einstein solutions. In certain regions one has two solutions (Rc—recollapses and Ex—expands forever. The expanding solutions in the two-solution parameter space have no big bang in the past.



Observationally allowed regions of Ω_M and Ω_Λ at 68.3, 95.4 and 99.7% confidence level obtained from the cosmic microwave background (CMB) temperature fluctuations, baryon acoustic oscillations (BAO) in large-scale structure data, and from the supernova (SN) brightness-distance relation. Also shown is the combined allowed region. Figure from Kowalski et al. 2008, `http://arxiv.org/abs/0804.4142`, published in Astrophysical Journal 686 (2008) 749–778.



2.5.4 Matter-only explicit solution

Rewrite Friedmann Eqn in terms of y and t as

$$H_0\,\mathrm{d}t = \frac{\mathrm{d}y}{\sqrt{1+\Omega_\mathrm{M}(y^{-1}-1)+\Omega_\Lambda(y^2-1)}} \qquad \text{where} \qquad y = \frac{a}{a_0} = \frac{1}{1+z}$$

First consider "traditional case" without vacuum energy

$$H_0\,\mathrm{d}t = \frac{\mathrm{d}y}{\sqrt{1-\Omega_\mathrm{M}+\Omega_\mathrm{M}\,y^{-1}}}$$

This can be integrated explicitly for the three curvature cases (negative $\Omega_\mathrm{M} < 1$, flat $\Omega_\mathrm{M} = 1$, positive $\Omega_\mathrm{M} > 1$).

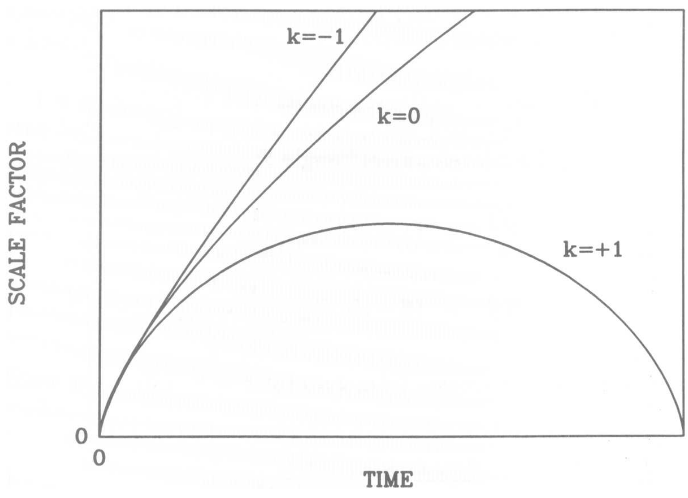
With the "development angles" $0 \le \psi < \infty$ and $0 \le \theta \le 2\pi$ the explicit solutions can be written in parametric form as

$$\begin{aligned}
\Omega_\mathrm{M} < 1: &\qquad \frac{a}{a_0} = \frac{\Omega_\mathrm{M}}{1-\Omega_\mathrm{M}}\,\frac{\cosh\psi - 1}{2}, \qquad H_0 t = \frac{\Omega_\mathrm{M}}{(1-\Omega_\mathrm{M})^{3/2}}\,\frac{\sinh\psi-\psi}{2} \\
\Omega_\mathrm{M} = 1: &\qquad \frac{a}{a_0} = \left(\frac{3H_0 t}{2}\right)^{2/3} \\
\Omega_\mathrm{M} > 1: &\qquad \frac{a}{a_0} = \frac{\Omega_\mathrm{M}}{\Omega_\mathrm{M}-1}\,\frac{1-\cos\theta}{2}, \qquad H_0 t = \frac{\Omega_\mathrm{M}}{2(\Omega_\mathrm{M}-1)^{3/2}}\,\frac{\theta-\sin\theta}{2}
\end{aligned}$$

In the positively curved case ($\Omega_\mathrm{M} > 1$) the scale factor reaches a maximum of

$$\frac{a_\mathrm{max}}{a_0} = \frac{\Omega_\mathrm{M}}{\Omega_\mathrm{M}-1}$$

and then re-collapses. In the other cases it expands forever. ("Geometry determines destiny.")



2.5.5 Explicit solution for flat universe

At present evidence, the matter-only case is only of historical interest because the universe seems to be flat with high precision and its dynamics is today dominated by vacuum energy. So we consider the case $\Omega_{\rm K} = 0$ and therefore after radiation is no longer important we use

$$H_0\,{\rm d}t = \frac{{\rm d}y}{\sqrt{\Omega_{\rm M}\,y^{-1} + \Omega_\Lambda\,y^2}} \qquad \text{where} \qquad \Omega_{\rm M} + \Omega_\Lambda = 1$$

With the transformation $x \equiv y^3$ this can be directly integrated and yields (with the lower boundary condition $t = 0$ and $y = 0$)

$$H_0\,t = \frac{2}{3}\,\frac{1}{\sqrt{\Omega_\Lambda}}\log\left(\frac{\sqrt{\Omega_\Lambda y^3} + \sqrt{\Omega_{\rm M} + \Omega_\Lambda y^3}}{\sqrt{\Omega_{\rm M}}}\right)$$

Invert explicitly

$$\frac{a}{a_0} = y = \left(\frac{\Omega_{\rm M}}{\Omega_\Lambda}\right)^{1/3}\sinh^{2/3}\left(\tfrac{3}{2}\sqrt{\Omega_\Lambda}\,H_0 t\right)$$

Recall that $\sinh x = \frac{1}{2}({\rm e}^x - {\rm e}^{-x})$.

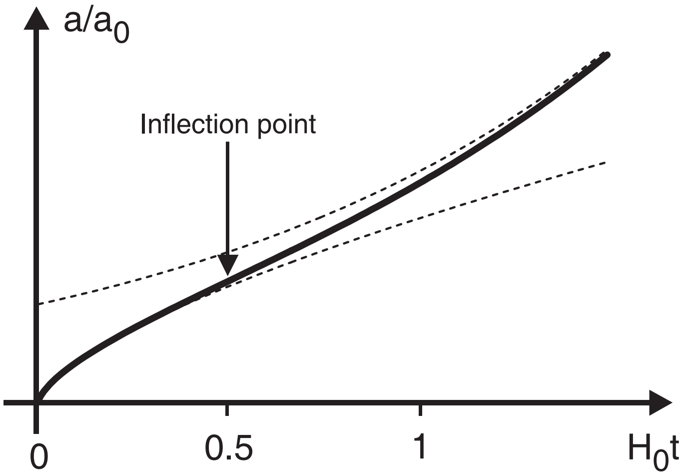
At early times this is approximately

$$y = \Omega_{\rm M}^{1/3}\,\left(\tfrac{3}{2}\,H_0 t\right)^{2/3}$$

Late times

$$y = \left(\frac{\Omega_{\rm M}}{4\Omega_\Lambda}\right)^{1/3}\exp\left(\sqrt{\Omega_\Lambda}\,H_0 t\right)$$

The universe first decelerates, later accelerates



There is an inflection point where the curvature (second derivative) vanishes and where the universe begins accelerating

$$\frac{\partial^2 y}{\partial t^2} = 0 \qquad \Rightarrow \qquad H_0 t_{\rm acc} = \frac{\mathrm{Acosh}(2)}{3\sqrt{\Omega_\Lambda}} = \frac{\log\left(2+\sqrt{3}\right)}{3\sqrt{\Omega_\Lambda}}$$

The corresponding scale factor and redshift is

$$y_{\rm acc} = \frac{a_{\rm acc}}{a_0} = \left(\frac{\Omega_{\rm M}}{2\Omega_\Lambda}\right)^{1/3}$$

$$z_{\rm acc} = \left(\frac{2\Omega_\Lambda}{\Omega_{\rm M}}\right)^{1/3} - 1$$

Expansion age today (set $y = 1$ in $H_0 t$ formula)

$$H_0\, t_0 = \frac{2}{3}\frac{1}{\sqrt{\Omega_\Lambda}} \log\left(\frac{1+\sqrt{\Omega_\Lambda}}{\sqrt{\Omega_{\rm M}}}\right) = \frac{1}{3\sqrt{\Omega_\Lambda}} \log\left(\frac{1+\sqrt{\Omega_\Lambda}}{1-\sqrt{\Omega_\Lambda}}\right)$$

For an empty universe (Milne case) the expansion is linear and the expansion age is exactly H_0^{-1}. In the flat case, which parameters reproduce this age?

The condition $H_0 t_0 = 1$ reads

$$e^{3\sqrt{\Omega_\Lambda}} = \frac{1+\sqrt{\Omega_\Lambda}}{1-\sqrt{\Omega_\Lambda}}$$

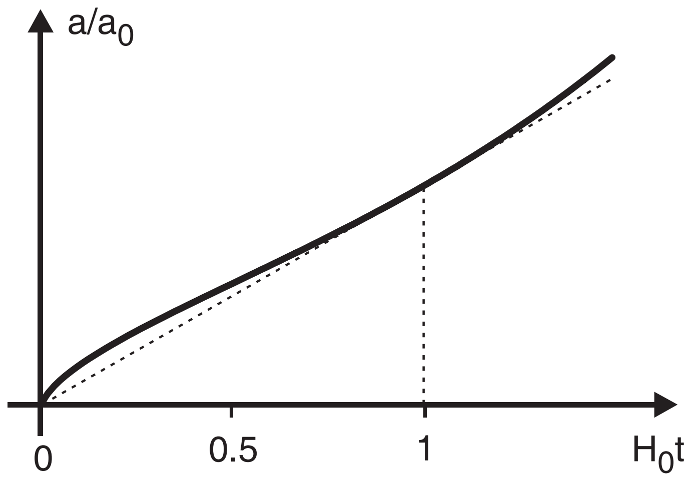
With the numerical solution

$$\Omega_\Lambda = 0.737125$$

Compare with the best observational estimate according to Komatsu et al. (2008), `http://arxiv.org/abs/0803.0547`, published in Astrophysical Journal Supplements 180 (2009) 330–376,

$$\Omega_\Lambda = 0.726 \pm 0.015$$

Therefore, within current errors the present-day values of Ω_M and Ω_Λ are such that the age of the universe is exactly H_0^{-1}. Of course, this must be a coincidence.



For $\Omega_\Lambda = 0.73$, the inflection point where the universe begins accelerating is at

$$t_\mathrm{acc} = 0.514\, H_0^{-1}$$
$$a_\mathrm{acc} = 0.570\, a_0$$
$$z_\mathrm{acc} = 0.755$$

Therefore, the transition occurred in the not very distant cosmic past.