2.3 Expanding or contracting space (2 Nov. 2009)

2.3.1 Robertson–Walker metric

A maximally symmetric (isotropic and homogeneous) space–time is finally given by the Robertson–Walker metric where $a \to a(t)$.

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\left(\frac{\mathrm{d}r^2}{1-k\,r^2} + r^2\mathrm{d}\Omega^2\right) \qquad \text{with} \qquad k = \begin{cases} +1 & \text{positive curvature} \\ 0 & \text{flat space} \\ -1 & \text{negative curvature} \end{cases}$$

The different elements mean

- Time t is clock time of a comoving observer.
- Comoving coordinates (r, θ, φ) remain fixed for an observer locally at rest (no peculiar velocity).
- Cosmic scale factor $a(t)$ changes with time if the space is not static. Except for a perfectly flat space, $a(t)$ is the instantaneous radius of curvature.

To the approximation that our physical universe is on average described by such a space, all the information is encoded in the time evolution of the cosmic scale factor $a(t)$. Its dynamics is given by the Friedmann equation (see below).

2.3.2 Do atoms grow?

Assuming all of space grows, how can we tell? Is everything growing with it, such as atoms or people?

Local physics can always be viewed in the tangent Minkowski space to our space–time and ordinary physics applies. In an atom, the local Coulomb interaction is much larger than the "pull" by cosmic expansion (see first homework assignment).

The properties and size of atoms is determined by quantum mechanics, particle masses and Coulomb's law.

Atomic frequencies provide an absolute scale, light waves an absolute length scale.

Local metric always dominated by local mass distributions, not by cosmic average metric.

2.3.3 Hubble's law

The cosmic scale factor $a(t)$ is an increasing function of time (we live in an expanding universe).

At some time t one defines two quantities describing the first derivative and curvature of this function

$$H = \frac{\dot{a}}{a} \qquad \text{Expansion or Hubble parameter}$$

$$q = -\frac{\ddot{a}a}{\dot{a}^2} = -\frac{\ddot{a}}{H^2 a} \qquad \text{Deceleration parameter}$$

where H, q, a, $\dot{a}$ and $\ddot{a}$ are all functions of t.

All quantities today receive an index 0. So t_0 is the present time after the big bang. In particular

$$H_0 = H(t_0) = \text{Hubble constant}$$

Measured value: 74.2 ± 3.6 km s^{-1} Mpc^{-1}

Taylor expansion of $a(t)$ around the present time t_0

$$\frac{a(t)}{a_0} = 1 + H_0\,(t - t_0) - \tfrac{1}{2}\,q_0\,[H_0\,(t - t_0)]^2 + \ldots$$

The notion of a "deceleration parameter" and the choice of a negative sign in the quadratic term derives from the assumption that the cosmic expansion should be slowing down (then q_0 would be a positive number). Today we know that q_0 is actually negative and the cosmic expansion is accelerating.

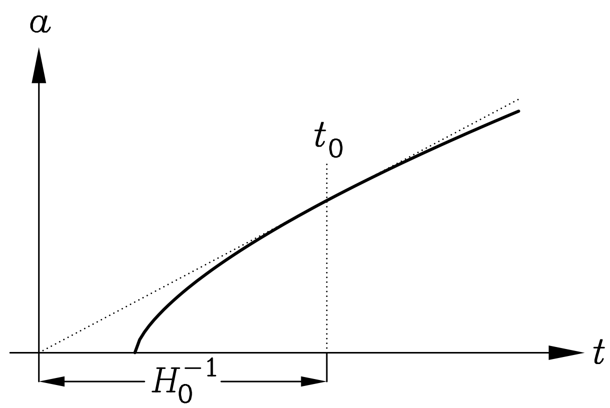


Figure 1: Evolution of the cosmic scale factor, assuming the expansion is always decelerating.

Let r be the comoving distance between two galaxies, each of them locally at rest. For $r \ll 1$ the coordinate distance is

$$D_\mathrm{c} = a_0\, r$$

It grows with time as

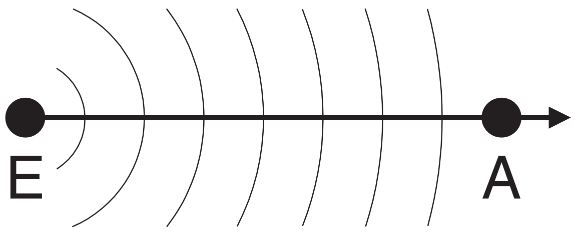
$$v_\mathrm{rec} = \dot{D}_\mathrm{c} = \dot{a}_0\, r = \frac{\dot{a}_0}{a_0}\, a_0 r = H_0 D_\mathrm{c}$$

The growth of coordinate distance is interpreted as a recession velocity of one galaxy from the other.

This simple interpretation is only true on small scales such that the recession velocity is small compared to the speed of light.

2.3.4 Cosmic reshift

If one galaxy recedes from another we expect a kinematical redshift (Doppler effect) of the light emitted in one galaxy (E) and absorbed in another (A).



How to interpret if both galaxies are locally at rest and space between them grows according to the Robertson-Walker metric?

Light propagation: $ds^2 = 0$

Assume E to be at coordinate origin, so light moves radially and $\mathrm{d}\Omega = 0$.

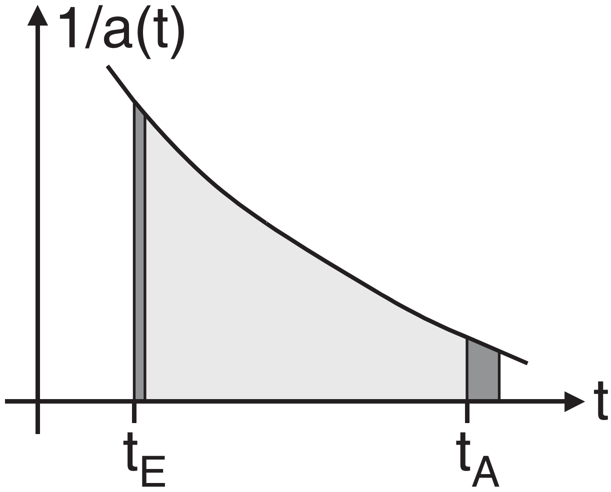
Using the comoving χ coordinate (for small distances $r \approx \chi$)

$$\mathrm{d}s^2 = 0 = \mathrm{d}t^2 - a^2(t)\, \mathrm{d}\chi^2 \quad \Rightarrow \quad \frac{\mathrm{d}t}{a(t)} = \mathrm{d}\chi$$

Coordinate distance between emitter and absorber

$$\chi = \int_0^\chi \mathrm{d}\chi' = \int_{t_\mathrm{E}}^{t_\mathrm{A}} \frac{\mathrm{d}t}{a(t)}$$

Is independent of t_E because coordinates are fixed in a co-moving system.



Look at the same integral some time $\Delta t_{\rm E}$ later

$$\chi = \int_{t_{\rm E}+\Delta t_{\rm E}}^{t_{\rm A}+\Delta t_{\rm A}} \frac{{\rm d}t}{a(t)}$$

Both integrals are the same, so the two end-point regions also must be the same

$$\int_{t_{\rm E}}^{t_{\rm E}+\Delta t_{\rm E}} \frac{{\rm d}t}{a(t)} = \int_{t_{\rm A}}^{t_{\rm A}+\Delta t_{\rm A}} \frac{{\rm d}t}{a(t)}$$

For small Δt the integrals are to lowest order

$$\frac{\Delta t_{\rm E}}{a(t_{\rm E})} = \frac{\Delta t_{\rm A}}{a(t_{\rm A})} \qquad \Rightarrow \qquad \frac{a_{\rm A}}{a_{\rm E}} = \frac{\Delta t_{\rm A}}{\Delta t_{\rm E}}$$

Interpret $\Delta t_{\rm E} \sim 1/\nu_{\rm E}$ of a light signal, i.e. $\Delta t_{\rm E}$ as the time difference between wave crests

$$\frac{a_{\rm A}}{a_{\rm E}} = \frac{\nu_{\rm E}}{\nu_{\rm A}} = \frac{\lambda_{\rm A}}{\lambda_{\rm E}} = 1 + z$$

Ratio of wavelengths between emission and absorption represents ratio of cosmic scale factors between those epochs.

Example: Highest-redshift object thus far observed is at $z = 8.3$. When the light was emitted the universe was a factor $1 + z = 9.3$ smaller than it is today.

2.3.5 Interpretation as Doppler effect

The definition of the Hubble constant and cosmic redshift imply

$$\frac{1}{1+z} = 1 - z + \ldots = \frac{a}{a_0} = 1 + H_0(t - t_0) + \ldots$$

and thus to linear order, using $D_c = t_0 - t$,

$$z = -H_0(t - t_0) = H_0 D_c \qquad \text{Hubble's law}$$

The r.h.s. was shown earlier to be the apparent recession velocity $v_{\text{rec}} = \dot{D}_c = H_0 D_c$ so that indeed to lowest order

$$z = v_{\text{rec}}$$

Therefore, it is perfectly consistent to interpret the cosmic redshift at small distances as a Doppler effect.

On small scales, the expanding universe can be interpreted as motion of galaxies in static (Minkowski) space, or equivalently as expansion of space with fixed objects.

2.3.6 Quantum-mechanical "derivation"

A body with energy E and momentum $\mathbf{p}$

$$E = \hbar\omega$$
$$\mathbf{p} = \hbar\mathbf{k}, \qquad |\mathbf{k}| = \lambda^{-1}$$

Wave pattern drawn on rubber balloon gets stretched: The wavelength of a photon gets redshifted.

$$\frac{\lambda_A}{\lambda_E} = \frac{a_A}{a_E}$$

Even true for nonrelativistic particles with $p = mv$

$$\frac{v_E}{v_A} = \frac{a_A}{a_E}$$

A galaxy with peculiar velocity v relative to the cosmic rest frame gets slowed down (Hubble friction)

$$\frac{\dot{v}}{v} = -\frac{\dot{a}}{a} = -H$$

Note: It is the momentum of a particle that gets redshifted, not its energy. For relativistic particles (photons) this is the same. For massive particles (e.g. neutrinos) or macroscopic bodies the distinction is important.

Of course, all of these results can be derived rigorously from the GR equations of motion.

2.3.7 Measures of distance

The universe expands monotonically. Therefore, any cosmological epoch is uniquely characterized by its redshift.

As measures of "distance," different quantities can be useful

- Lookback time Δt_z
- Coordinate distance D_c
- Brightness distance D_L
- Angle distance D_A

Assume the cosmological model is given, i.e. we know $a(t)$ where $a_0 = a(t_0)$ is the present-day scale factor and t_0 the age of the universe, $t = 0$ the big bang. The redshift is

$$z(t) + 1 = \frac{a_0}{a(t)}$$

Lookback time

Inverting the monotonically decreasing function $z(t)$ yields the lookback-time

$$\Delta t_z = t_0 - t_z \qquad \text{where} \qquad t_z = t(z)\,.$$

This is the time of travel taken by a light signal to reach us if it was emitted at a cosmic epoch z.

Coordinate distance

Light travel is characterized by $\mathrm{d}s = 0$. Taking the observer to be at the origin of polar coordinates we have $\mathrm{d}\Omega = 0$ along the light ray from the source to the observer. From the Robertson-Walker metric follows

$$\frac{\mathrm{d}r}{\sqrt{1 - kr^2}} = \frac{\mathrm{d}t}{a(t)}$$

Multiplying with a_0 provides on the l.h.s. the differential of the *coordinate distance*

$$D_\mathrm{c} = a_0 \int_0^{r_z} \frac{\mathrm{d}r}{\sqrt{1 - kr^2}} = \int_{t_0}^{t_z} \mathrm{d}t\, \frac{a_0}{a(t)} = \int_{t_0}^{t_z} \mathrm{d}t\, [z(t) + 1]$$

Flat space ($k = 0$): $D_\mathrm{c} = a_0 r$

Brightness (luminosity) distance

An object of known intrinsic luminosity $L_{\text{intrinsic}}$ appears dimmer in the sky if it is more distant. In the Newtonian case the energy flux (energy per unit area and unit time) received by an observer at distance D is

$$F = \frac{L_{\text{intrinsic}}}{4\pi D^2} \qquad \Rightarrow \qquad D = \sqrt{\frac{L_{\text{intrinsic}}}{4\pi F}}$$

If we know $L_{\text{intrinsic}}$ (for an astrophysical "standard candle" such as supernovae of type Ia) and we measure F, we infer the *brightness distance* or *luminosity distance* D_{L} from this formula.

Expanding universe: Every photon reaching us is redshifted by the factor $(1+z)$. The rate of photon emission at the source is also redshifted by the same factor

$$L_{\text{apparent}} \propto L_{\text{intrinsic}}/(1+z)^2$$

From Robertson-Walker metric and $\mathrm{d}t = 0$, the differential of an area transverse to the radial direction is

$$\mathrm{d}A = a^2(t) r^2 \, \mathrm{d}\theta \, \sin\theta \, \mathrm{d}\varphi$$

and therefore the surface area of a sphere at our distance

$$A = 4\pi \, a_0^2 \, r^2 \qquad \Rightarrow \qquad F = \frac{L_{\text{intrinsic}}}{4\pi \, (1+z)^2 \, r^2 \, a_0^2}$$

Brightness distance therefore

$$D_{\text{L}} = (1+z) \, r \, a_0 \qquad \Rightarrow \qquad \text{Flat space:} \quad D_{\text{L}} = (1+z) \, D_{\text{c}}$$

Angle distance

In Newtonian physics the angle θ subtended by an object shrinks with distance. If $R_{\text{intrinsic}}$ is the known size of an object ("standard rod") and if it is oriented transverse to the direction of an observer, the angle distance is

$$D_{\text{A}} = \frac{R_{\text{intrinsic}}}{\theta}$$

Use Robertson-Walker metric with $\mathrm{d}t = 0$. Transverse to radial direction: $\mathrm{d}r = 0$. Without loss of generality take $\mathrm{d}\varphi = 0$. Then $\mathrm{d}\ell = a(t) \, r \, \mathrm{d}\theta$ and

$$R_{\text{intrinsic}} = a(t) \, r \, \theta$$

where t is the time of emission of the light

$$a(t) = a_0/(1+z)$$

Therefore

$$D_{\mathrm{A}} = \frac{r\, a_0}{1+z} \qquad \Rightarrow \qquad \text{Flat space:} \quad D_{\mathrm{A}} = D_{\mathrm{c}}/(1+z)$$

Angle distance always smaller than brightness distance

$$\frac{D_{\mathrm{A}}}{D_{\mathrm{L}}} = \frac{1}{(1+z)^2}$$

Flat universe:

$$D_{\mathrm{L}} = (1+z)\, D_{\mathrm{c}}$$
$$D_{\mathrm{A}} = \frac{1}{1+z}\, D_{\mathrm{c}}$$

In contrast to the luminosity distance, the angle distance does not increase monotonically with redshift. Beyond some redshift, the angular size of objects increases with redshift ("gravitational lensing by the universe").

However, this effect has not been unambiguously observed because in astrophysics there are no reliable standard rods. Galaxies evolve and are not the same in the early universe.

Concrete examples for different distance measures is specific cosmological models will be calculated in the second homework assignment.

Tolman's test

Consider surface brightness of an object: Energy emitted per unit time and unit surface area of the object.

$$\text{Surface brightness} \;\propto\; \frac{D_{\mathrm{L}}^2}{D_{\mathrm{A}}^2} = \frac{1}{(1+z)^4}$$

Does not depend on cosmological model, but only on redshift. First proposed by Richard Tolman (1930).

Difficult to observe: No good standard sources (evolution of galaxies!).

2.4 Dynamics: Friedmann Equation

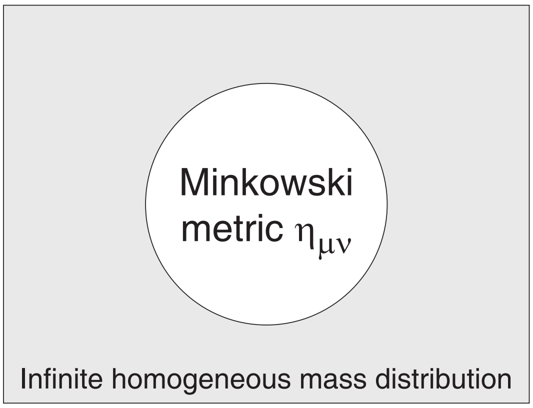
2.4.1 Birkhoff's Theorem

The dynamics of the universe is determined by its energy content (energy curves space–time). Need to know how matter affects space–time.

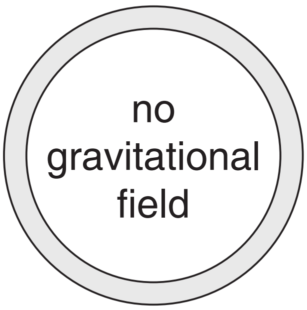
In GR, the connection between energy-momentum tensor and curvature tensor given by Einstein equation.

For maximally symmetric case, a heuristic Newtonian derivation is possible using Birkhoff's theorem.

Birkhoff: Assume a spherical cavity in a homogeneous mass distribution. Within the cavity the metric is given by the Minkowski metric.



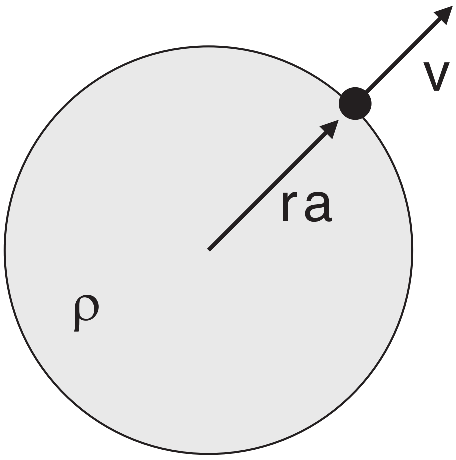
Analogous to the Gauss argument that within a spherical mass shell there is no gravitational field. (Consequence of $1/r$ potential.)



In cosmology: Assuming Robertson-Walker metric (isotropy and homogeneity) we may use Newtonian mechanics and Newtonian gravity on scales that are small compared with the Hubble distance.

2.4.2 Friedmann equation from Newtonian argument

Choose arbitrary point of universe as the center of an isotropic and homogeneous mass distribution with density ρ. Use a dimensionless radial variable r and an arbitrary scale factor a with dimension of length.



Consider radial motion of a test mass ("galaxy") m.

Radial velocity:	$v = r\,\dot{a}$
Kinetic energy:	$T = \frac{mv^2}{2} = \frac{m\,(r\,\dot{a})^2}{2}$
Enclosed mass:	$M = \frac{4\pi}{3}\,\rho\,(ra)^3$
Potential energy:	$\Phi = -G_{\mathrm{N}}\,\frac{Mm}{ra} = -\frac{4\pi}{3}\,G_{\mathrm{N}}\,\rho\,(ra)^2\,m$
Total energy:	$E_{\mathrm{tot}} = -\frac{k\,m\,r^2}{2}$

Here k is an arbitrary parameter chosen to reproduce the total energy.

Conservation of energy

$$T + V = E_{\mathrm{tot}}$$

$$\frac{m\,(r\dot{a})^2}{2} - \frac{4\pi}{3}\,G_{\mathrm{N}}\rho\,(ra)^2 m = -\frac{k\,m\,r^2}{2}$$

From this follows directly

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\,G_{\mathrm{N}}\,\rho - \frac{k}{a^2} \qquad \text{Friedmann Equation}$$

The scale a is arbitrary, so choose it such that $k = \pm 1$ or 0.

In Newtonian picture:

- $k = +1$: Total energy negative, m will eventually fall back.
- $k = -1$: Total energy positive, m is not bound, will escape.
- $k = 0$: Total energy zero, m has exactly escape velocity.

In GR picture:

- k is curvature parameter of Robertson-Walker metric.

The cosmic expansion rate is larger for a larger cosmic density that exerts greater gravity. Un-intuitive? Actually is a question of initial condition. Take $k = 0$, i.e. flat case, or in Newtonian physics, body with exactly escape velocity. For larger gravitating mass, the escape velocity is larger, so the system must have received a larger initial kick.

2.4.3 Sketch of GR derivation (3 Nov. 2009)

Curved space–time influences motion of particles and gravitating mass–energy curves space–time.

In general relativity (GR), all physics encoded in the metric and its derivatives as a function of the chosen coordinates.

Metric
Roughly corresponds to gravitational potential)

$$g_{\mu\nu}$$

Affine connection (Christoffel sympols)
Roughly corresponds to gravitational fields. With $\partial_\gamma = \partial/\partial x^\gamma$

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}\, g^{\alpha\beta}\left(\partial_\nu g_{\beta\mu} + \partial_\mu g_{\beta\nu} - \partial_\beta g_{\mu\nu}\right)$$

Curvature tensor
Space–time "flat" exactly when $R_{\alpha\beta\gamma\delta} = 0$

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2}\left(\partial_\alpha\partial_\delta g_{\beta\gamma} + \partial_\beta\partial_\gamma g_{\alpha\delta} - \partial_\alpha\partial_\gamma g_{\beta\delta} - \partial_\beta\partial_\delta g_{\alpha\gamma}\right) + g_{\mu\nu}\left(\Gamma^\mu_{\beta\gamma}\Gamma^\nu_{\alpha\delta} - \Gamma^\mu_{\alpha\gamma}\Gamma^\nu_{\beta\delta}\right)$$

Local frame can always be chosen such that $g_{\alpha\beta} = \eta_{\alpha\beta}$ and $\Gamma^\gamma_{\alpha\beta} = 0$, but not $R_{\alpha\beta\gamma\delta} = 0$. Real gravitational fields can not be globally transformed away and show up as curvature. Other measures of curvature

Ricci tensor	$R_{\alpha\gamma} = g^{\beta\delta} R_{\alpha\beta\gamma\delta}$ (symmetric)
Scalar curvature	$R = g^{\alpha\gamma} R_{\alpha\gamma} = g^{\beta\delta} g^{\alpha\gamma} R_{\alpha\beta\gamma\delta}$
Einstein tensor	$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta}\, R$

Einstein tensor the only rank-2 tensor that obeys the "contracted Bianchi identities" and is therefore "divergence free."

Motion of particles

In special relativity: Without external forces, particles move uniformly on straight lines (no acceleration)

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}s^2} = 0$$

Remains true in a local inertial frame. In the presence of real gravitational fields one needs to account for transition from one inertial frame to another by "affine connections"

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}s^2} + \Gamma^\alpha_{\mu\nu}\,\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}x^\nu}{\mathrm{d}s} = 0$$

Can be derived by principle of least action

$$\delta \int_1^2 \mathrm{d}s = 0$$

Proper time along trajectory is extreme.

Einstein equation

Matter influences space–time, so assume

Measure of curvature = $\kappa\, G_{\mathrm{N}}$ × Measure of gravitating mass-energy density

Candidate for r.h.s. is energy-momentum tensor $T_{\alpha\beta}$, is symmetric and divergence free. An observer with four-velocity U measures

Density of four momentum	$T_{\alpha\beta} U^{\beta}$
Energy density	$\rho = T_{\alpha\beta} U^{\alpha} U^{\beta}$

L.h.s. should also be a tensor of rank 2 that has the properties

- Symmetric
- Divergence free
- Vanishes for flat space time
- Constructed from $R_{\alpha\beta\gamma\delta}$ and $g_{\alpha\beta}$ and nothing else
- Linear in $R_{\alpha\beta\gamma\delta}$ for a natural measure of curvature

Einstein tensor $G_{\alpha\beta}$ the only candidate.

Comparison with Newtonian limit yields constant of proportionality and thus the *Einstein equation*

$$G_{\alpha\beta} = \frac{8\pi G_{\mathrm{N}}}{c^4}\, T_{\alpha\beta} = 8\pi G_{\mathrm{N}}\, T_{\alpha\beta} \quad \text{in natural units}$$

Friedmann equation special case for maximally symmetric case (isotropic and homogeneous).

Cosmological term

Einstein equation unique if we insist that r.h.s. vanishes in vacuum (Newtonian experience). If not, one more tensor available for r.h.s., i.e. metric itself times a constant

$$G_{\alpha\beta} = 8\pi G_{\mathrm{N}}\, T_{\alpha\beta} + \Lambda\, g_{\alpha\beta}$$

Cosmological term would not influence local physics, only relevant on cosmic scales.

Einstein initially admitted this term to be able to construct static solutions for universe. After discovery of cosmic expansion called this his "greatest blunder." However, today it seems that a cosmological term dominates the expansion of the universe.

Interpretation as vacuum energy

The energy-momentum tensor of a perfect fluid in isotropic, homogeneous space is (in terms of energy density ρ and pressure p)

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu - p\, g_{\mu\nu}$$

where $U = (1, 0, 0, 0)$ for a comoving observer.

Minkowski space: $T = \text{diag}(\rho, p, p, p)$.

Express r.h.s. of Einstein Eqn locally with Minkowski metric

$$8\pi G_\text{N} \begin{pmatrix} \rho & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix} + \Lambda \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

Cosmological term can be interpreted as vacuum energy with properties

$$\rho_\text{vac} = \frac{\Lambda}{8\pi G_\text{N}} \qquad \text{and} \qquad p_\text{vac} = -\rho_\text{vac}$$

Vacuum energy density can be positive or negative.

In quantum field theory, vacuum energy is expected to arise from vacuum fluctuations, but value much too large (or even infinite).

Yakov Zel'dovich (1914–1987), a leading Soviet cosmologist (and theoretical brain behind the Soviet atomic and hydrogen bombs) first pointed out that the observed properties of the universe constrain the allowed value for vacuum energy caused by quantum fluctuations.

2.4.4 Critical density and Ω parameter

For $k = 0$ (flat universe) Friedmann Eqn implies a unique relationship between ρ and H

$$H^2 = \frac{8\pi}{3} G_{\rm N}\, \rho_{\rm crit}$$
$$\rho_{\rm crit} = \frac{H^2}{8\pi G_{\rm N}/3}$$

Present-day universe: $H_0 = h\, 100\ {\rm km\ s^{-1}\ Mpc^{-1}}$

$$\rho^0_{\rm crit} = \frac{H_0^2}{8\pi G_{\rm N}/3} = h^2\, 1.88 \times 10^{-29}\ {\rm g\ cm^{-3}} = h^2\, 1.054 \times 10^4\ {\rm eV\ cm^{-3}}$$

Best measured value $h = 0.742 \pm 0.036$

$$\rho^0_{\rm crit} = (1.04 \pm 0.10) \times 10^{-29}\ {\rm g\ cm^{-3}} = (0.580 \pm 0.056) \times 10^4\ {\rm eV\ cm^{-3}}$$

Or roughly 6 atomic mass units per cubic meter.

Express all densities in terms of critical density or in terms of expansion rate as

$$\Omega = \frac{\rho}{\rho_{\rm crit}} = \frac{8\pi}{3} \frac{G_{\rm N}\rho}{H^2}$$

Often the Ω parameter is only used at the present epoch, in which case one uses $\rho^0_{\rm crit}$ and H_0 on the r.h.s., but for now we consider Ω to be a function of cosmic time t.

From Friedmann equation follows

$$\Omega = 1 + \frac{k}{(aH)^2} = 1 + \frac{k}{\dot{a}^2}$$

One consequence is that a flat universe always stays flat, $\Omega = 1$ is a fixed point.

If $k = \pm 1$, and if the expansion is decelerating, $\dot{a}^2$ shrinks, therefore $1/\dot{a}^2$ grows, i.e., the deviation of $\Omega_{\rm tot}$ from unity grows.

For accelerated expansion the opposite is true and $\Omega_{\rm tot}$ approaches unity: accelerated expansion "flattens" the curvature.

Correspondence between the following cases

$\rho < \rho_{\rm crit}$	$\Omega < 1$	$k = -1$	negatively curved
$\rho = \rho_{\rm crit}$	$\Omega = 1$	$k = 0$	flat
$\rho > \rho_{\rm crit}$	$\Omega > 1$	$k = +1$	positively curved

Friedmann equation can also be written as

$$H^2 a^2 = \frac{k}{\Omega - 1}$$

If Hubble parameter and gravitating mass-energy density are measured separately, radius of curvature can be determined as

$$a = \frac{1}{H\sqrt{|\Omega - 1|}}$$

Today we know from the CMB temperature fluctuations, to be discussed later, that today

$$\Omega_0 = 1.011 \pm 0.012 \qquad \Rightarrow \qquad |1 - \Omega_0| \lesssim 0.023 \qquad \Rightarrow \qquad a_0 \gtrsim 7\, H_0^{-1}$$

2.4.5 Empty space (Milne universe)

In empty space $\rho = 0$ and the Friedmann equation is

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = -\frac{k}{a^2} \qquad \Rightarrow \qquad \dot{a}^2 = -k$$

Flat geometry: $k = 0 \rightarrow$ Static Euclidean space.

Positively curved empty space not possible.

$k = -1$ (Milne universe)

$$\dot{a} = \pm 1$$

Scale factor grows or shrinks with the speed of light.

Expanding Milne universe:

$$a(t) = t$$

In this case the expansion is linear and the age of the universe is exactly $t_0 = H_0^{-1}$.