Homework Set (Week 06)
Introduction to Astroparticle Physics

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1 Glauber states

A natural basis for possible states of the radiation field are the number states $|n\rangle_{\mathbf{k},\epsilon}$ for each mode $\mathbf{k}$ and polarization ϵ. Let us consider a single mode and drop the indices $\mathbf{k}$ and ϵ. Macroscopic fields, such as a radio wave or a laser beam, are highly occupied and have nonvanishing expectation values of the electric or magnetic field. They are represented by Glauber states, a coherent superposition of number states. With a complex number α, a Glauber state is

$$|\alpha\rangle = \mathrm{e}^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\,|n\rangle = \mathrm{e}^{-|\alpha|^2/2}\,\mathrm{e}^{\alpha a^\dagger}|0\rangle$$

(i) Show that $|\alpha\rangle$ is normalized: $\langle\alpha|\alpha\rangle = 1$. (ii) Show that $|\alpha\rangle$ is an eigenstate of the destruction operator with eigenvalue α. (iii) What is the average occupation number $\langle\alpha|a^\dagger a|\alpha\rangle$. (iv) What is the root mean square (rms) variation of the occupation number? (v) Can you imagine why a laser beam comes out as a Glauber state and not as a number eigenstate?

2 Electron-positron annihilation

Consider the process $e^+e^- \to 2\gamma$. An exact QED calculation yields the cross section

$$\sigma_{\rm ann} = \frac{\pi\alpha^2}{m_e^2}\,\frac{1-v^2}{2v}\left[\frac{3-v^4}{2v}\ln\left(\frac{1+v}{1-v}\right) - 2 + v^2\right]$$

where v is the velocity of the e^- or e^+ in the CM frame. (i) Express v in terms of the CM squared energy s. (ii) Derive the cross section in the non-relativistic limit ($v \ll 1$) and ultrarelativistic limit ($s \gg m_e^2$). (iii) The nonrelativistic cross section is found to diverge with v^{-1}. Show that in an electron-positron plasma the annihilation rate remains finite. (Consider the inverse mean free path of a positron in a gas of electrons with density n_e.)

3 Pair creation

Consider now the reverse process $2\gamma \to e^+e^-$ where the cross section is found to be

$$\sigma_{\text{pair}} = 2v^2\,\sigma_{\text{ann}}$$

where v is the velocity of the final-state e^- or e^+ in the CM frame. Of course, the initial-state photons must have at least the threshold energy $\omega \geq m_e$ to be able to produce a pair. (i) Derive the cross section in the non-relativistic limit ($v \ll 1$) and ultrarelativistic limit ($s \gg m_e^2$). (ii) Consider a high-energy photon ($E \gg m_e$) propagating in a bath of low-energy ones ($\omega \ll m_e$). Which energy E is required to be able to produce an electron-positron pair? (iii) Assuming the low-energy photon is a typical photon from the cosmic microwave background ($T = 2.725$ K), what is E? (iv) How far will photons with energies around this value travel in the universe? (v) The MAGIC high-energy gamma-ray telescope has recently observed the quasar 3C 279 at a redshift of $z = 0.536$ in energies 100–300 GeV.[1] Which distance does this correspond to? The energy range corresponds to which typical background photon range, assuming we are around the maximum of the pair-creation cross section? Which approximate upper limit on the exta-galactic background light (EBL) can be inferred?

4 Transparency of the universe at early times

Returning to the early universe, at which redshift would it have become transparent to thermal photons, assuming all electrons are free? In reality it becomes transparent earlier at redshift $z \sim 1100$. Can you imagine what happened at that redshift?

[1] J. Albert et al. (MAGIC Collaboration), "Very-high-energy gamma rays from a distant quasar: How transparent is the universe?", Science 320, 1752 (2008) and http://arXiv.org/abs/0807.2822