Homework Set (Week 01)
Introduction to Astroparticle Physics

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1 Cosmic expansion without center

According to Hubble's law all galaxies show on average a recession velocity that increases linearly with distance: $v_{\rm rec} = H_0 D$. Use the linearity of this relation to show that an observer on any other galaxy will likewise see himself in the apparent center of the expansion. The Hubble flow looks the same to all observers.

2 Doppler effect (kinematical redshift)

Consider light of wavelength λ emitted by a source that moves relative to an observer. Which wavelength or frequency is measured by the observer? Consider in particular the limiting cases of (i) parallel motion (the source moves along the line of sight) and (ii) transverse motion. Interpretation of the results?

3 Redshift by gravitation

(i) A photon moves along a gravitational field (acceleration g, approximately homogeneous). After overcoming a height difference H, what is the photons's redshift? [Hint: Use the equivalence between a homogeneous gravitational field and an accelerated system of reference ("Einstein elevator"). In the freely falling frame the absorber acquires a velocity during the time that passes between emission and absorption of the light wave.] Express the result as a difference between the gravitational potentials at the emission and absorption points.

(ii) How large is therefore the redshift of a spectral line emitted from a star of radius R and mass M, observed at a large distance? The solar mass is $M_\odot = 2 \times 10^{30}$ kg and its radius $R_\odot = 6.96 \times 10^5$ km. How large is the redshift here? For a neutron star, typical values are $M_{\rm NS} = 1.4\, M_\odot$ und $R_{\rm NS} = 12$ km. Redshift here in the Newtonian approximation?

4 Friction by Hubble expansion

Consider a body (e.g. a galaxy) moving with a nonrelativistic speed v relative to the Hubble flow ("peculiar velocity"). It will slow down relative to the Hubble flow by cosmic expansion. (i) How large is the deceleration as a function of v and of the Hubble parameter H_0? (ii) Compare the result for the Earth on its orbit with the gravitational acceleration caused by the Sun? ($H_0 = 74$ km s^{-1} Mpc^{-1}, $M_\odot = 2 \times 10^{30}$ kg, average Earth-Sun distance 150 million km). (iii) How do these numbers compare in a hydrogen atom relative to the Coulomb acceleration, assuming a typical distance and velocity of the electron?

Units and Dimensions

In the astrophysical context, frequently occurring units of length are centimeters, (light) seconds, light years, and parsecs. Conversion factors are given in Table 1. For example, 1 pc = 3.26 ly. Moreover, the cgs system remains in common use.

Using both centimeters and (light) seconds as units of length implies a system of units where the speed of light c is dimensionless and equal to unity. In these lectures I always use natural units where Planck's constant $\hbar$ and Boltzmann's constant k_B are also dimensionless and equal to unity. This implies that $(\text{length})^{-1}$, $(\text{time})^{-1}$, mass, energy, and temperature can all be measured in the same units by virtue of $x = ct$, $E = mc^2$, $E = \hbar\omega$, $\omega = 2\pi/t$, and $E = k_\mathrm{B}T$. In Table 2 conversion factors are given. For example, $1\ \mathrm{K} = 0.862\times10^{-4}$ eV or $1\ \mathrm{erg} = 0.948\times10^{27}\ \mathrm{s}^{-1}$. Also note that $1\ \mathrm{erg} = 10^{-7}$ Joule. In cosmology, the only constant of nature that remains dimensionful is Newton's constant. In natural units it has the dimension of an inverse squared mass,

$$G_\mathrm{N} = \frac{1}{m_\mathrm{Pl}^2} \tag{1}$$

with

$$m_\mathrm{Pl} = 1.221 \times 10^{19}\ \mathrm{GeV} \tag{2}$$

the "Planck mass" m_Pl. In principle, one can also choose units where all energies, masses etc. are measured in units of the Planck mass. Sometimes the Planck mass is defined as the "reduced Planck mass" by $m_\mathrm{Pl} \to m_\mathrm{Pl}/\sqrt{8\pi} = 2.44 \times 10^{18}$ GeV.

Tabelle 1: Conversion factors between different units of length.

	cm	s	ly	pc
cm	1	0.334×10^{-10}	1.06×10^{-18}	0.325×10^{-18}
s	2.998×10^{10}	1	0.317×10^{-7}	0.973×10^{-8}
ly	0.946×10^{18}	3.156×10^{7}	1	0.307
pc	3.08×10^{18}	1.028×10^{8}	3.26	1

Tabelle 2: Conversion factors in the system of natural units.

	s^{-1}	cm^{-1}	K	eV	amu[a]	erg	g
s^{-1}	1	0.334×10^{-10}	0.764×10^{-11}	0.658×10^{-15}	0.707×10^{-24}	1.055×10^{-27}	1.173×10^{-48}
cm^{-1}	2.998×10^{10}	1	0.2289	1.973×10^{-5}	2.118×10^{-14}	3.161×10^{-17}	0.352×10^{-37}
K	1.310×10^{11}	4.369	1	0.862×10^{-4}	0.926×10^{-13}	1.381×10^{-16}	1.537×10^{-37}
eV	1.519×10^{15}	0.507×10^{5}	1.160×10^{4}	1	1.074×10^{-9}	1.602×10^{-12}	1.783×10^{-33}
amu	1.415×10^{24}	0.472×10^{14}	1.081×10^{13}	0.931×10^{9}	1	1.492×10^{-3}	1.661×10^{-24}
erg	0.948×10^{27}	0.316×10^{17}	0.724×10^{16}	0.624×10^{12}	0.670×10^{3}	1	1.113×10^{-21}
g	0.852×10^{48}	2.843×10^{37}	0.651×10^{37}	0.561×10^{33}	0.602×10^{24}	0.899×10^{21}	1

[a] Atomic mass unit.

The most confusing issue is that of electromagnetic field strengths. The square of a field strength is an energy density (erg/cm^3) which, in natural units, is (energy)^4 or (length)$^{-4}$. Thus, an electric or magnetic field may be measured, for example, in eV^2 or cm^{-2}. In natural units, electric charges are dimensionless numbers.

However, there is a general ambiguity in the definition of charges and field strengths because only their product (a force on a charged particle) is operationally defined. All physical quantities stay the same if the charges are multiplied with an arbitrary number and the field strengths are divided by it. However, the fine-structure constant $\alpha \approx 1/137$ is dimensionless in all systems of units, and its value does not depend on this arbitrary choice. If e is the charge of the electron one has $\alpha = e^2/4\pi$ in the rationalized system of (natural) units which is always used in modern works on field theory. The energy density of an electromagnetic field is then $\frac{1}{2}(E^2 + B^2)$. In the older literature and also in the plasma physics literature, unrationalized units are used where $\alpha = e^2$ and the energy density is $(E^2 + B^2)/8\pi$.

In the astrophysical literature the cgs system remains very popular where magnetic fields are measured in Gauss (G). Confusingly, this system happens to be an unrationalized one. Field strengths given in Gauss can be translated into our rationalized natural units by

$$1\ \mathrm{G} \to \sqrt{\frac{1\ \mathrm{erg/cm^3}}{4\pi}} = 1.953\times10^{-2}\ \mathrm{eV}^2 = 0.502\times10^{8}\ \mathrm{cm}^{-2}, \tag{3}$$

where I have converted erg and cm^{-1} into eV according to Table 2. The energy density of a magnetic field of strength 1 G is, therefore, $\frac{1}{2}\,(1.953\times10^{-2}\ \mathrm{eV}^2)^2 = 1.908\times10^{-4}\ \mathrm{eV}^4 = 3.979\times10^{-2}\ \mathrm{erg\ cm^{-3}} = (1/8\pi)\ \mathrm{erg\ cm^{-3}}$. Note also that a field strength of 1 G corresponds, in SI units, to 10^{-4} Tesla.

It is sometimes useful to measure very strong magnetic fields in terms of a critical field strength $B_{\rm crit}$ which is defined by the condition that the quantum energy corresponding to the classical cyclotron frequency $\hbar\,(eB/m_e c)$ of an electron equals its rest energy $m_e c^2$ so that in natural units

$$B_{\rm crit} = m_e^2/e. \tag{4}$$

Note that the Lorentz force on an electron in this field is proportional to $eB_{\rm crit}$ so that the electron charge cancels. Hence, Eq. (4) is the same in a rationalized or unrationalized system of units. In our rationalized units $e = \sqrt{4\pi\alpha} = 0.303$ so that $B_{\rm crit} = (0.511\ \mathrm{MeV})^2/0.303 = 0.862 \times 10^{12}\ \mathrm{eV}^2$ which, with Eq. (3), corresponds to 4.413×10^{13} G.