An Introduction to the String Theory Swampland (Lectures for BUSSTEPP 2018)

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Abstract

The aim of these lecture notes is to present a basic introduction to the string theory Swampland. The first two lectures will be a quick introduction to string theory. The third lecture will cover a simple circle compactification and will form the first encounter with the Swampland Distance Conjecture. In the fourth lecture we will consider the gauge field part of the compactification and will encounter the Weak Gravity Conjecture.

Contents

1 Introduction

String theory postulates that at sufficiently high energy scales the relevant degrees of freedom for describing the universe are extended in nature rather than point-like. The best understood, but not necessarily the most fundamental, of these extended objects are one-dimensional strings. These strings posses the remarkable property that after quantisation they lead to the force of gravity. In this sense string theory is a theory which combines quantum physics with gravitational physics, and is therefore a valuable window into the nature of quantum gravity. It is a very rich theory, and we certainly do not fully understand it. Nonetheless, from what we do understand we have learned a huge amount about the nature of quantum gravity.

Strings are expected to be very small, so with an associated energy scale around the Planck scale

$$M_p \sim 10^{18}\text{GeV} . \tag{1.1}$$

One thing which this implies is that we do not expect to probe strings directly, for example, by creating them in accelerators. One might be tempted to dismiss them completely by saying that we can always work with an Effective Quantum Field Theory (EQFT) at energy scales far below the Planck mass, and by the principle of separation of scales string effects should only induce small corrections to this theory. This is certainly true to some extent, we do not need to know string theory to calculate processes in the Standard Model. The Standard Model is a special EQFT because we can determine its parameters experimentally.[1] When we think about physics beyond the Standard Model, there is a vast space of effective theories that we could consider. In that case we would like some guidance as to which type of theories are theoretically consistent. One such guide is the fact that whatever theory we consider it should be able to consistently couple to quantum gravity. To determine this we need a sufficiently detailed theory of quantum gravity where we can ask such quantitative questions. String theory is the only framework where we can do this.

One might have hoped that string theory would have been sufficiently constrained to single out some very specific low-energy effective theory as the only one compatible with quantum gravity. This may still be the case, remember that we do not yet fully understand string theory. However, within our current understanding, there is no principle which we know that can pick out a specific such theory. Rather, it appears that the range of low-energy effective theories that can arise in string theory is huge. This is the so-called *Landscape* of string theory. Ironically, while the Landscape is huge, there is still not a single known way to embed the Standard Model, of particle physics and cosmology, in string theory. So while our universe is in principle consistent with string theory, in practice we still do not know how its embedding could work in detail. It is therefore an important and interesting task to work out the 'details' of this embedding, and this is a large part of the research field of String Phenomenology.

In these lectures we will be concerned with a related, though qualitatively different, question. Namely, are there low-energy effective field theories which appear perfectly self-consistent within the usual rules of quantum field theory but yet cannot be in the Landscape of string theory? So theories which could never be consistent within string theory, and therefore quite likely, in quantum gravity. Such inconsistent theories are termed to be in the string theory *Swampland* [1]. There is good evidence to suggest that the Swampland is infinitely larger than the Landscape, or in other words, there are infinitely more self consistent effective quantum field theories than those which could arise from string theory. An illustration of the set of effective quantum field theories with respect to the Landscape and Swampland is shown in figure 1. It is clear that determining which type of theories are in the Swampland, rather than the Landscape, would be a very valuable guide to physics beyond the Standard Model. If an effective theory cannot

[1] Even though we can measure the parameters it is still, of course, an interesting and worthwhile question to ask why they have the values that they do.

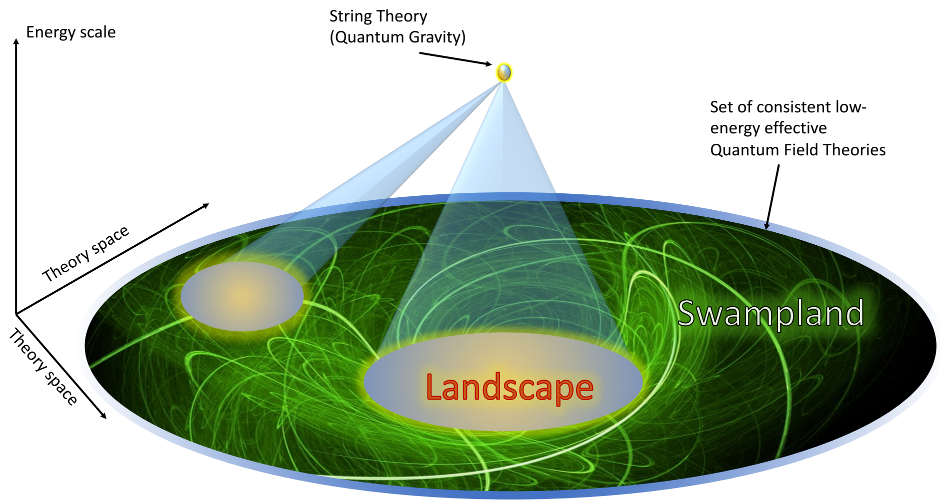

Figure 1: A schematic illustration of the space of self-consistent effective quantum field theories. The sub-set which could arise from string theory is the string Landscape, while all the other theories are in the string Swampland.

be consistently coupled to quantum gravity it is much less likely to be on the right path to describing nature.

Determining the rules for which theories are in the string theory Landscape and which are in the Swampland is a difficult task. Not least because we do not fully understand string theory itself, and so how could we claim that an effective low-energy theory would be incompatible with it? For this reason many of the Swampland criteria, so criteria for an effective theory to be consistent with string theory, are termed only as conjectures rather than proven rules. Nonetheless, there are good reasons to still take them seriously. The first is that often, while not proven, there is significant evidence for some of the conjectures[2]. For example, one Swampland criterium is that an EQFT can not have an exact continuous global symmetry. While this is still not proven, it is widely accepted within theoretical physics. The second reason to seriously study Swampland criteria is that they could lead us to new microscopic principles. So we can view them as interesting properties of string theory that seem to be hinting at some new microscopic physics. Indeed, AdS/CFT was first discovered as a particular setup in string theory, but is by now a much more general microscopic principle of quantum gravity. Yet another reason is that Swampland criteria are useful guides for where to search in the Landscape for particular types of theories. So even if we cannot prove that a certain type of EQFT could not arise from string theory, in studying this problem we can gain insights into what are the obstructions to embedding in string theory and what are the more promising avenues.

In practice, the lectures will only touch upon the subject of the string theory Swampland. The first two lectures will provide a very quick introduction to string theory. In the third lecture we will encounter the first Swampland criterium which is known as the Swampland Distance Conjecture [2, 3]. It states that when a scalar field has a very large expectation value, at least as large as M_p, there must be a tower of states which become exponentially light as the expectation

[2]There are a number of Swampland-type conjectures in the literature, some with more evidence for them than others. In these lectures we will only consider conjectures which have significant evidence for them.

value is increased further. In the fourth lecture we will encounter the Swampland criterium known as the Weak Gravity Conjecture [4]. This says that in an effective theory with a $U(1)$ gauge symmetry there must exist a charged particle with a charge larger than its mass in Planck units. So theories which do not respect these two properties are said to be in the Swampland rather than the Landscape.

The bulk of the lectures will cover basic aspects of string theory. This is standard material and is covered by almost all textbooks on the subject, see for example [5]. I also recommend the very nice online notes in [6]. The subject of the Swampland is much more advanced. There is some discussion of it in the textbook [7]. There are also some reviews [8], but they are rather advanced.

In these notes we set $\hbar = c = 1$. We work in mostly-plus signature for the metric, so the flat-space metric is

$$\eta_{\mu\nu} = \text{diag}\ (-,+,+,...,+)\ . \tag{1.2}$$

Our units are chosen such that, in whatever number of dimensions we are working in, the (reduced) Planck mass is set to one $M_p = 1$. We will sometimes reinstate it when it plays an important role.

2 Lecture 1: The classical bosonic string

To begin let us consider a point particle. We let it propagate in D-dimensional spacetime, so on $\mathbb{R}^{1,D-1}$. To describe the motion of the particle we can split the coordinates into $X^0 = t$ and X^i, where $i = 1, ..., D-1$. Then its motion is associated to a world-line γ which specifies the $X^i\left(X^0\right)$ as a function of X^0. We can also describe the particle world-line in relativistic coordinates by using a world-line parameter τ so that γ specifies $X^\mu\left(\tau\right)$, where $\mu = 0, 1, ..., D-1$,

$$\gamma\ :\ \tau \hookrightarrow X^\mu\left(\tau\right) \in \mathbb{R}^{1,D-1}\ . \tag{2.1}$$

These ways of describing the particle motion are illustrated in figure 2.

2.1 The Nambu-Goto action for a particle

In Minkowski space the line element is taken as

$$ds^2 = \eta_{\mu\nu} dX^\mu dX^\nu\ . \tag{2.2}$$

We can then write an action for the particle, called the Nambu-Goto action, which is just the length of its (time-like) worldline

$$S_{\text{NG}} = -m \int_\gamma \sqrt{-ds^2} = -m \int_\gamma \left(-\dot{X}^2\right)^{\frac{1}{2}} d\tau\ , \tag{2.3}$$

where $\dot{X}^2 = \eta_{\mu\nu}\dot{X}^\mu\dot{X}^\nu$ and $\dot{X}^\mu = \frac{\partial X^\mu}{\partial\tau}$. The constant parameter m will be related to the mass of the particle.

To see that this action correctly describes the motion of the particle we can associate to it a Lagrangian

$$S_{\text{NG}} = -m \int_\gamma d\tau L\left(\tau\right)\ . \tag{2.4}$$

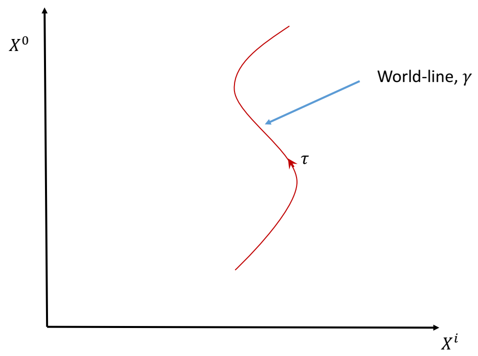

Figure 2: Figure illustrating the world-line of a particle.

The canonical momentum is then

$$p_\mu = \frac{\partial L}{\partial \dot{X}^\mu} = \frac{m\dot{X}_\mu}{\left(-\dot{X}^2\right)^{\frac{1}{2}}} \ . \tag{2.5}$$

We therefore arrive at the constraint

$$p^2 + m^2 = 0 \ , \tag{2.6}$$

which shows that m is indeed the mass of the particle. The equation of motion for X^μ in turn gives

$$m\ddot{X}^\mu = 0 \ , \tag{2.7}$$

so the particle is freely propagating.

2.2 The Polyakov action for a particle

We can also consider a different action for the particle, called the Polyakov action. It is defined as

$$S_P = \frac{1}{2}\int_\gamma d\tau e\left(\tau\right)\left[\frac{1}{e\left(\tau\right)^2}\dot{X}^2 - m^2\right] \ . \tag{2.8}$$

The degree of freedom $e\left(\tau\right)$ is called the world-line metric. Its equation of motion gives

$$\dot{X}^2 + m^2 e\left(\tau\right)^2 = 0 \ . \tag{2.9}$$

Since this is an algebraic constraint we can use it to eliminate the world-line metric in the Polyakov action which gives

$$S_P = \frac{1}{2}\int_\gamma d\tau e\left(\tau\right)\left[-2m^2\right] = -m\int_\gamma \left(-\dot{X}^2\right)^{\frac{1}{2}} d\tau = S_{\mathrm{NG}} \ . \tag{2.10}$$

Therefore, the Polyakov and Nambu-Goto actions are classically equivalent. However, the utility of the Polyakov action is that it is much easier to quantise.

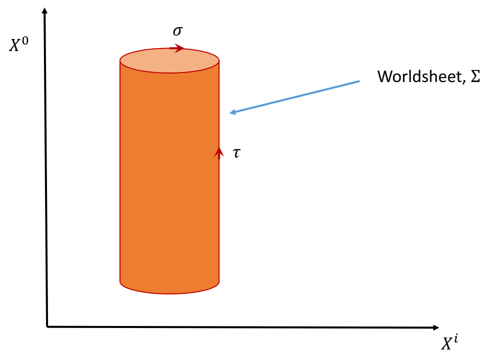

Figure 3: Figure illustrating the worldsheet of a string.

2.3 The String Worldsheet

We now go through the same process we did for the particle but for a string. A string sweeps out a two-dimensional worldsheet Σ parameterised by two coordinates (σ, τ). So we have

$$\Sigma \; : \; (\sigma, \tau) \hookrightarrow X^{\mu}(\sigma, \tau) \in \mathbb{R}^{1,D-1} \; . \tag{2.11}$$

We take the coordinates to have the ranges

$$0 \leq \sigma \leq 2\pi \; , \quad \tau \in \mathbb{R} \; , \tag{2.12}$$

We will mostly be concerned with closed, rather than open, strings. We therefore identify

$$\sigma \simeq \sigma + 2\pi \; . \tag{2.13}$$

The coordinates therefore parameterise the string worldsheet as shown in figure 3. We will often denote

$$\{\sigma, \tau\} \equiv \xi^a \; , \quad a = 0, 1 \; . \tag{2.14}$$

We want to describe the dynamics of the string through an action. The associated Polyakov action is

$$S_P = -\frac{T}{2} \int_{\Sigma} d^2\xi \left(-\det h\right)^{\frac{1}{2}} h^{ab}(\xi)\, \partial_a X^{\mu}(\xi)\, \partial_b X^{\nu}(\xi)\, \eta_{\mu\nu} \; . \tag{2.15}$$

Here $h_{ab}(\xi)$ is the worldsheet metric. T is the string tension, which is often denoted in terms of a parameter α' as

$$T \equiv \frac{1}{2\pi\alpha'} \; . \tag{2.16}$$

Note that other similar scales are the string length l_s and the string scale M_s, defined as

$$l_s \equiv \sqrt{\alpha'} \; , \quad M_s \equiv \frac{1}{2\pi\sqrt{\alpha'}} \; . \tag{2.17}$$

Object	Degrees of freedom
h_{ab}	$\frac{1}{2}D\left(D+1\right)$
Diffeomorphisms	D
Weyl	1

Table 2.1: Table showing the degrees of freedom in the metric and symmetries in D-dimensions.

The mass dimensions of the coordinates are $[X^\mu] = -1$ and $[\xi^a] = 0$.

We should think of the worldsheet action (2.15) as specifying a two-dimensional theory with scalar fields $X^\mu\left(\xi\right)$. Such theories are called sigma models. The space-time in which the string propagates, parameterised by the X^μ, is known as the Target space of the worldsheet theory. The metric on that spacetime, here $\eta_{\mu\nu}$, is the metric on the field space of the scalar fields X^μ. So strings propagating in different target spaces have different metrics on the scalar field spaces.

2.4 The worldsheet symmetries

The worldsheet theory (2.15) is invariant under local diffeomorphisms

$$\xi^a \rightarrow \tilde{\xi}^a\left(\xi\right) \ . \tag{2.18}$$

It is also invariant under Weyl transformations, which are defined as

$$\delta X^\mu = 0 \ , \quad h_{ab} \rightarrow \tilde{h}_{ab} = e^{2\Lambda(\xi)} h_{ab} \ . \tag{2.19}$$

To see this directly note that under (2.19) we have $\sqrt{-\det h} \rightarrow e^{2\Lambda(\xi)}\sqrt{-\det h}$.

The worldsheet symmetries can be used to completely fix the worldsheet metric h_{ab}. It is worth looking at this generally. For a D-dimensional theory, we can count the number of degrees of freedom in the metric, a symmetric tensor, and in the diffeomorphism and Weyl symmetries, these are shown in table 2.1. We see that for $D = 2$, so a string, the number of symmetry parameters is the same as the degrees of freedom of the metric. Using the symmetries we can therefore set

$$h_{ab} = \eta_{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \ . \tag{2.20}$$

This is called flat gauge.

It is important to note that even though we can gauge away the metric we must still impose its equations of motion. The equations of motion for the metric correspond to the vanishing of the energy momentum tensor of the theory

$$T_{ab} = 0 \ , \tag{2.21}$$

where

$$T_{ab} \equiv \frac{4\pi}{\sqrt{-\det h}} \frac{\delta S_P}{\delta h^{ab}} \ . \tag{2.22}$$

The resulting constraint (2.21) is called a Virasoro constraint, and it will play an important role when we quantise the string.

From table 2.1 we see that strings are special extended objects. For $D > 2$ we cannot use the symmetries of the Polyakov action to remove the metric degrees of freedom. This makes the extended objects with $D > 2$ much more difficult to quantise. From a modern perspective, we do not think of strings as any more 'fundamental' than other extended objects, but what makes them special is that they can be readily quantised.

In flat gauge the Polyakov action reduces to the action of a set of free scalar fields

$$S_P = \frac{T}{2}\int_\Sigma d\sigma d\tau \left[(\partial_\tau X)^2 - (\partial_\sigma X)^2\right] . \tag{2.23}$$

It is convenient to go to so-called light-cone coordinates

$$\xi^\pm \equiv \tau \pm \sigma \ , \quad \partial_\pm \equiv \frac{1}{2}\left(\partial_\tau \pm \partial_\sigma\right) . \tag{2.24}$$

In light-cone coordinates the Polyakov action reads

$$S_P = T\int_\Sigma d\xi^+ d\xi^- \partial_+ X \partial_- X . \tag{2.25}$$

The equations of motion for the X^μ are readily obtained

$$\partial_+\partial_- X^\mu = 0 . \tag{2.26}$$

We can therefore write X^μ as a sum of left-moving and right-moving waves along the string

$$X^\mu = X^\mu_L\left(\xi^+\right) + X^\mu_R\left(\xi^-\right) . \tag{2.27}$$

And we must impose periodic boundary conditions $X^\mu\left(\tau, \sigma = 0\right) = X^\mu\left(\tau, \sigma = 2\pi\right)$. The most general solution is

$$\begin{aligned} X^\mu_R\left(\xi^-\right) &= \frac{1}{2}\left(x^\mu + c^\mu\right) + \frac{1}{2}\alpha' p^\mu_R \xi^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}\ ,\ n\neq 0}\frac{1}{n}\alpha^\mu_n e^{-in\xi^-} \ , \\ X^\mu_L\left(\xi^+\right) &= \frac{1}{2}\left(x^\mu - c^\mu\right) + \frac{1}{2}\alpha' p^\mu_L \xi^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}\ ,\ n\neq 0}\frac{1}{n}\tilde{\alpha}^\mu_n e^{-in\xi^+} . \end{aligned} \tag{2.28}$$

Here x^μ, c^μ, p^μ_L, p^μ_R, α^μ_n and $\tilde{\alpha}^\mu_n$ are constants. Periodicity in σ implies

$$p^\mu_L = p^\mu_R \equiv p^\mu . \tag{2.29}$$

If we average X^μ over the string we have

$$q^\mu \equiv \frac{1}{2\pi}\int_0^{2\pi} d\sigma X^\mu = x^\mu + \alpha' p^\mu \tau . \tag{2.30}$$

So x^μ is the centre of mass position, and p^μ is the target space momentum.

It will be important for later to note that even after the gauge fixing the worldsheet metric, there are still residual symmetries

$$\xi^\pm \to \tilde{\xi}^\pm\left(\xi^\pm\right) . \tag{2.31}$$

These are associated to so-called conformal Killing vectors.[3]

[3]Note that these are an infinitesimal subset of diffeomorphisms because the transformations restrict to only one coordinate. This is consistent with the earlier counting argument in table 2.1.

3 Lecture 2: The string spectrum

In this lecture we study the spectrum of excitations of the string. So far we have considered the string in a classical sense, but in order to study the spectrum of excitations we need to quantise it. We will do this using so-called light-cone quantisation. The starting point is to introduce target-space light-cone coordinates

$$X^{\pm} \equiv \frac{1}{\sqrt{2}}\left(X^0 \pm X^{D-1}\right) \;,\;\; X^i \;,\;\; i=1,...,D-2 \;. \tag{3.1}$$

The target-space metric then becomes

$$\eta_{+-}=\eta_{-+}=-1 \;,\;\; \eta_{ij}=\delta_{ij} \;. \tag{3.2}$$

And this gives an inner product

$$X^2 = -2X^+X^- + \dot{X}^i\dot{X}^i \;. \tag{3.3}$$

Consider now the expansion for X^+, this reads

$$X^+\left(\tau,\sigma\right) \;\; = \;\; x^+ + \alpha' p^+ \tau + i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}\;,\;n\neq 0}\frac{1}{n}\alpha^+_n e^{-in\xi^-} + i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}\;,\;n\neq 0}\frac{1}{n}\tilde{\alpha}^+_n e^{-in\xi^+} \;. \tag{3.4}$$

Recall that we have a residual infinite dimensional symmetry (2.31) after going to light-cone gauge. We can use this to set all the oscillator modes of the X^+ to zero. In that gauge we then have

$$X^+\left(\tau,\sigma\right) = x^+ + \alpha' p^+ \tau \;. \tag{3.5}$$

Now recall that we must impose the Virasoro constraints (2.21) on the theory. It can be shown that these imply

$$\partial_{\pm}X^- = \frac{1}{\alpha' p^+}\left(\partial_{\pm}X^i\right)^2 \;. \tag{3.6}$$

Therefore, we see that also the X^- oscillators are given in terms of the transverse oscillators in X^i. So only the transverse oscillators are independent degrees of freedom.

The usefulness of the target-space light-cone gauge is therefore that only the X^i contain physically independent oscillators. This is useful because it automatically projects out two polarizations of the string which are unphysical. This is completely analogous to how a Maxwell field in four dimensions only has two physical polarizations.

The action in light-cone gauge reads

$$\begin{aligned} S_{LC} &= \frac{1}{4\pi\alpha'}\int_{\Sigma} d\tau d\sigma \left[\left(\partial_\tau X^i\right)^2 - \left(\partial_\sigma X^i\right)^2 + 2\left(-\partial_\tau X^+\partial_\tau X^- + \partial_\sigma X^+\partial_\sigma X^-\right)\right] \\ &= \frac{1}{4\pi\alpha'}\int_{\Sigma} d\tau d\sigma \left[\left(\partial_\tau X^i\right)^2 - \left(\partial_\sigma X^i\right)^2\right] - \int d\tau p^+ \partial_\tau q^- \\ &\equiv \int d\tau L \;, \end{aligned} \tag{3.7}$$

where we define

$$q^- \equiv \frac{1}{2\pi}\int_0^{2\pi} d\sigma X^- \;. \tag{3.8}$$

From this Lagrangian we can define canonical momenta

$$p_- \equiv \frac{\partial L}{\partial \dot{q}^-} = -p^+ \ , \ \ \Pi_i \equiv \frac{\partial L}{\partial \dot{X}^i} = \frac{\dot{X}_i}{2\pi\alpha'} \ . \tag{3.9}$$

We then quantize the theory by introducing the canonical commutation relations

$$\left[X^\mu\left(\tau,\sigma\right), \Pi^\mu\left(\tau,\sigma'\right)\right] = i\eta^{\mu\nu}\delta\left(\sigma-\sigma'\right) \ , \tag{3.10}$$

which give

$$\begin{aligned} \left[x^i, p^i\right] &= i\delta_{ij} \ , \\ \left[p^+, q^-\right] &= i \ , \\ \left[\alpha^i_m, \alpha^j_n\right] &= m\delta_{n+m,0}\delta_{ij} \ , \\ \left[\tilde{\alpha}^i_m, \tilde{\alpha}^j_n\right] &= m\delta_{n+m,0}\delta_{ij} \ . \end{aligned} \tag{3.11}$$

We therefore follow the usual procedure for quantisation, as in quantum field theory, by promoting the oscillator modes to operators acting on a Hilbert space. The α^i_{-n} with $n>0$ are creation operators acting on a vacuum state $|0,p\rangle$. While the α^i_n with $n>0$ are annihilation operators.

Recall that there are no oscillators to quantise for X^+, while the X^- oscillators are given in terms of the X^i. Explicitly this reads

$$\alpha^-_n = \frac{1}{2\sqrt{2\alpha'}p^+}\sum_{m=-\infty}^{m=\infty}\alpha^i_{n-m}\alpha^i_m \ . \tag{3.12}$$

When we quantise the theory the ordering of the α's matters, and so we should write things in terms of normal ordered products and a normal ordering constant a which we needs to determine

$$\alpha^-_n = \frac{1}{2\sqrt{2\alpha'}p^+}\left(\sum_{m=-\infty}^{m=\infty} :\alpha^i_{n-m}\alpha^i_m: -a\delta_{n,0}\right) \ , \tag{3.13}$$

where

$$:\alpha^i_m\alpha^i_n:\equiv \begin{cases} \alpha^i_m\alpha^i_n & \text{for } m\le n \\ \alpha^i_n\alpha^i_m & \text{for } n<m \end{cases} \ . \tag{3.14}$$

This is the canonical quantisation procedure.

3.1 Criticality and Lorentz Invariance

The quantisation of the theory was performed in special target-space light-cone coordinates. It is therefore not clear that the quantum theory respects Lorentz invariance. Indeed, we will see that requiring the preservation of target-space Lorentz invariance also in the quantum theory will place rather stringent constraints on the theory.

In general, the generators of Lorentz transformations are

$$J^{\mu\nu} = \int_0^{2\pi} d\sigma\left(X^\mu\Pi^\nu - X^\nu\Pi^\mu\right) \equiv l^{\mu\nu} + E^{\mu\nu} + \tilde{E}^{\mu\nu} \ , \tag{3.15}$$

where

$$\begin{aligned} l^{\mu\nu} &= x^\mu p^\nu - x^\nu p^\mu \ , \\ E^{\mu\nu} &= -i\sum_{n=1}^{\infty}\frac{1}{n}\left(\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n\right) \ , \\ \tilde{E}^{\mu\nu} &= -i\sum_{n=1}^{\infty}\frac{1}{n}\left(\tilde\alpha^\mu_{-n}\tilde\alpha^\nu_n - \tilde\alpha^\nu_{-n}\tilde\alpha^\mu_n\right) \ . \end{aligned} \tag{3.16}$$

Now the Lorentz algebra reads

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\eta^{\mu\rho}J^{\nu\sigma} + i\eta^{\nu\sigma}J^{\mu\rho} - i\eta^{\mu\sigma}J^{\nu\rho} - i\eta^{\nu\rho}J^{\mu\sigma} \ . \tag{3.17}$$

In particular,

$$\left[J^{-i}, J^{-j}\right] = i\eta^{--}J^{ij} + i\eta^{ij}J^{--} - i\eta^{-j}J^{i-} - i\eta^{i-}J^{-j} = 0 \ . \tag{3.18}$$

However, an explicit calculation yields

$$\left[J^{-i}, J^{-j}\right] = -\frac{1}{(p^+)^2}\sum_{m=1}^{\infty}\Delta_m\left(\alpha^i_{-m}\alpha^j_m - \alpha^j_{-m}\alpha^i_m\right) + (\tilde{...}) \ , \tag{3.19}$$

where

$$\Delta_m \equiv m\frac{26-D}{12} + \frac{1}{m}\left[\frac{D-26}{12} + 2\,(1-a)\right] \ . \tag{3.20}$$

Therefore, we find that maintaining Lorentz invariance at the quantum level requires

$$D = 26 \ , \quad a = 1 \ . \tag{3.21}$$

This is a rather remarkable result. It shows that while the classical string is consistent in any number of dimensions, the quantum bosonic string is only consistent in 26 dimensions. The restriction on the number of dimensions is called criticality. In fact, there are many different ways to arrive at this criticality result.

We will not go into this during these lectures but for the Superstring, so a string theory incorporating supersymmetry, the critical number of dimensions changes to 10.

3.2 The quantum string spectrum

Having quantised the string we can now examine its spectrum. The classical Hamiltonian is given by

$$\begin{aligned} H &= p_-\dot{q}^- + \int_0^{2\pi} d\sigma \Pi_i \dot{X}^i - L \\ &= \frac{1}{4\pi\alpha'}\int_0^{2\pi} d\sigma\left[\left(\partial_\tau X^i\right)^2 + \left(\partial_\sigma X^i\right)^2\right] \\ &= \frac{\alpha'}{2}p^ip^i + \frac{1}{2}\sum_{n=-\infty}^{\infty}\left(\alpha^i_{-n}\alpha^i_n + \tilde\alpha^i_{-n}\tilde\alpha^i_n\right) \ . \end{aligned} \tag{3.22}$$

This is related to X^- through the Virasoro constraint (3.6)

$$\partial_\tau X^- = \frac{1}{2p^+\alpha'}\left[\left(\partial_\tau X^i\right)^2 + \left(\partial_\sigma X^i\right)^2\right] \ , \tag{3.23}$$

so that

$$p^- = \frac{1}{2\pi\alpha'}\int_0^{2\pi} d\sigma \partial_\tau X^- = \frac{H}{\alpha' p^+} \ . \tag{3.24}$$

In quantising we need to normal order the α, so we have

$$p^+p^- = \frac{1}{\alpha'}\left[N_\perp + \tilde{N}_\perp - 2a + \frac{\alpha'}{2}p^ip^i\right] \ , \tag{3.25}$$

where we define

$$N_\perp \equiv \sum_{n=1}^{\infty} : \alpha^i_{-n}\alpha^i_n : \ , \quad \tilde{N}_\perp \equiv \sum_{n=1}^{\infty} : \tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n : \ . \tag{3.26}$$

The mass in the target-space is given by

$$M^2 = -p^2 = 2p^+p^- - p^ip^i = \frac{2}{\alpha'}\left(N_\perp + \tilde{N}_\perp - 2a\right) \ . \tag{3.27}$$

Finally, we note that for the closed string we have a symmetry of translations along σ, and this can be shown to imply the level matching condition

$$N_\perp = \tilde{N}_\perp \ , \tag{3.28}$$

so the expression for the mass becomes

$$M^2 = \frac{4}{\alpha'}\left(N_\perp - 1\right) \ . \tag{3.29}$$

Here we used the fact that the normal ordering constant is fixed by criticality to $a = 1$ (3.21).

Now we can examine the spectrum on the string according to how many oscillators are present:

$N_\perp = 0$

Here we have

$$M^2 = -\frac{4}{\alpha'} \ . \tag{3.30}$$

This is a tachyonic mode, which means that it is signalling an instability in the bosonic string. For superstrings this mode will be absent, and so such strings are stable. It will not play an important role in these lectures and so we will not discuss it further.

$N_\perp = 1$

In this case we have

$$M^2 = 0 \ . \tag{3.31}$$

This is therefore the massless spectrum of the quantum bosonic string. It is given by

$$\xi_{ij}\tilde{\alpha}^i_{-1}\alpha^j_{-1}\,|0,p\rangle \ , \quad i,j=1,...,24 \ . \tag{3.32}$$

We can decompose the tensor ξ_{ij} into irreducible representations of $SO(24)$ as

$$\xi_{ij} = g_{(ij)} + B_{[ij]} + \Phi \ , \tag{3.33}$$

where $g_{(ij)}$ is traceless symmetric, $B_{[ij]}$ is anti-symmetric and Φ is a scalar (corresponding to the trace).

We therefore find a massless, transversely polarised, symmetric tensor field g_{ij}. This is a graviton! Indeed, it can be shown that it has spin 2. So the quantum bosonic string contains gravitational modes in its spectrum.

We also find an anti-symmetric massless tensor $B_{[ij]}$ termed the Kalb-Ramond field. We will return to this field later when we discuss compactifications.

The massless scalar Φ is called the dilaton. It actually determines the coupling constant for string interactions.

$N_{\perp} > 1$

These are massive oscillator string modes, with a mass starting at M_s. They are crucial for showing the finiteness of string theory scattering amplitudes. But we will not study them further.

3.3 The low-energy effective action

We see that the string has a set of massless fields and some very massive fields. We can therefore write a low-energy effective theory describing the massless fields. The effective action for this theory is given by

$$S_D = 2\pi M_s^{D-2}\int d^DX\sqrt{-G}e^{-2\Phi}\left(R - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} + 4\partial_\mu\Phi\partial^\mu\Phi\right) \ . \tag{3.34}$$

We have written the action for general dimension D so that we can treat both the bosonic string $D=26$ and the superstring $D=10$, since both contain this massless spectrum. We have also restored Lorentz invariance. We have written the metric for the gravity theory as G, with an associated Ricci scalar R. The H is defined as

$$H_{\mu\nu\rho} \equiv \partial_{[\mu}B_{\nu\rho]} \ , \tag{3.35}$$

where the square brackets denote anti-symmetrisation of the indices. The action is therefore composed purely of the kinetic terms for the fields, as expected since they are massless.

Note that in the case of the bosonic string there is also a tachyon mode, which we neglect here. We are only studying the dynamics of the massless modes. This is not really consistent, but can be done completely consistently for the superstring with $D=10$.

The action allows for vacuum configurations of the fields $G_{\mu\nu}$, $B_{[\nu\rho]}$ and Φ. These can be thought of as coherent states of the string excitation modes. In such a non-trivial background the string worldsheet theory is modified to

$$\begin{aligned} S_P &= -\frac{T}{2}\int_\Sigma d^2\xi\,(-\det\, h)^{\frac{1}{2}}\, h^{ab}(\xi)\,\partial_a X^\mu(\xi)\,\partial_b X^\nu(\xi)\,\eta_{\mu\nu} \\ &\to -\frac{T}{2}\int_\Sigma d^2\xi\,(-\det\, h)^{\frac{1}{2}}\Big[h^{ab}(\xi)\,\partial_a X^\mu(\xi)\,\partial_b X^\nu(\xi)\,G_{\mu\nu} \\ &\quad + iB_{[\mu\nu]}\partial_a X^\mu(\xi)\,\partial_b X^\nu\ \epsilon^{ab} + \alpha'\Phi R^{(2)}\Big]\ . \end{aligned} \tag{3.36}$$

Here ϵ^{ab} is the anti-symmetric unit tensor, and $R^{(2)}$ is the two-dimensional Ricci scalar.

The resulting worldsheet theory is much more difficult to quantise. This is why we can only study the string from the worldsheet perspective in full detail for small curvature backgrounds (as well as some specific other backgrounds for which we can solve the worldsheet theory).

4 Lecture 3: Compactification and the Swampland Distance Conjecture

We have seen that the bosonic string lives in 26 dimensions. The superstring lives in 10 dimensions. These both seem to be directly incompatible with the observed universe. However, this need not be the case. The point is that the additional dimensions may be compact and small, so that they have yet to be observed. This naturally leads to thinking about string theory in a space-time which has a compact direction. The simplest such setting is the case where one of the dimensions is in the shape of a circle. We will study this in this section and this will lead to our first encounter with a Swampland criterium.

4.1 Compactification of field theory on a circle

We consider $D = d+1$ dimensional space-time. The spatial direction X^d is taken to be compact in the shape of a circle so is periodically identified

$$X^d \simeq X^d + 1\ . \tag{4.1}$$

We are interested in looking at the effective theory in the d non-compact dimensions.

First, recall that we are working in Planck units, which in this case we therefore set as $M_p^d = 1$, where M_p^d denotes the d-dimensional Planck mass. The periodicity of X^d is set to one in those units.

We can write the metric on the D-dimensional space as

$$ds^2 \equiv G_{MN}dX^M dX^N = e^{2\alpha\phi}g_{\mu\nu}dX^\mu dX^\nu + e^{2\beta\phi}\left(dX^d\right)^2\ . \tag{4.2}$$

So here we have introduced the coordinates X^M which are D-dimensional, so $M = 0, ..., d$, while $\mu = 0, ..., d-1$. The D-dimensional metric is G_{MN} and we take it as a product metric. The d-dimensional metric is $g_{\mu\nu}$. In practice we will take this to be $\eta_{\mu\nu}$ but we keep it general for now. The metric has a parameter ϕ which can be regarded as a d-dimensional scalar field. The constants α and β are chosen to be

$$\alpha^2 = \frac{1}{2\left(d-1\right)\left(d-2\right)} \;, \quad \beta = -\left(d-2\right)\alpha \;. \tag{4.3}$$

Let us look at the circumference of the circle, denoted $2\pi R$, it is given by

$$2\pi R \equiv \int_0^1 \sqrt{G_{dd}} dX^d = e^{\beta\phi} \;. \tag{4.4}$$

We see that the radius of the circle is a dynamical field in d-dimensions. We will be interested in the behaviour of the d-dimensional theory under variations of the expectation value of the field ϕ, which amounts to variations of the size of the circle.

The first thing we want to do is decompose the D-dimensional Ricci scalar R^D for the metric (4.2). We have

$$\int d^D X \sqrt{-G} R^D = \int d^d X \sqrt{-g} \left[R^d - \frac{1}{2} \left(\partial\phi\right)^2 \right] \;. \tag{4.5}$$

We observe that indeed ϕ picks up dynamics, and that it is canonically normalised.

Now consider introducing a massless D-dimensional scalar field Ψ. Since the d^{th} dimension is periodic so must Ψ be, therefore we can decompose it as

$$\Psi\left(X^M\right) = \sum_{n=-\infty}^{\infty} \psi_n \left(X^\mu\right) e^{2\pi i n X^d} \;. \tag{4.6}$$

The modes ψ_n are d-dimensional scalar fields. The mode ψ_0 is called the zero-mode of Ψ, while the ψ_n are called Kaluza-Klein (KK) modes. Note that the momentum is quantized along the compact direction

$$-i \frac{\partial}{\partial X^d} \Psi = 2\pi n \Psi \;. \tag{4.7}$$

For simplicity we now restrict to $g_{\mu\nu} = \eta_{\mu\nu}$. Since Ψ is massless in D-dimensions, it equation of motion is

$$\partial^M \partial_M \Psi = \left(e^{-2\alpha\phi} \partial^\mu \partial_\mu + e^{-2\beta\phi} \partial^2_{X^d} \right) \Psi = 0 \;. \tag{4.8}$$

This gives the equations of motion for the ψ_n modes

$$\left[\partial^\mu \partial_\mu - \left(\frac{1}{2\pi R}\right)^2 \left(\frac{1}{2\pi R}\right)^{\frac{2}{d-2}} \left(2\pi n\right)^2 \right] \psi_n = 0 \;. \tag{4.9}$$

We can therefore read off the mass of the KK modes as

$$M_n^2 = \left(\frac{n}{R}\right)^2 \left(\frac{1}{2\pi R}\right)^{\frac{2}{d-2}} \;. \tag{4.10}$$

So in the d-dimensional theory the KK modes are a massive tower of states with increasing masses as in (4.10).

4.2 Compactification of string theory on a circle

Now let us repeat this exercise in string theory by considering strings on a circle of radius R. We would like to connect with our results in sections 2 and 3, but those were performed for a metric

$$ds^2 = \eta_{MN} dX^M_{(s)} dX^N_{(s)} \ , \tag{4.11}$$

rather than (4.2). The subscripts on $X^M_{(s)}$ are to remind us that we are working with the metric (4.11). For now we will proceed with the metric (4.11) and take the $X^d_{(s)}$ direction as R-periodic

$$X^d_{(s)} \simeq X^d_{(s)} + 2\pi R \ . \tag{4.12}$$

We will reconnect to the metric (4.2) later.

We consider the bosonic mode expansion, as in (2.28), but now we will not impose yet $X^M_{(s)}(\sigma + 2\pi, \tau) = X^M_{(s)}(\sigma, \tau)$ on the linear terms in σ. So we have

$$X^M_{(s)}(\tau, \sigma) = x^\mu + \alpha' p^M \tau + \frac{\alpha'}{2}\left(p^M_L - p^M_R\right)\sigma + \text{oscillators} \ . \tag{4.13}$$

We have allowed here for independent left-moving and right-moving momenta, and the overall momentum of the string is half their sum

$$p^M = \frac{1}{2}\left(p^M_R + p^M_L\right) \ . \tag{4.14}$$

Recall that because the X^d direction is compact this is quantised. The appropriate quantisation, as we will soon see, is

$$p^d = \frac{n}{R} \ . \tag{4.15}$$

In non-compact space we imposed $X^\mu_{(s)}(\sigma + 2\pi, \tau) = X^\mu_{(s)}(\sigma, \tau)$ which lead to $p^\mu_R = p^\mu_L$, but for a circle we may have a winding string

$$X^d_{(s)}(\sigma + 2\pi, \tau) = X^d_{(s)}(\sigma, \tau) + w 2\pi R \ , \tag{4.16}$$

with $w \in \mathbb{Z}$. The string is wrapping around the circle w times, as illustrated in figure 4. For such a winding string we therefore have

$$\frac{\alpha'}{2}\left(p^d_L - p^d_R\right) = wR \ . \tag{4.17}$$

Now consider the mass spectrum for the string on such a background. We again go to target-space light-cone gauge. The Hamiltonian (3.22) now reads

$$H = \frac{\alpha'}{2}\left[\frac{1}{4}\left(p^d_L - p^d_R\right)^2 + p^\alpha p^\alpha + \left(p^d\right)^2\right] + \left(N_\perp + \tilde{N}_\perp - 2\right) \ , \tag{4.18}$$

where we split the index $i = \{\alpha, d\}$. Note that we no longer have the level matching condition (3.28), but instead have

$$N_\perp - \tilde{N}_\perp = nw \ . \tag{4.19}$$

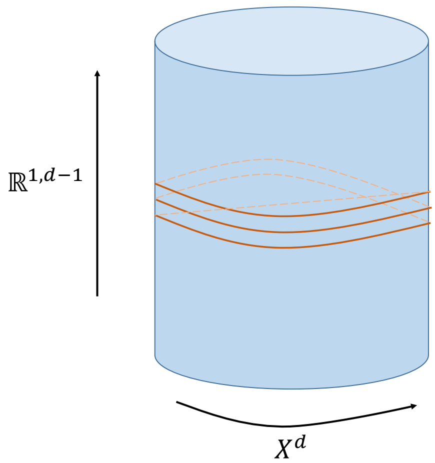

Figure 4: Figure showing a string winding around a circler dimension 3 times.

Then the d-dimensional mass is given by $-p_\mu p^\mu = 2p^+p^- - p^\alpha p^\alpha$ which, for states with no oscillators excited, leads to

$$\left(M^{(s)}_{n,w}\right)^2 = \left(\frac{n}{R}\right)^2 + \left(\frac{wR}{\alpha'}\right)^2 \ . \tag{4.20}$$

We would like to connect this result with the effective action (4.5). However, to do that we need to change from the metric (4.11) to the metric (4.2). This is called going from the string frame to the Einstein frame. The difference is the factor of $e^{2\alpha\phi}$ multiplying the $g_{\mu\nu}$ directions. To get the Einstein frame mass for the states we simply note that $p^\mu p_\mu$ has one inverse factor of the metric and so we need to multiply masses by a factor of $e^{2\alpha\phi} = \left(\frac{1}{2\pi R}\right)^{\frac{2}{d-2}}$.

This does not quite do the full job. If we look at the effective action coming from string theory (3.34) we see that there is an overall factor of the exponential of the dilaton $e^{-2\Phi}$. An important object is the d-dimensional dilaton Φ^d defined as

$$\Phi^d \equiv \Phi - \frac{1}{2}\log\left(2\pi R M_s\right) \ . \tag{4.21}$$

We would like to look at variations of R which keep Φ^d fixed. This means that we must vary

$$e^{-2\Phi} \sim \frac{1}{2\pi R M_s} \ . \tag{4.22}$$

But if we consider the definition of the d-dimensional Planck mass M_p^d coming from the string effective action (3.34)

$$\frac{\left(M_p^d\right)^{d-2}}{2} \equiv 2\pi M_s^{D-2} e^{-2\Phi} \ , \tag{4.23}$$

we see that in order to stay in the Einstein frame $M_p^d = 1$ we have to choose our units such that

$$M_s \sim \left(2\pi R\right)^{\frac{1}{d-2}} \ . \tag{4.24}$$

This will then affect the mass of the winding modes in (4.20) because of the factor of α'.

Performing the change of frames then finally leads to the Einstein frame mass

$$(M_{n,w})^2 = \left(\frac{1}{2\pi R}\right)^{\frac{2}{d-2}} \left(\frac{n}{R}\right)^2 + (2\pi R)^{\frac{2}{d-2}} \left(\frac{wR}{\alpha'}\right)^2 . \tag{4.25}$$

We see that this indeed matches the simple field theory calculation for the KK masses (4.10).

4.3 The Swampland Distance Conjecture

We can now study the d-dimensional effective theory. The action is given in (4.5), and this must be supplemented by the spectrum (4.25). We are particularly interested in how the spectrum of states behaves under variations of the expectation value of the field ϕ. This is easy to determine from the simple relation (4.4). The possible expectation values of field ϕ define a field space, which in this case has one infinite real dimension. So we can consider

$$-\infty < \phi < \infty . \tag{4.26}$$

Let us define a variation of ϕ from some initial value ϕ_i to some final value ϕ_f as

$$\Delta\phi = \phi_f - \phi_i . \tag{4.27}$$

We now note that there are two infinite towers of massive states in this theory. The tower of KK modes, with masses given by $M_{n,0}$ in (4.25), and a tower of winding modes given by $M_{0,m}$. We can associate to each tower a mass scale, which is the universal factor multiplying the integers n and m. Using (4.4) we can write these mass scales as

$$M_{KK} \sim e^{\gamma\phi} , \quad M_w \sim e^{-\gamma\phi} , \tag{4.28}$$

where

$$\gamma = \sqrt{2}\left(\frac{d-1}{d-2}\right)^{\frac{1}{2}} > 0 . \tag{4.29}$$

We therefore can make the following observation. For any $\Delta\phi$ there exists an infinite tower of states, with some associated mass scale M, which becomes light at an exponential rate in $\Delta\phi$

$$M(\phi_i + \Delta\phi) \sim M(\phi_i)\, e^{-\gamma|\Delta\phi|} . \tag{4.30}$$

This is illustrated in figure 5. There are some important things to note about this observation

- The tower of states which becomes light is the KK tower if $\Delta\phi < 0$ while it is the winding tower if $\Delta\phi > 0$. So some tower of states always become light no matter what the sign of $\Delta\phi$ is.
- The behaviour (4.30) is deeply string theoretic. It is not true in quantum field theory because one set of states are winding states which are absent in field theory.
- The product of the mass scales of the two towers is independent of ϕ.
- The exponent γ in the mass is a constant of order one.
- The field ϕ is canonically normalised, so the behaviour of the mass scales is exponential in the proper distance in ϕ field space.

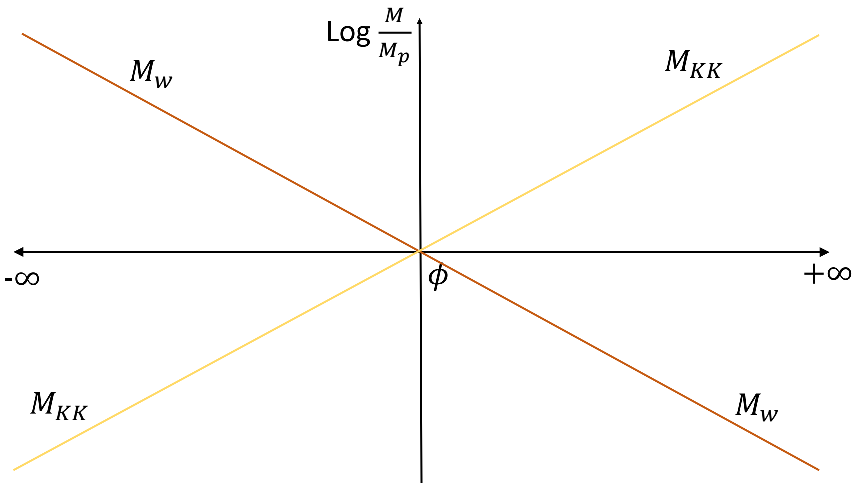

Figure 5: Figure showing the mass scale, on a log plot, for the KK and winding towers as a function of the scalar field ϕ expectation value. The gradient of the slope is the exponent γ. The $\mathbb{Z}_2$ symmetry in the figure is due to T-duality.

- If $|\Delta\phi| \to \infty$ then an infinite number of states become massless, which means that there is no description of that locus in a d-dimensional quantum field theory.[4]

The last point has a continuous analogue. If we consider the effective field theory which has some finite number of states below a cutoff Λ, then this field theory can only hold for a finite range of expectation values of ϕ.

The behaviour (4.30) is very interesting and it is natural to wonder if there is a deep reason behind it, and if so, then if it is a general property of string theory. There is good reason to expect that the answer is positive to both of these questions. One clue is in the origin of the two towers, the KK and winding modes. These towers a deeply related, indeed there is a $\mathbb{Z}_2$ symmetry which interchanges them. This is called T-duality, and it is most directly seen in the string frame where we observe that the mass spectrum (4.20) is invariant under the action

$$\mathrm{T-duality} \;:\; R \leftrightarrow \frac{\sqrt{\alpha'}}{R}\;. \tag{4.31}$$

It can be shown that this is not only a symmetry of the mass spectrum, but of the full string theory. In fact it can be embedded into a gauge symmetry which becomes manifest at the self-dual radius $R = \frac{\sqrt{\alpha'}}{R}$. Duality is a very deep property of string theory. There are many more dualities than T-duality. Indeed, all known string theories are themselves related by dualities. It is then natural to expect that there are many different towers of states which are dual, and this duality is such that as one moves in the parameter space of the theory, which in string theory means in the scalar field space, the product of the mass scale of the dual towers stays constant and so one must become light in any direction. As we move an infinite distance in parameter space the tower must become massless.

[4]We can describe it as a D-dimensional theory. Remarkably, this is true for either a very large or very small radius of the circle.

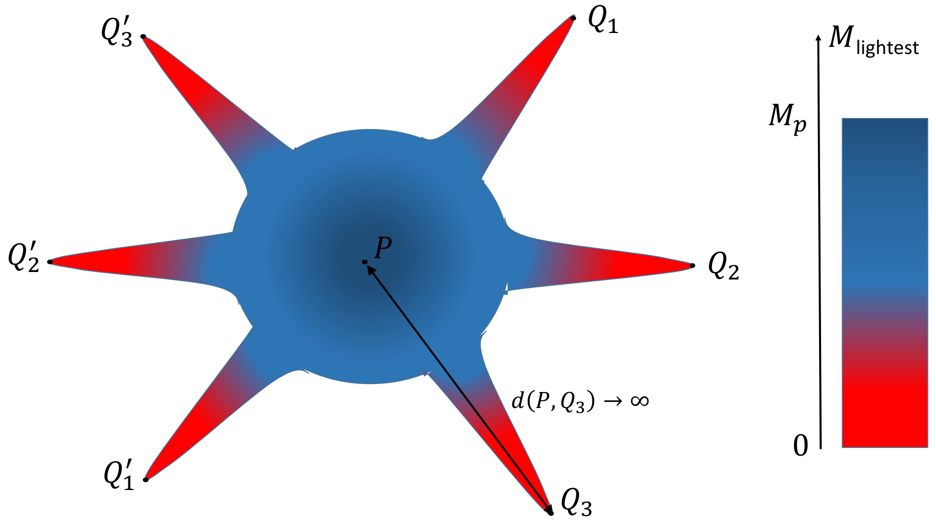

Figure 6: Figure showing a schematic illustration of string moduli space. The distance from any point P in the bulk of moduli space to any point Q_i, or Q_i', is infinite. M_{lightest} denotes the mass scale of the lightest tower of states in the theory, and this goes to zero at any point Q_i. Points Q_i and Q_i' are related by duality and the light towers of states are interchanged between them.

This kind of reasoning, and various simple examples in string theory, let to the proposal of the Swampland Distance Conjecture (SDC) in [2]. The conjecture is at heart analogous to (4.30) but can be phrased more generally and precisely as follows.

- Consider a theory with a moduli space $\mathcal{M}$ which is parametrised by the expectation values of some field ϕ^i which have no potential. Starting from any point $P \in \mathcal{M}$ there exists another point $Q \in \mathcal{M}$ such that the geodesic distance between P and Q, denoted $d(P, Q)$, is infinite.
- There exists an infinite tower of states, with an associated mass scale M, such that as $d(P, Q) \to \infty$ we have
$$M(Q) \sim M_p e^{-\gamma d(P,Q)} , \tag{4.32}$$
where γ is some positive constant.

Note that because this is an asymptotic statement about infinite distance $d(P, Q) \to \infty$ the mass scale value at P is not important. The behaviour of the conjecture is illustrated schematically in figure 6

This is our first encounter with a Swampland conjecture. It typifies many of the general properties of conjectures about the Swampland.

- It is supported by constructions in string theory.
- There are no known counter-examples to it in string theory.
- There are some general, but imprecise, arguments for why it may be expected to hold generally.

- It goes beyond quantum field theory.

Following further investigation in string theory, a stronger version of the SDC was proposed as the Refined SDC in [3]. We will not motivate it here but only state it. It states that

- The constant γ is always of $\mathcal{O}(1)$.
- The exponential behaviour of the mass scale of the tower of states sets in already for $d(P,Q) \sim \mathcal{O}(1)$. In particular this means that the conjecture amounts to a stronger statement
$$M(Q) \sim M(P)\, e^{-\gamma d(P,Q)} \; . \tag{4.33}$$
- It holds also for scalar fields which have a potential, so not only moduli.

It is worth thinking generally about the SDC and its refined version. First, from the perspective of quantum field theory it is quite surprising. We have seen that the conjecture can only hold due to string theory, or more generally, quantum gravity. Typically, the mass scale associated to such physics is M_p, and one might expect that working at energy scales far below the Planck mass would mean that we lose sensitivity to such physics. But the energy scale associated to the conjecture, M, is exponentially lower than M_p. Therefore, it claims that the usual rules of effective quantum field theory break down at an exponentially lower energy scale than expected whenever a field develops a large expectation value.

There are a number of interesting phenomenological implications of the SDC. In particular within the context of early universe cosmology. One central example is in relation to the magnitude of primordial gravitational waves produced during inflation. Here there exists an important relation between an observable, the so-called tensor-to-scalar ratio r, which measures the magnitude of gravitational waves produced during inflation, and the variation of the inflaton expectation value. The Lyth bound states that

$$\frac{\Delta\phi}{M_p} > \mathcal{O}(1)\sqrt{\frac{r}{0.01}} \; . \tag{4.34}$$

Where $\Delta\phi$ is the variation of the inflaton expectation value during inflation. The current experimental bounds on r are $r < 0.07$. Therefore, it can very well be that a future measurement of r would imply that the inflaton varied its expectation value by more than M_p which would mean that the Refined SDC could be applied to it. In that case the conjecture would imply that the cut-off scale of the model of inflation must be exponentially smaller than the Planck mass. However, the tensor-to-scalar ratio is also related to the energy scale of inflation M_{inf} as

$$M_{\text{inf}} \sim \left(\frac{r}{0.1}\right)^{\frac{1}{4}} 10^{-3} M_p \; . \tag{4.35}$$

So a larger r also needs a high energy scale of inflation. We conclude that the (Refined) SDC implies an exponential tension between the two ingredients required for a large tensor-to-scalar ratio and therefore may place an upper bound on its possible magnitude in string theory.

5 Lecture 4: The gauge fields and the Weak Gravity Conjecture

In the previous lecture we considered the reduction of string theory on a metric (4.2). The metric encoded the parameter ϕ which became a dynamical field in the lower d-dimensional theory. There is another degree of freedom in the metric which becomes a d-dimensional gauge, rather than scalar, field A_μ. It is associated to mixed terms as

$$ds^2 = e^{2\alpha\phi} g_{\mu\nu} dX^\mu dX^\nu + e^{2\beta\phi} \left(dX^d + A_\mu dX^\mu \right)^2 \ . \tag{5.1}$$

Dimensionally reducing the Ricci scalar now gives

$$\int d^D X \sqrt{-G} R^D = \int d^d X \sqrt{-g} \left[R^d - \frac{1}{2} \left(\partial\phi\right)^2 - \frac{1}{4} e^{-2(d-1)\alpha\phi} F_{(A),\mu\nu} F^{\mu\nu}_{(A)} \right] \ , \tag{5.2}$$

where $F_{(A),\mu\nu} = \frac{1}{2}\partial_{[\mu} A_{\nu]}$ is the gauge field kinetic term. We therefore see that the lower dimensional theory has a propagating $U(1)$ gauge field with gauge coupling

$$g_{(A)} = e^{(d-1)\alpha\phi} = \frac{1}{2\pi R} \left(\frac{1}{2\pi R} \right)^{\frac{1}{d-2}} \ . \tag{5.3}$$

The gauge symmetry associated to the gauge field, with a local gauge parameter $\lambda\left(X^\nu\right)$, is coming from the circle isometry

$$A_\mu \rightarrow A_\mu - \partial_\mu \lambda\left(X^\nu\right) \ , \ \ X^d \rightarrow X^d + \lambda\left(X^\nu\right) \ . \tag{5.4}$$

Recall the KK expansion for the a D-dimensional field (4.6). From the gauge symmetry transformation (5.4) we therefore see that the KK modes ψ_n are charged under the KK $U(1)$ gauge field A_μ. Their charge is

$$q_n^{(A)} = 2\pi n \ , \tag{5.5}$$

which, as expected, is quantised. We now note that there is a relation between the charge and mass of the KK modes

$$g_{(A)} q_n^{(A)} = M_{n,0} \ , \tag{5.6}$$

where the KK mass is as in (4.25).

Let us now consider the String theory effective action (3.34). If we compactify this action on a circle we will obtain a gauge field coming from the gravitational sector as in (5.2). However, we will also obtain a second gauge field V_μ coming from the Kalb-Ramond B-field with one index along the X^d direction

$$V_\mu \equiv B_{[\mu d]} \ . \tag{5.7}$$

The kinetic terms for V_μ come from dimensional reduction of the kinetic terms of the Kalb-Ramond field. This gives

$$\int d^d X \sqrt{-g} \left[R^d - \frac{1}{4} e^{-2(\alpha+\beta)\phi} F_{(V),\mu\nu} F^{\mu\nu}_{(V)} \right] \ . \tag{5.8}$$

The factor in front of kinetic terms comes from reducing $\sqrt{-G} H_{\mu\nu d} H^{\mu\nu d}$ so that

$$e^{-2(\alpha+\beta)\phi} = \underbrace{e^{2\alpha\phi}}_{\sqrt{-G}} \underbrace{e^{-4\alpha\phi}}_{\left(G^{\mu\nu}\right)^2} \underbrace{e^{-2\beta\phi}}_{G^{dd}} \ . \tag{5.9}$$

We therefore find a gauge coupling of

$$g_{(V)} = e^{(\alpha+\beta)\phi} = 2\pi R \left(\frac{1}{2\pi R}\right)^{\frac{1}{d-2}} \; . \tag{5.10}$$

The states that are charged under V_μ are the winding modes of the string. To see this we can evaluate directly the Polyakov action (3.36) for a string wrapping the X^d direction w times (in the Einstein frame). Because the string wraps X^d we can set $\sigma = \frac{2\pi}{w} X^d$ and so

$$\begin{aligned} S_P &= -\frac{T}{2}\int_\Sigma d\tau d\sigma \left[2iV_\mu \partial_\tau X^\mu \partial_\sigma \left(\frac{w\sigma}{2\pi}\right)\right] \\ &= -i\frac{w}{2\pi\alpha'}\int_\gamma d\tau \left(\partial_\tau X^\mu\right) V_\mu \; . \end{aligned} \tag{5.11}$$

This is just the world-line action of a particle with charge

$$q_w^{(V)} = \frac{w}{2\pi\alpha'}\left(2\pi R\right)^{\frac{2}{d-2}} \; , \tag{5.12}$$

where we have included the extra factor of $(2\pi R)^{\frac{2}{d-2}}$ coming from (4.24). We therefore have

$$g_{(V)} q_w^{(V)} = M_{0,w} \; , \tag{5.13}$$

where the winding mass is as in (4.25).

We therefore find that there are two gauge fields in d-dimensions A_μ and V_μ and there are states charged under them, KK modes and winding modes respectively. Further, the charged states have interesting relations between their charges and masses (5.13) and (5.6). We will investigate these properties in the context of the Swampland below. But before that we note that there is a symmetry in the theory where we exchange the gauge fields A_μ and V_μ and the KK and winding modes. This is in fact just the T-duality symmetry we encountered already.

5.1 The Completeness Conjecture

The first thing we notice is that each of the gauge fields A_μ and V_μ has states charged under it. In fact all the possible integer charges with respect to each gauge field are populated by states. This is not something one would necessarily expect from a QFT perspective. For example, we can easily have a QFT gauge theory with no charged matter at all, or only two charged states of charges 3 and 157. The fact that we find that all the possible charges under the gauge fields have states associated to them is conjectured to be general in string theory. This is termed the Completeness Conjecture [9]. It states that in a quantum theory of gravity all the possible charges under a gauge symmetry are populated by states. Of course, these states may be very heavy and not accessible from a low energy effective field theory perspective.

5.2 The No Global symmetries Conjecture

We see that if we send the gauge coupling to zero, $g_A \to 0$, or $g_V \to 0$, the mass scale of the full infinite tower goes to zero.[5] This means that an infinite number of states become massless.

[5]This is true for the appropriately normalised gauge coupling, which is such that the charges of the states are integers.

This signals the complete breakdown of any effective QFT since it would need to include an infinite number of degrees of freedom.

If we start with a gauge symmetry and send its gauge coupling g to zero, then that symmetry effectively becomes a global symmetry. One way to see this is to note that the kinetic term of the gauge field goes to infinity

$$\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} \xrightarrow{g\to 0} \infty. \tag{5.14}$$

Therefore, the actual dynamical gauge field stops propagating and only the gauge symmetry selection rules remain, which therefore is the same as a global symmetry.

Connecting the two observations we see that if we try to turn a gauge symmetry into a global symmetry we find an obstruction with an infinite number of states becoming massless. In fact, we can understand the role of these states quite nicely. The infinite tower of KK modes becoming massless implies that we need to describe the theory in terms of an additional dimension. In the higher dimensional theory the $U(1)$ is actually part of the gravitational dynamics and is therefore associated to the local symmetry of diffeomorphisms. Indeed, we can see this also by the fact that the gauge symmetry is associated to the isometry of the circle (5.4), but this is a space-time symmetry which is embedded in diffeomorphisms. Therefore, in this sense, quantum gravity is obstructing the existence of a global symmetry by gauging it. Indeed, this is the only way that such a global symmetry can be obstructed because the other possibility, that it is broken, is not allowed since it is a limit of a gauge symmetry.

Again we note that this is a property of quantum gravity. In QFT there are no light states as we take $g_V \to 0$, and therefore there is no obstruction to this global symmetry limit.

These are examples of a general conjecture which states that in a quantum theory of gravity there are no global symmetries, see for example [10] for a review.

5.3 The Weak Gravity Conjecture

We have seen that string theory, or more generally quantum gravity, obstructs the global symmetry limit of gauge theories $g \to 0$ by making states light. The Weak Gravity Conjecture (WGC) [4] is an attempt to quantify this by relating the magnitude of the gauge coupling of $U(1)$ gauge fields with certain mass scales.

We will first study the WGC in four space-time dimensions. It is a Swampland constraint on $U(1)$ gauge theories coupled to gravity, so we will consider the action

$$\int d^4X\sqrt{-g}\left[\frac{R}{2} - \frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu}\right] \ . \tag{5.15}$$

The WGC is a statement about the charged spectrum of this theory, it has two parts, the so-called electric and magnetic WGC.

- The electric WGC says that a theory with a $U(1)$ gauge symmetry must contain a charged particle with a mass m and charge q such that the following inequality is satisfied

$$m \leq \sqrt{2}gqM_p \ . \tag{5.16}$$

- The magnetic WGC says that the cutoff scale of the effective theory Λ is approximately bounded by the gauge coupling
$$\Lambda \lesssim g M_p \ . \tag{5.17}$$

In our string theory example we can see that both (5.16) and (5.17) are satisfied. The first is actually satisfied by an infinite number of states, for the gauge field A_μ by the KK tower (5.6) and for V_μ by the winding modes tower (5.13). Actually, since the string theory example is not in four dimensions the quantitative relation to the four-dimensional statement is slightly modified and is discussed in section 5.5. The qualitative relations are still correct.

The electric WGC (5.16) is also responsible for the name of the conjecture. There are two forces present in the theory (5.15), gravity and a $U(1)$ electromagnetic force. A charged particle will couple to both of these forces. The coupling strength to the gravitational force is $\frac{m}{\sqrt{2}M_p}$, while the coupling strength to the $U(1)$ force is gq.[6] The inequality (5.16) is therefore the statement that the WGC particle should couple to the $U(1)$ force at least as strong as it does to gravity. So that for that particle, gravity is always the weakest force acting on it. It is rather important to emphasise that this does not imply that gravity should be the weakest force acting on *any* particle in the theory. Rather, it is the weakest force on at least one particle.

We also see that the magnetic WGC (5.17) is satisfied because Λ is associated with the mass scale of the KK or winding towers. So it is the mass scale where an infinite number of modes begin to appear. Note that this is not a sharp cutoff statement, one can go above Λ by including some of the KK or winding modes in the effective theory. However, it is a cutoff in the sense that going to higher energies than gM_p necessitates introducing an increasing number of degrees of freedom into the theory.

5.4 The Weak Gravity Conjecture and Black Holes

The WGC is also a good point to introduce a different aspect of Swampland conjectures, which is that while the initial evidence for them comes from string theory it is sometimes possible to present general arguments for why they should hold. The WGC is a nice example of this since it first arose as a (related) observation in string theory [12] but later formulated in a manner independent of string theory in [4]. It is important to state though, that while the WGC is a swampland constraint which comes close to having a general argument for it, in fact, the argument we will present is not strong enough to convincingly imply both (5.16) and (5.17). Indeed, there is no general very convincing argument for the WGC. Nonetheless, since the argument presented in [4] comes close it is certainly worth discussing it.

The starting point is to consider black hole solutions to the theory (5.15). In particular one can construct charged semi-classical black holes called Reissner-Nordstrom (RN) black holes. We will not discuss these solutions here, but will state one property of them. Their Arnowitt-Deser-Misner (ADM) mass M_{ADM} is the black hole mass as measured at infinity in its flat space asymptotics. Similarly, their charge Q is the charge measured by surrounding the black hole by a sphere and measuring the electric flux through it. Then the solutions satisfy the so called extremality bound
$$M_{\mathrm{ADM}} \geq \sqrt{2} g Q M_p \ . \tag{5.18}$$

[6] Recall that gravitational attraction is $\frac{m^2}{8\pi M_p^2 r^2}$ while electromagnetic repulsion is $\frac{g^2q^2}{4\pi r^2}$.

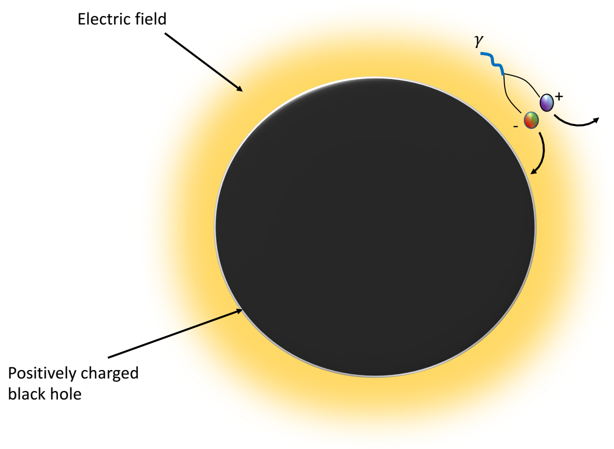

Figure 7: Figure illustrating the discharge process of black holes. A pair of positively charged particle and its negatively charged anti-particle are created in the electric field outside the black hole. The anti-particle falls into the black hole, decreasing its charge, while the particle escapes to infinity.

Black holes which saturate the inequality are termed extremal black holes.

Charged black holes can, in general, discharge themselves by emitting charged particles. This is a quantum process, and is analogous, though slightly different, to how black holes can lose mass by emitting photons as Hawking radiation. One way to see this is that because the black hole is charged there is an electric field outside its horizon, and in this field it is possible to Schwinger pair create a charged particle anti-particle pair, and the anti-particle can fall into the black hole decreasing its charge while the particle escapes to infinity. This is illustrated in figure 7.

Imagine that the black hole was able to completely discharge itself by emitting N particles of mass m and charge q. Then by charge and energy conservation we would have

$$Q = Nq \; , \quad M_{\mathrm{ADM}} \geq Nm \; . \tag{5.19}$$

Eliminating N in this we have

$$\frac{q}{m} \geq \frac{Q}{M_{\mathrm{ADM}}} \; . \tag{5.20}$$

So by charge and energy conservation the black hole can only discharge if there exists a particle for which the charge-to-mass ratio is larger than than of the black hole. The black hole with the maximal charge-to-mass ratio is an extremal black hole where the inequality (5.18) is saturated. Therefore, extremal black holes can only discharge if there exists a particle which satisfies the WGC (5.16). Or in other words, the WGC is equivalent to the statement that extremal black holes should be able to discharge through emitting particles.

If we could motivate the idea that black holes should be able to discharge, we could motivate the WGC. Unfortunately, there is no convincing argument that RN black holes must be able to discharge. So while the WGC can be tied to black hole physics in this way, it does not lead to a strong argument for the WGC.[7]

In the original paper [4] an argument was presented for black hole discharge which is based on a classic argument against global symmetries in quantum gravity (see, for example [10]). It is worthwhile going through this to see that it does not quite work, but that it does provide an interesting perspective on some of our earlier results.

Let us first outline the argument for the absence of global symmetries in quantum gravity. The idea is to consider black holes charged under a $U(1)$ global, rather than gauge, symmetry. For a global symmetry there is no extremality bound relating the charge and mass of the black hole (5.18). This means that the black hole will continue to lose mass through Hawking radiation until it leaves the semi-classical regime where Hawkings' calculation is valid. This occurs when its horizon is of order the Planck length, and its mass is of order the Planck mass. Further, because there is no electric field outside the black hole, there is no semi-classical way for the black hole to discharge itself from the global symmetry charge. By the time it reaches the Planck regime it can no longer discharge its global symmetry charge because it does not have enough mass left to emit a sufficient number of charged particles. Therefore, what remains is a Planck mass remnant state which is completely stable due to its global symmetry charge. Since the initial black hole could have an arbitrary global charge, the theory contains an infinite number of stable remnant states with a mass below the Planck mass. So black holes can turn one charged particle below M_p into an infinite number of charged particles below M_p. Having an infinite number of states below a fixed mass scale is considered to be inconsistent. This is not theorem, but, for example, any lopp computation would diverge from an infinite number of states running in the loop. Another argument is that such a situation would lead to a violation of the covariant entropy bound [10]. The process leading to an infinite number of remnants in the presence of a global symmetry is illustrated in figure 8.

Now in [4] a modification of this argument was applied to a $U(1)$ gauge, rather than global, symmetry. It was assumed that black holes are not able to discharge, so the WGC is violated. Then the only difference between the gauge and global symmetry case is the extremality bound (5.18). The bound means that one cannot increase the charge of the black hole and still have it decay to the same mass. Nonetheless, if say $g \sim 10^{-100}$ then we would still obtain 10^{100} stable remnants below M_p. the number of remnants goes to infinity as $g \to 0$. It is therefore tempting to argue that this would be catastrophic and so we should insist that black holes must be able to discharge and thereby avoid the remnants in the first place.

However, recall that in the string theory setting when we sent $g \to 0$ we indeed actually did find an infinite number of massless charged states. So there cannot be a fundamental issue with the remnants existing. We interpreted these states as signalling the breakdown of the effective theory description and instead one had to utilise some other theory (in the example string theory case this theory was a higher-dimensional theory). Indeed, we can see that the magnetic WGC (5.17) tells us the effective theory cutoff is going to zero $\Lambda \to 0$. Therefore, it is plausible that the better way to interpret the infinite remnants, or states, as $g \to 0$ is not as a fundamental inconsistency but rather as a statement that the effective theory must break down at any finite cutoff scale $\Lambda > 0$, and must be replaced by some other theory. Another way to put it is that for any finite value of g one can consistently utilise some effective field theory below a cutoff scale Λ which is sufficiently low. The global symmetry limit is the one where $\Lambda \to 0$. So the

[7] At least not for RN black holes. In [13] it was argued that for black holes coupled to scalar fields there may exist limits in parameter space where the discharge of extremal black holes can be argued for.

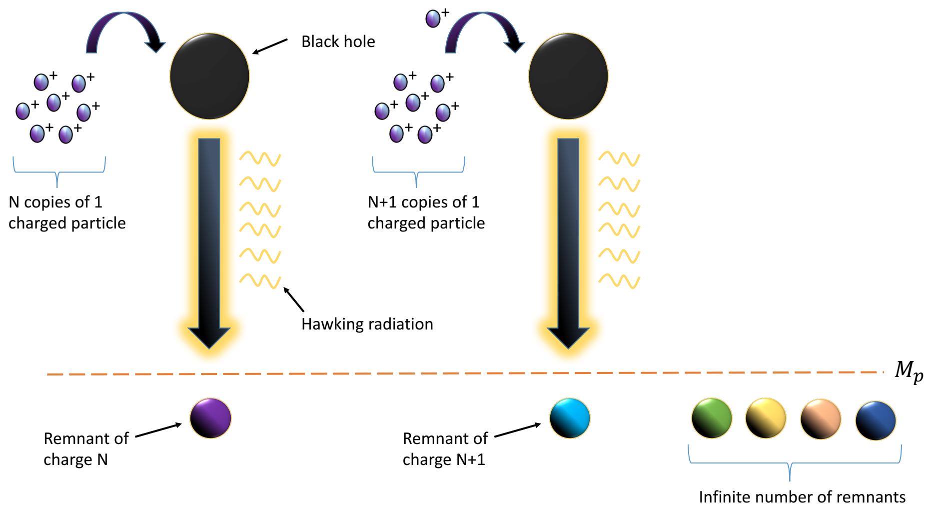

Figure 8: Figure illustrating how in the presence of a $U(1)$ global symmetry, black hole formation and decay would lead to an infinite number of stable remnant states below a fixed mass scale.

statement that there are no global symmetries in quantum gravity is better interpreted as the statement that a global symmetry would imply the cutoff of the theory must be at zero energy.

With these subtleties explained, we see that there are currently no strong black-hole based arguments for the Weak Gravity Conjecture and this remains an open problem. This is a first order picture, there are various interesting ideas in the literature about general arguments for different interpretations of the WGC, but covering them is beyond the scope of these notes.

5.5 The Weak Gravity Conjecture more precisely* (optional)

In this section we will give a more precise and general relation between the WGC and (5.6). The starting point is the action

$$\int d^dX\sqrt{-g}\left[\frac{R^d}{2}-\frac{1}{4}\left(\partial\phi\right)^2-\frac{1}{4e^2}e^{-\delta\phi}F_{\mu\nu}F^{\mu\nu}\right] , \tag{5.21}$$

The WGC is based on the requirement for Black Holes to be able to decay. The charge to mass ratio of an extremal black hole depends on the number of dimensions that the black hole lives in. Further, the particular black hole of interest in this theory actually has a dilatonic hair form the scalar field ϕ. The WGC therefore is modified from the four-dimensional version in the absence of scalar fields. For this particular setup this modification was calculated in [11] and reads

$$q^2g_A^2 \geq \left[\frac{d-3}{d-2}+\frac{\delta^2}{2}\right]m^2 . \tag{5.22}$$

Here $g_A^2 = e^2e^{\delta\phi}$, m is the mass of the WGC particle and q is its charge. Let us apply this to our setup (5.2) with the WGC particle being the KK modes. We have

$$\delta = 2\left(d-1\right)\alpha \ , \quad e^2 = 2 \ . \tag{5.23}$$

We therefore find (5.22) gives

$$2q^2 e^{\delta\phi} \geq \left[\frac{d-3}{d-2} + \frac{d-1}{d-2}\right] q^2 e^{\delta\phi} = 2q^2 e^{\delta\phi} \ . \tag{5.24}$$

We therefore see that the KK modes saturate the more general WGC bound. This is actually the more precise interpretation of the equality (5.6).

6 Problem set 1: Associated to lecture 1

Problem 1

In the lectures we utilised the Polyakov action for the string. In this problem we expand on some properties of this action, in particular relating to the energy-momentum tensor.

- Calculate the energy momentum tensor, defined as

$$T_{ab} \equiv \frac{4\pi}{\sqrt{-\det h}} \frac{\delta S_P}{\delta h^{ab}} \ , \tag{6.1}$$

  for the Polyakov action (2.15).

- The equation of motion for the worldsheet metric h_{ab} implies the energy-momentum tensor should vanish $T_{ab} = 0$. Use this to show the classical equivalence of the Polyakov and Nambu-Goto actions for the string.

- Show that the equations of motion for the X^μ are

$$h^{ab} \nabla_a \left(\partial_b X^\mu\right) = 0 \ . \tag{6.2}$$

  Use this to show that the conservation of the energy momentum tensor $\nabla^a T_{ab} = 0$.

- Use the conformal invariance of the Polyakov action to show that the energy-momentum tensor is traceless (without using any equations of motion).

Problem 2

This problem studies the symmetries of the Polyakov action.

- Show that the Polyakov action (2.15) is invariant under worldsheet diffeomorphisms (2.18).

- Consider the transformations

$$\xi^a \rightarrow \tilde{\xi}^a = \xi^a - \epsilon^a \ , \quad h_{ab} \rightarrow \tilde{h}_{ab} = h_{ab} e^{2\Lambda} \ . \tag{6.3}$$

  Show that for ϵ^a satisfying

$$\nabla_a \epsilon_b + \nabla_b \epsilon_a - \nabla^c \epsilon_c h_{ab} = 0 \ , \tag{6.4}$$

  there is a choice of Λ which leaves the metric invariant. Vectors ϵ_a satisfying (6.4) are called Conformal Killing Vectors.

- Write down the conformal Killing vectors in light-cone coordinates.

7 Problem set 2: Associated to lectures 2 and 3

Problem 1

- The Virasoro constraints in light-cone coordinates read

$$T_{++} = T_{--} = 0 \; . \tag{7.1}$$

In target-space light-cone coordinates, show that this implies

$$\partial_\pm X^- = \frac{1}{\alpha' p^+} \left(\partial_\pm X^i\right)^2 \; . \tag{7.2}$$

- In terms of oscillators, show that this leads to (3.12).

Problem 2

- Show how the decomposition of the $N = 1$ states of the closed string (3.33) is written explicitly in terms of the oscillator α^i_{-1} and $\tilde{\alpha}^i_{-1}$.
- Take the angular momentum operator (3.15) and consider a string state with no centre of mass angular momentum $l_{\mu\nu} = 0$. The resulting angular momentum is then associated to the spin of the state. Consider polarizing the string oscillators along the $i = 2, 3$ directions, so we excite only $\alpha^{2,3}$. Show that the relevant spin operator for massless states is then

$$S^{23} = -i\left(\alpha^2_{-1}\alpha^3_1 - \alpha^3_{-1}\alpha^2_1\right) - i\left(\tilde{\alpha}^2_{-1}\tilde{\alpha}^3_1 - \tilde{\alpha}^3_{-1}\tilde{\alpha}^2_1\right) \; . \tag{7.3}$$

- Use this to calculate the spin of the different massless fields in (3.33).

Problem 3

- Show that the metric ansatz (4.2) leads to the effective d-dimensional action (4.5).

Problem 4

The Swampland Distance Conjecture is associated to the proper distance traversed in field space. Consider parameterising the path between P and Q in terms of a scalar field t which need not be canonically normalised

$$P \; : \; t = 1 \; , \quad Q \; : \; t = \infty \; . \tag{7.4}$$

Take the kinetic term of t to be $g_{tt}(t)\left(\partial t\right)^2$.

- For $g_{tt} \sim t^\epsilon$, what are the bounds on the constant ϵ for the proper distance in field space $d(P, Q)$ to be infinite?
- Calculate g_{tt} for the case where $t = R$, the radius of the circle. Are $R \to 0$ and $R \to \infty$ at finite or infinite distance?

- Consider the dilaton field Φ in the effective action (3.34). Are the points $\Phi = 0$ and $\Phi = \pm\infty$ separated by infinite or finite distance? What is the physical meaning of $\Phi = \pm\infty$ in string theory? (Bonus: Can you name a Hollywood movie where (3.34) appears?)
- (Hard, advanced) What are the two dual infinite towers of states which become light for type IIA superstring theory in the weak and strong coupling limits?

Problem 5* (Optional, for string theory enthusiasts)

- Show equation (3.19) in the notes.

8 Problem set 3: Associated to lecture 4

Problem 1

- Show that the metric ansatz (5.1) leads to the effective d-dimensional action (5.2).

Problem 2

- Consider the string on a circle of radius R. By explicitly acting with oscillator creation operators on the vacuum, show that the massless spectrum for general values of R contains two gauge bosons A_μ and V_μ.
- Show that at the special value of $R = \sqrt{\alpha'}$ there are an additional 4 new gauge bosons.
- By calculating the charges of the new states under the original $U(1)$ gauge bosons, show that at $R = \sqrt{\alpha'}$ the gauge symmetry enhances from $U(1) \times U(1)$ to $SU(2) \times SU(2)$.

Problem 3

- Write the metric and electric field profile for an electrically charged Reissner-Nordstrum black hole. Explain the meaning of the extremality bound (5.18).
- The extremality bound (5.18) can be derived in an approximate way as follows. Write down the natural length scale r associated to a black hole of mass M. What is the energy stored in the electric field resulting from bringing in a charge Q from infinity to a sphere of the radius r? The mass of the black hole should be at least that energy, show that this leads to the extremality bound.
- Consider not a RN black hole but a charged black hole solution to the four-dimensional version of the theory (5.2) which has the additional coupled scalar ϕ. With the previous derivation of the extremality bound in mind, would you expect that the charge-to-mass ratio of an extremal black hole in this theory to be smaller or larger than that of a RN black hole with the same charge? How does this relate to the more general proposal for the WGC (5.22)?

Problem 4

- The WGC states that gravity should be the weakest force. Consider a four-dimensional theory which has a real scalar field ϕ rather than a gauge field

$$\int d^4X\sqrt{-g}\left[\frac{R}{2}-(\partial\phi)^2\right] . \tag{8.1}$$

A scalar field also acts as a force. What is the coupling constant, as appears in a Coulomb type force, of a particle of mass $m(\phi)$ to the scalar ϕ?

- Show that the statement that gravity should be the weakest force for a particle implies that for that particle we have

$$\partial_\phi m(\phi) \geq m(\phi) . \tag{8.2}$$

- What can we deduce form (8.2) regarding the form of $m(\phi)$ as we send $\phi \to \infty$? How does this relate to the behaviour of a mass scale in a different Swampland conjecture?

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