Exercise 4.2: $D$-dimensional phase space integrals

Show that the $D$-dimensional three-particle phase space for $Q \rightarrow p_1 + p_2 + p_3$, with $p_i^2 = 0$, can be expressed in terms of the kinematic invariants $s_{ij} = (p_i + p_j)^2$ as

$$d\Phi_3 = \frac{(2\pi)^{3-D}}{2^{D+1}} (Q^2)^{\frac{D-3}{2}} d\Omega_{D-2} d\Omega_{D-3} ds_{12} ds_{13} ds_{23} \left( s_{12} s_{13} s_{23} \right)^{\frac{D-4}{2}} \delta \left( Q^2 - s_{12} - s_{13} - s_{23} \right).$$

Solution:

We start from eq.(3.76) in the lecture notes:

$$d\Phi_3 = \frac{1}{4} (2\pi)^{3-D} dE_1 dE_2 d\theta_1 \left[ E_1 E_2 \sin \theta \right]^{D-3} d\Omega_{D-2} d\Omega_{D-3} \delta \left( (Q - p_1 - p_2)^2 \right),$$

(1)

based on the following parametrisation of the momenta:

$$Q = (E, \vec{0}(D-1)),$$

$$p_1 = E_1 (1, \vec{0}(D-2), 1),$$

$$p_2 = E_2 (1, \vec{0}(D-3), \sin \theta, \cos \theta),$$

$$p_3 = Q - p_2 - p_1.$$

(2)

The kinematic invariants $s_{ij} = (p_i + p_j)^2$ are given by

$$s_{12} = 2E_1 E_2 (1 - \cos \theta),$$

$$s_{13} = 2E_1 (E - E_2 (1 - \cos \theta)) = E (E - 2E_2),$$

$$s_{23} = 2E_2 (E - E_1 (1 - \cos \theta)) = E (E - 2E_1).$$

(3)

The Jacobian for the transformation is

$$| \det J | = \left| \det \left( \frac{\partial(s_{12}, s_{13}, s_{23})}{\partial(E_1, E_2, \theta)} \right) \right| = 8E^2 E_1 E_2 \sin \theta.$$

(4)

We can save a lot of algebra if we relate $| \det J |$ to the determinant of the Gram matrix $G_{ij}$, which we can define either using the vectors $Q, p_1, p_2$, or the vectors $p_1, p_2, p_3$. The determinant will be the same in both cases. In the first case, the Gram matrix is given by

$$G^{(1)} = \begin{pmatrix} Q^2 & Qp_1 & Qp_2 \\ Qp_1 & 0 & p_1 p_2 \\ Qp_2 & p_1 p_2 & 0 \end{pmatrix}$$

(5)

On the other hand, using $p_1, p_2, p_3$ to form frame-independent variables, we have

$$G^{(2)} = \begin{pmatrix} 0 & s_{12} & s_{13} \\ s_{12} & 0 & s_{23} \\ s_{13} & s_{23} & 0 \end{pmatrix}$$

(6)

such that

$$\det G^{(1)} = \det G^{(2)} = \det G = 2s_{12} s_{13} s_{23} = 8E^2 E_1^2 E_2^2 \sin^2 \theta.$$

(7)

and therefore

$$\det J = 2E \sqrt{2 \det G} = 4E \sqrt{s_{12} s_{13} s_{23}}.$$

(8)

This leads to

$$\int d\theta_1 dE_1 dE_2 \left[ E_1 E_2 \sin \theta \right]^{D-3} = \int d\theta_1 dE_1 dE_2 \left[ \frac{\det J}{8E^2} \right]^{D-3}$$

$$= \int ds_{12} ds_{13} ds_{23} (8E^2)^{3-D} (\det J)^{D-4}$$

$$= E^{2-D} 2^{1-D} \int ds_{12} ds_{13} ds_{23} \left( \frac{\det G}{2} \right)^{\frac{D-4}{2}}.$$


Combining everything one obtains

\[ \int d\Phi_3 = \frac{(2\pi)^{3-D}}{2^{D-1}}(Q^2)^{1-D/2} \int d\Omega_{D-2} d\Omega_{D-3} ds_{12} ds_{13} ds_{23} \left( s_{12}s_{13}s_{23} \right)^{\frac{D-4}{2}} \delta(Q^2 - s_{12} - s_{13} - s_{23}). \]

Defining dimensionless variables by

\[ y_1 = s_{12}/Q^2, y_2 = s_{13}/Q^2, y_3 = s_{23}/Q^2 \]

we arrive at

\[ \int d\Phi_3 = \frac{(2\pi)^{3-D}}{2^{D-1}}(Q^2)^{D-3} \int d\Omega_{D-2} d\Omega_{D-3} dy_1 dy_2 dy_3 \left( y_1 y_2 y_3 \right)^{\frac{D-4}{2}} \delta(1 - \sum_{i=1}^{3} y_i). \] (9)