Exercise 2.1: Four-gluon vertex

The 4-gluon vertex can be written as in a form where colour and kinematic parts of the Feynman rules factorise by introducing an auxiliary field with propagator

\[
\bar{\alpha} = \frac{\gamma}{\alpha} = \frac{\delta}{\beta} = b = -\frac{i}{2} \delta^{ab} (g^{\alpha\beta} g^{\gamma\delta} - g^{\alpha\delta} g^{\beta\gamma}) \quad \text{which couples only to the gluon, via}
\]

\[
\bar{\gamma} = \frac{\xi}{\zeta} = \frac{x}{c} = \xi = i \sqrt{2} g_s f^{xac} g^{\alpha_{\xi}} g^{\gamma_{\zeta}}.
\]

Show that a single four-gluon vertex can be written as a sum of the three graphs shown below, where colour and Lorentz indices factorise.

Exercise 2.2: Renormalisation schemes

The QCD $\beta$-function is of the form

\[
\beta(\alpha_s) = \mu^2 \frac{\partial \alpha_s}{\partial \mu^2} = -b_0 \alpha_s^2 \left[ 1 + b_1 \alpha_s + b_2 \alpha_s^2 + O(\alpha_s^3) \right] .
\] (1)

Consider two renormalisation schemes A and B, where the couplings are related by

\[
\alpha_s^B = \alpha_s^A \left[ 1 + c_1 \alpha_s^A + c_2 (\alpha_s^A)^2 + O((\alpha_s^A)^3) \right] .
\] (2)

Show that the first two coefficients, $b_0$ and $b_1$, are scheme-independent, while the third one in scheme B fulfills the relation

\[
b_2^B = b_2^A + c_2 - b_1 c_1 c_2^2 .
\] (3)