Stringy Non-Invertible Symmetries

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Based on [2402.00118], w/ J. Heckman, M. Montero, A. Sharon, C. Vafa, and I. Valenzuela

See also [2402.00105], by J. Kaidi, Y. Tachikawa, and H. Y. Zhang

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Last Time at Ringberg Castle…

Non-invertible symmetries were very new. I described a gravitational effect: the **breaking of non-invertible symmetries by gravitational solitons**. [JM ’21]

QFT Fact: Selection rules for non-invertible global symmetries fail to hold on nontrivial topology.

Thus: Once we couple to semiclassical gravity and sum over topologies, non-invertible global symmetries are broken.

This is in line with “No Global Symmetries”: every potential symmetry in QG must either be **broken** or **gauged**.

A Natural Question

Soon after, at my thesis defense, Cumrun raised a natural question:

Can non-invertible symmetries be gauged in QG instead?

At the time, we did not know what “non-invertible gauge symmetry” meant, but they seemed easy to construct in string theory, due to the following logic:

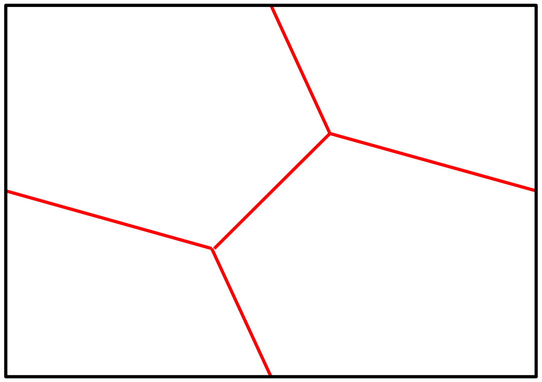
Non-Invertible Global Symmetry on the Worldsheet $\Rightarrow$ **Non-Invertible Gauge Symmetry in Target Space**

It is very easy to find examples of the LHS.

Today: What have we learned since then? Is this picture correct?

Non-Invertible Gauge Symmetries in QFT

Since then, non-invertible symmetries have seen an enormous amount of activity.



We now know exactly what it means to gauge a non-invertible symmetry: **sum over a network of non-invertible extended operators**.

This defines a QFT notion of "non-invertible gauge theory." (For experts: this includes SymTFTs.)

So, we ask: is this QFT notion the target space physics of non-invertible global symmetries on the worldsheet?

Stringy Non-Invertible Symmetries

Answer: Nope! The target space physics of non-invertible global symmetries of the worldsheet CFT is something else. Call it "**stringy non-invertible symmetry**," whatever it might be.

The QFT notion of "non-invertible gauge symmetry" does appear in QG (such as in AdS/CFT) but does not arise the way we thought.

Key Observation: String perturbation theory is not just 2D CFT; you couple to **2D worldsheet gravity** and sum over worldsheet topologies.

Stringy non-invertible symmetries are generically **broken by string loop effects**, and only restored in the limit $g_s \to 0$. They can still be useful, so long as we are near such a limit. Relevant for swampland!

Plan for the Talk

1. Review non-invertible symmetries in 2D CFT.
2. Illustrate the breaking effect in concrete examples.
3. General story and discussion.

Review: Non-Invertible Symmetries in 2D CFT

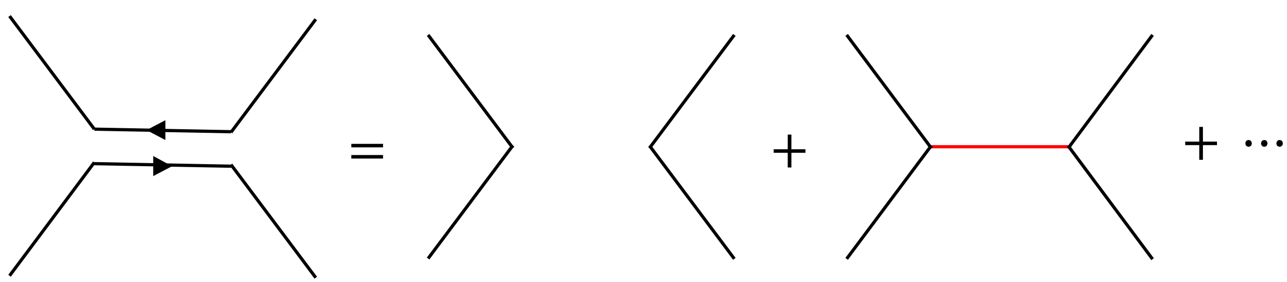
Non-Invertible Symmetries in 2D CFT

Given a 2D CFT, a (0-form) **non-invertible symmetry** is a collection $\{\mathcal{N}_i\}$ of topological defect line operators (TDLs), with fusion algebra:

$$\mathcal{N}_i \otimes \mathcal{N}_j = \sum_k \mathcal{T}_{ij}^k \mathcal{N}_k \,, \qquad \mathcal{T}_{ij}^k \in \mathbb{N}.$$

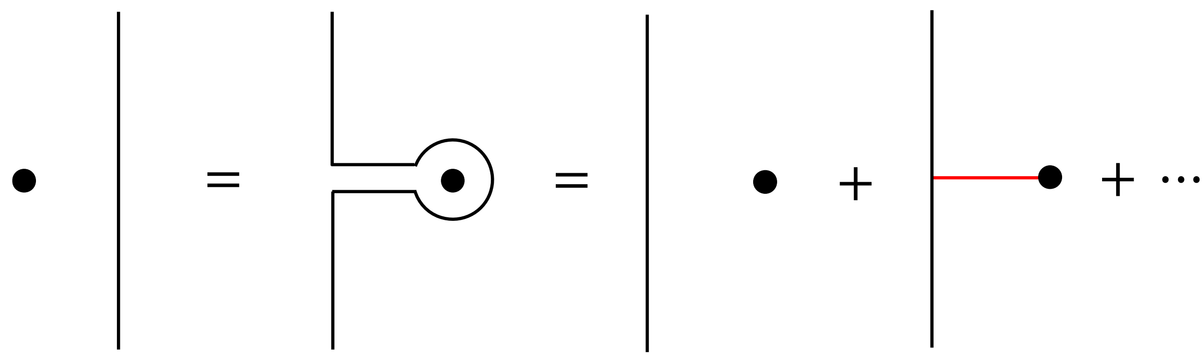
Generalization of group law. For any TDL $\mathcal{N}$, we have $\mathcal{N} \otimes \mathcal{N}^\dagger = 1 + \cdots$, a sum over other TDLs. Invertible if and only if this sum contains only the identity.

Non-invertible TDLs **cannot be freely reconnected**:



Action on Local Operators

TDLs acts on local operators by **sweeping** (or lassoing):

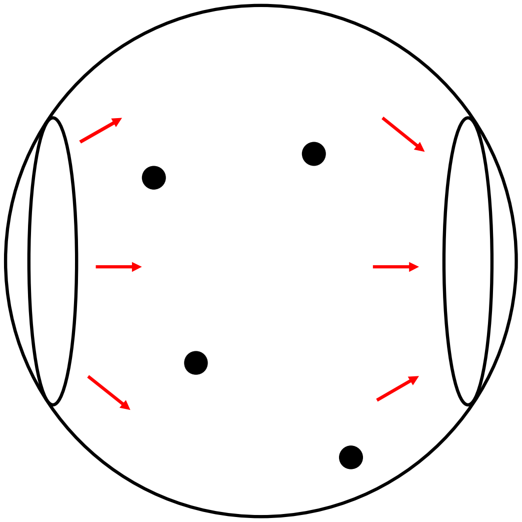


A non-invertible symmetry maps local operators to superpositions of local and/or disorder operators.

Def: a **representation (charge) of the non-invertible symmetry** is a collection of local and/or disorder operators closed under the non-invertible action. (For experts: representation of Ocneanu's tube algebra.)

Selection Rules at String Tree Level

Consider a sphere n-point function $\langle \mathcal{O}_1 \ldots \mathcal{O}_n \rangle_{\mathbb{S}^2}$.
Standard argument: nucleate TDL, sweep it past local operators, and annihilate it.

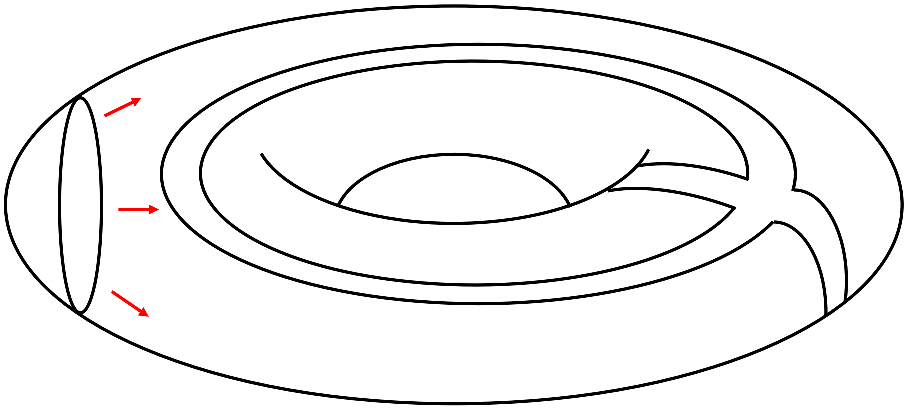


Invertible: Implies $\langle \mathcal{O}_1 \ldots \mathcal{O}_n \rangle_{\mathbb{S}^2}$ can only be nonzero if charges cancel.

Non-Invertible: Obtain system of equations relating order/disorder correlators. Solving system leads to **selection rules**: $\langle \mathcal{O}_1 \ldots \mathcal{O}_n \rangle_{\mathbb{S}^2}$ can only be nonzero if the (non-invertible symmetry) charges cancel. [Lin, Okada, Seifnashri, Tachikawa '22]

Selection Rules at Higher Genus

What changes at string loop level? If we try to repeat the argument on a Riemann surface, $\mathcal{N}$ gets caught on the handles, in addition to getting caught on the local operators.



Obtain a **network of TDLs wrapping every cycle**. Additional network spoils derivation of selection rules.

Concrete Examples

Non-Invertible Momentum Symmetry of $S^1/\mathbb{Z}_2$

Consider the orbifold CFT $S^1/\mathbb{Z}_2$, by the action $X \to -X$. Before orbifolding, we have a $U(1)^{\mathrm{KK}}$ **momentum symmetry**, generated by $\mathcal{U}_\theta = e^{\frac{i\theta R}{2\pi}\int *\mathrm{d}X}$.

Under the $\mathbb{Z}_2$ action, we have $\mathcal{U}_\theta \leftrightarrow \mathcal{U}_{-\theta}$, so only $\mathcal{U}_0, \mathcal{U}_\pi$ are gauge-invariant after orbifolding, generating $\mathbb{Z}_2^{\mathrm{KK}}$.

We have a larger **non-invertible momentum symmetry**, generated by the topological operators $\mathcal{L}_\theta = \mathcal{U}_\theta + \mathcal{U}_{-\theta}$.

Charges: KK momentum p, up to $p \to -p$. Charged operators are cosine wavefunctions $\mathcal{O}_p = \cos(pX/R)$, linear combinations of plane waves of KK momentum $\pm p$.

$S^1/\mathbb{Z}_2$ at Tree Level

Consider a sphere correlation function $\langle \mathcal{O}_{p_1} \dots \mathcal{O}_{p_n} \rangle_{\mathbb{S}^2}$ of cosine operators.

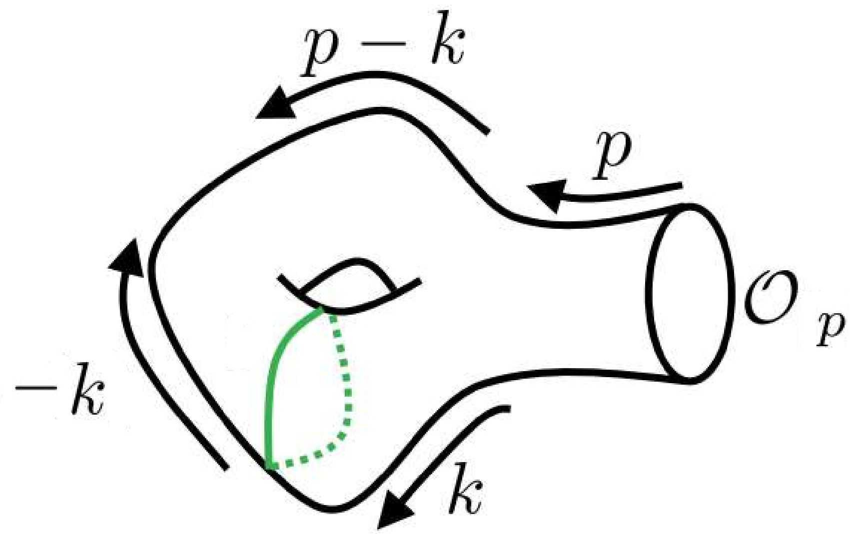
General Fact: sphere correlation functions of untwisted operators in an orbifold can be computed "upstairs," before taking the orbifold. But upstairs, KK momentum is conserved!

Selection Rule: Conservation of KK momentum modulo $p \leftrightarrow -p$. A sphere correlation function $\langle \mathcal{O}_{p_1} \dots \mathcal{O}_{p_n} \rangle_{\mathbb{S}^2}$ can only be nonzero if we can choose signs such that $\pm p_1 \pm \cdots \pm p_n = 0$.

$S^1/\mathbb{Z}_2$ at Loop Level

Consider the torus one-point function $\langle \mathcal{O}_k \rangle_{\mathbb{T}^2}$ of a cosine operator. On the torus, the $\mathbb{Z}_2$ orbifold instructs us to sum over $\mathbb{Z}_2$ line insertions on the two cycles.

Tracking the flow of KK momentum through $\mathbb{T}^2$ with a single $\mathbb{Z}_2$ line inserted, we learn that $\langle \mathcal{O}_p \rangle_{\mathbb{T}^2}$ can be nonzero if there exists p such that $p - k = k$, or in other words, if $p = 2k$.



Remaining Selection Rule: Conservation of KK momentum mod 2. At loop level, the non-invertible momentum symmetry is **broken to its invertible sub-symmetry** $\mathbb{Z}_2^{\mathrm{KK}}$.

$\mathbf{Rep}(\Gamma)$ Symmetry of Non-Abelian Orbifolds

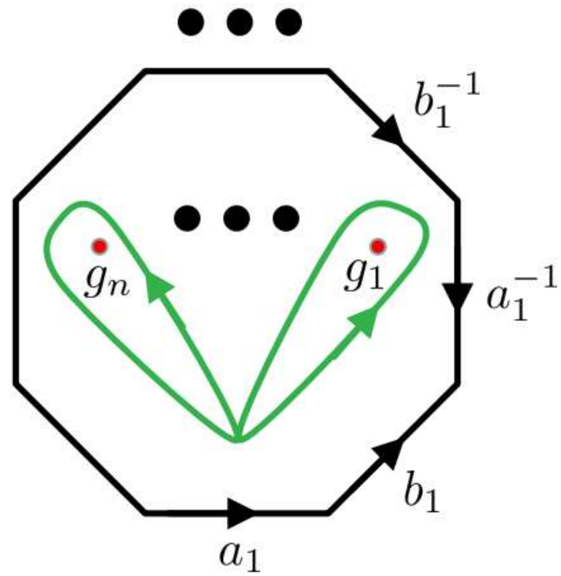
Consider a worldsheet CFT with a non-abelian symmetry Γ, and gauge Γ on the worldsheet to obtain a non-abelian orbifold. Have **twisted sectors** labeled by conjugacy classes $[g] \subset \Gamma$.

Conjugacy class is the charge under a **non-invertible $\mathbf{Rep}(\Gamma)$ symmetry** generated by topological Γ Wilson lines.

Selection Rule: Conservation of conjugacy class $[g]$. A sphere n-point function $\langle \mathcal{O}_{[g_1]} \ldots \mathcal{O}_{[g_n]} \rangle_{\mathbb{S}^2}$ of twisted sector operators can only be nonzero if we can choose representatives $g_i \in [g_i]$ such that the product is trivial: $g_1 \cdots g_n = 1$.

Non-Abelian Orbifolds at Loop Level

What happens at higher genus? A Riemann surface Σ_g of genus g can be obtained by gluing the edges of a $4g$-gon as illustrated.



On Σ_g, a correlation function can only be nonzero if the holonomy is a product of commutators. As a result, operators in the **commutator subgroup** $[\Gamma, \Gamma]$ can get a vev. [Hamidi, Vafa '87]

Remaining Selection Rule: Conservation of the image of $[g]$ in the abelianization $\Gamma_{\mathrm{ab}} = \Gamma/[\Gamma, \Gamma]$. At string loop level, the non-invertible $\mathrm{Rep}(\Gamma)$ symmetry is **broken to its invertible sub-symmetry** $\mathrm{Rep}(\Gamma_{\mathrm{ab}}) = \Gamma_{\mathrm{ab}}^{\vee}$.

Discussion

Universal Breaking of Non-Invertible Symmetries

In both examples, we saw that that the non-invertible symmetry was **broken to its invertible sub-symmetry** by string loop effects.

Math Conjecture: this pattern is general. A precise statement about fusion categories.

Could not quite prove, but:

- Can prove for Verlinde lines of diagonal RCFT.
- Checked in examples as exotic as Haagerup symmetry. This sort of non-abelian non-invertible symmetry is where one might have expected a counterexample.

Connections to Swampland Conjectures

Besides the obvious connection to No Global Symmetries, stringy non-invertible symmetries show up in at least a few other Swampland contexts:

1. **Tower of States for $S^1/\mathbb{Z}_2$ Decompactification:** KK modes are charged under the non-invertible, continuous KK momentum symmetry. Can (re)argue for their existence using non-invertible spectral flow, at least when g_s is small.

2. **(Sub)lattice WGC:** Some counterexamples to Lattice WGC (such as toroidal orbifolds) have continuous non-invertible symmetries that enhance discrete symmetries under which the non-extremal states are charged.

3. **Hidden Higher SUSY:** In the Green-Schwarz formalism, 4D $\mathcal{N}=2$ orbifolds of $\mathcal{N}=4$ backgrounds have non-invertible fermionic symmetries that realize a non-invertible version of target space $\mathcal{N}=4$ SUSY. (Re)explain extra SUSY protection in (tree-level exact) prepotential when orbifold acts freely.

Conclusion

In this talk, I discussed the physics of non-invertible symmetries on the string worldsheet.

We saw that these “stringy non-invertible symmetries” do not correspond to ordinary non-invertible gauge symmetries in target space.

Instead, I argued that they are **always broken by string loop effects** (I would love to see this proven as a theorem). As a result, stringy non-invertible symmetries are a type of symmetry that can **only emerge in the limit of a tensionless, perturbative string**, well outside local EFT.

Nevertheless, stringy non-invertible symmetries can still be a useful tool in the perturbative regime, and their applications should be studied further.

Thank you for listening!

Bonus: AdS/CFT

Non-Invertible Symmetries in AdS/CFT

Under the AdS/CFT dictionary, we usually say that

Global Symmetry in the CFT $\Longleftrightarrow$ **Gauge Symmetry in AdS**

Given a holographic CFT with a non-invertible global symmetry, we expect the bulk to have a non-invertible gauge symmetry. More precisely, the bulk will admit a topological sector (the SymTFT) which can be viewed as the **gauging of the boundary non-invertible symmetry**. Many examples have been described in the literature!

How does this interact with our story? The non-invertible symmetry in the CFT is exact, so why isn't it broken by string loop effects in the bulk?

To answer these questions, let's look at an example.

$\mathbf{AdS_3 \times S^3 \times T^4}$

Consider the Type IIB NS background $\mathrm{AdS}_3 \times S^3 \times T^4$. Tuning moduli, we can **preserve a non-abelian group Γ** of discrete isometries of T^4_{tuned}.

At this point, the dual CFT_2 has Γ global symmetry, which may be gauged to obtain the orbifold CFT_2/Γ. This orbifold admits a non-invertible $\mathrm{Rep}(\Gamma)$ global symmetry, so there should be a **propagating $\mathrm{Rep}(\Gamma)$ gauge theory** in AdS_3.

Fact: in $3\mathrm{D}$, $\mathrm{Rep}(\Gamma)$ gauge theory is electromagnetically dual to ordinary Γ gauge theory. The Dirichlet boundary condition for $\mathrm{Rep}(\Gamma)$ is mapped to the Neumann boundary condition for Γ.

Thus: The $\mathrm{Rep}(\Gamma)$ gauge theory in AdS_3 is just the Γ gauge theory arising from discrete isometries of T^4_{tuned}, now with Neumann boundary conditions. The bulk worldsheet σ-model into T^4_{tuned} has global symmetry Γ, an invertible symmetry which is not broken by string loops.

Invertible Gauge Symmetries in Disguise

In every example of AdS/CFT with an Einstein gravity bulk, we find that the bulk “non-invertible gauge theory” has an **alternative presentation in terms of ordinary, invertible gauge fields**.

This can arise due to electromagnetic duality, or directly due to a presentation of the bulk topological sector as invertible gauge fields + Chern-Simons terms.

While such examples certainly qualify as “non-invertible gauge symmetry in QG,” they are of a more benign sort than the “stringy non-invertible gauge symmetries” I described before.

To find examples of stringy non-invertible symmetries in AdS/CFT, we need to move beyond theories whose bulk is semiclassical.

An Exception that Proves the Rule

Let us return to $\mathrm{AdS}_3 \times S^3 \times T^4$, in the special limit of $N_5 = 1, N_1 = N$, as considered by [L. Eberhardt, M. R. Gaberdiel, R. Gopakumar '18, '19, '20].

This is a **tensionless string theory with $\boldsymbol{g_s > 0}$**, whose holographic dual is the symmetric orbifold CFT $\mathrm{Sym}^N(T^4) = (T^4)^N/S_N$. Thus, the CFT has an exact, non-invertible $\mathrm{Rep}(S_N)$ symmetry.

The large-N limit $\mathrm{Rep}(S_\infty)$ is realized as a non-invertible symmetry of the bulk string worldsheet CFT. Why isn't it broken by string loops?

The tensionless worldsheet CFT **localizes on branched covers of the boundary $\boldsymbol{S^2}$ of $\boldsymbol{\mathrm{AdS}_3}$**. At each branch point, we have some permutation of sheets, defining a conjugacy class in S_∞. Though the worldsheet Σ may have any genus, the existence of a branched cover $\Sigma \to S^2$ ensures that the conjugacy classes fuse to the identity, imposing the selection rule at all orders in g_s.

Bonus: String Field Theory?

What are Stringy Non-Invertible Symmetries?

We have seen that “stringy non-invertible symmetries” are generically broken, and only restored in the limit $g_s \to 0$ in which a string becomes tensionless. How should we think about them?

Proposal: stringy non-invertible symmetries are a non-invertible extension of the **gauge symmetries of string field theory (SFT)** (which are only restored as $g_s \to 0$).

Recall that sweeping a non-invertible TDL inserts a network of TDLs on each cycle of a Riemann surface. This can be viewed as a generalized orbifold. Thus, perhaps **the orbifold procedure can be viewed as a gauge symmetry** of SFT.

A string theory and its orbifold contain (essentially) the same information at tree level, while beginning to differ more substantially for $g_s > 0$.