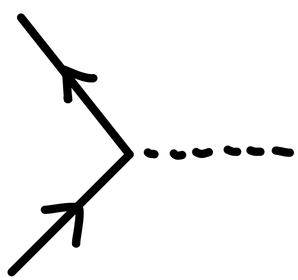
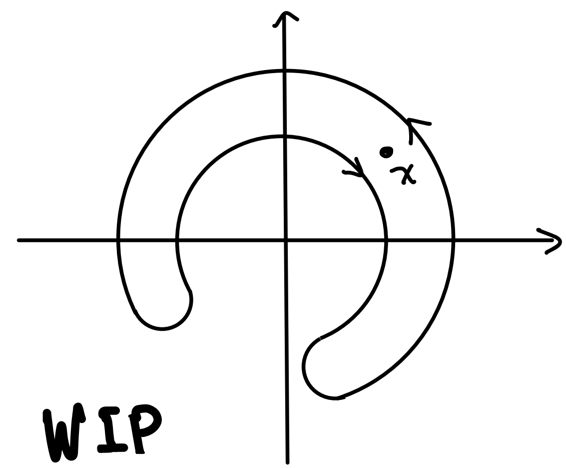
Proving the Weak Gravity Conjecture in Perturbative (Bosonic) String Theory

Ben Heidenreich
UMass Amherst





Based on 2401.14449 & WIP
with Matteo Lotito

Geometry, Strings, and the Swampland, Mar. 2024

The Weak Gravity Conjecture

(Arkani-Hamed, Motl, Nicolis, Vafa '06)

In a quantum gravity theory with a massless photon, there is a (light) charged particle with

$$\frac{|q|}{m} \geq \frac{|Q|}{M}\bigg|_{\text{large, extremal black hole}}$$

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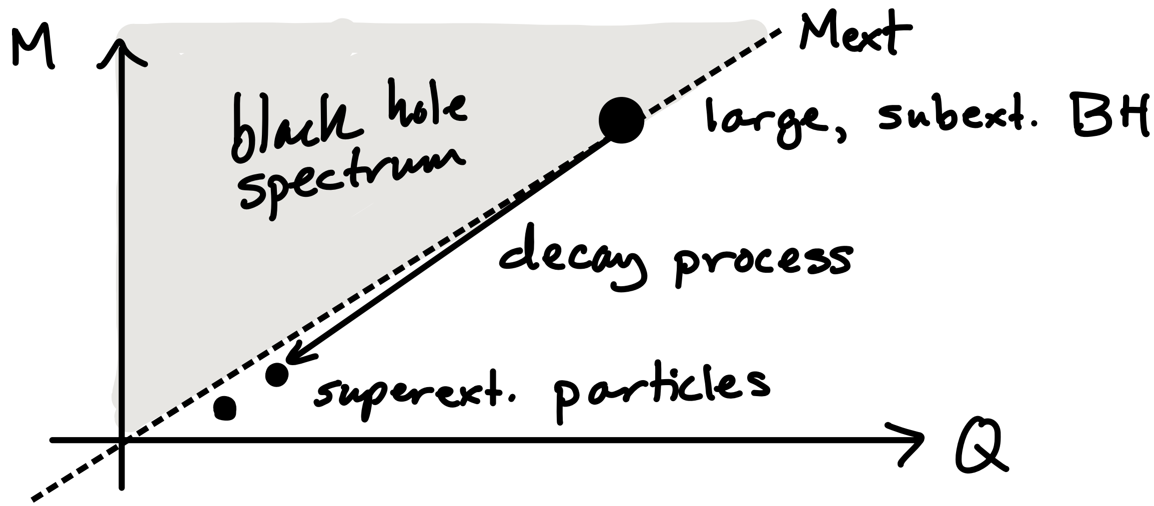
<u>Extremal</u> BH is one that saturates <u>extremality bound</u>:

$$M_{BH} \geq M_{ext}(Q) \equiv \inf\left\{\begin{array}{l}\text{masses of all BHs} \\ \text{of charge } Q\end{array}\right\}$$

"<u>large</u>" means $M, |Q| \to \infty$ $\Rightarrow$ $|Q|/M\,|_{ext} \to$ const. $\swarrow$

(assuming $\Lambda_{c.c.} = 0$) determined by 2-deriv. EFT

WGC $\Rightarrow$ black holes are <u>unstable</u>



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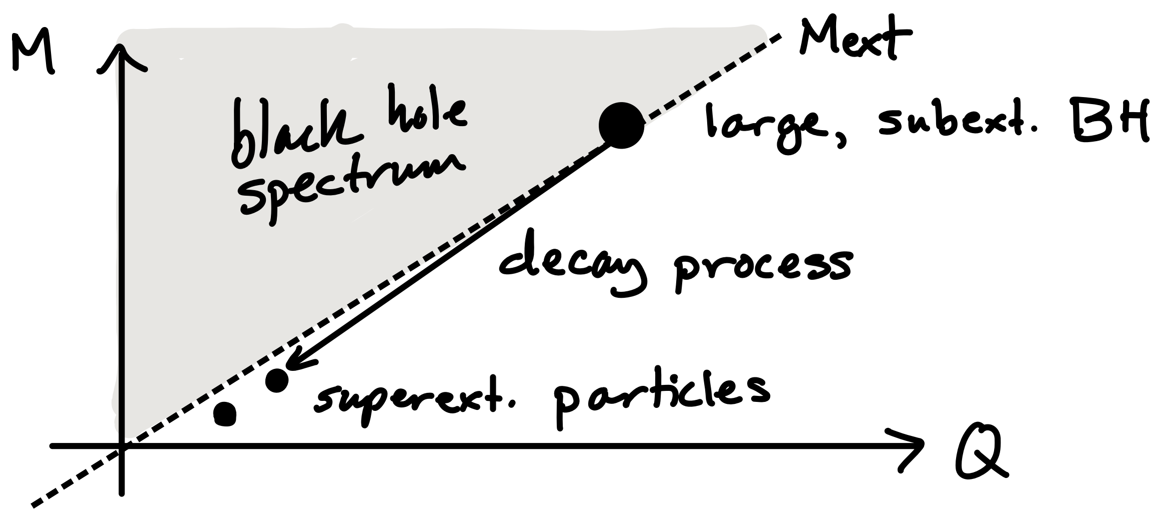
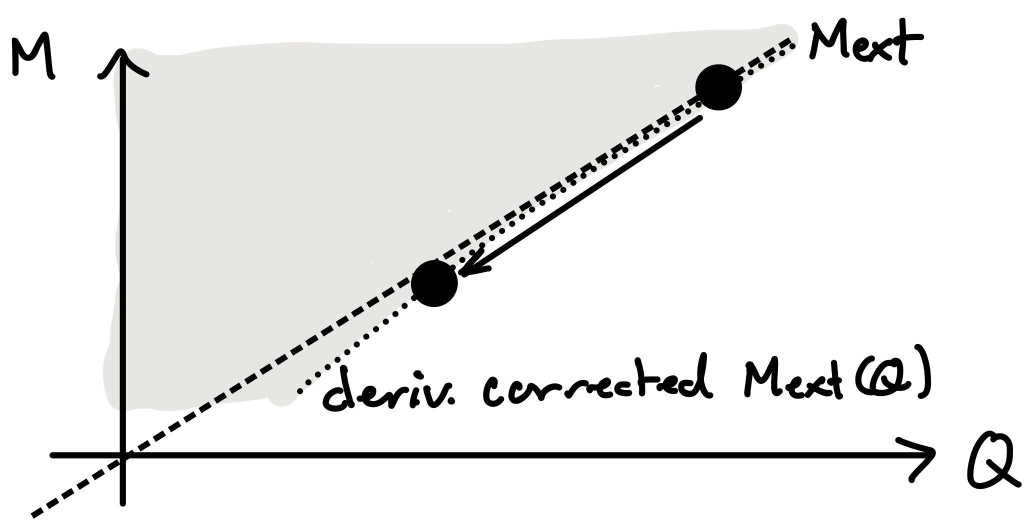


... possibly they decay to <u>other</u> BHs !



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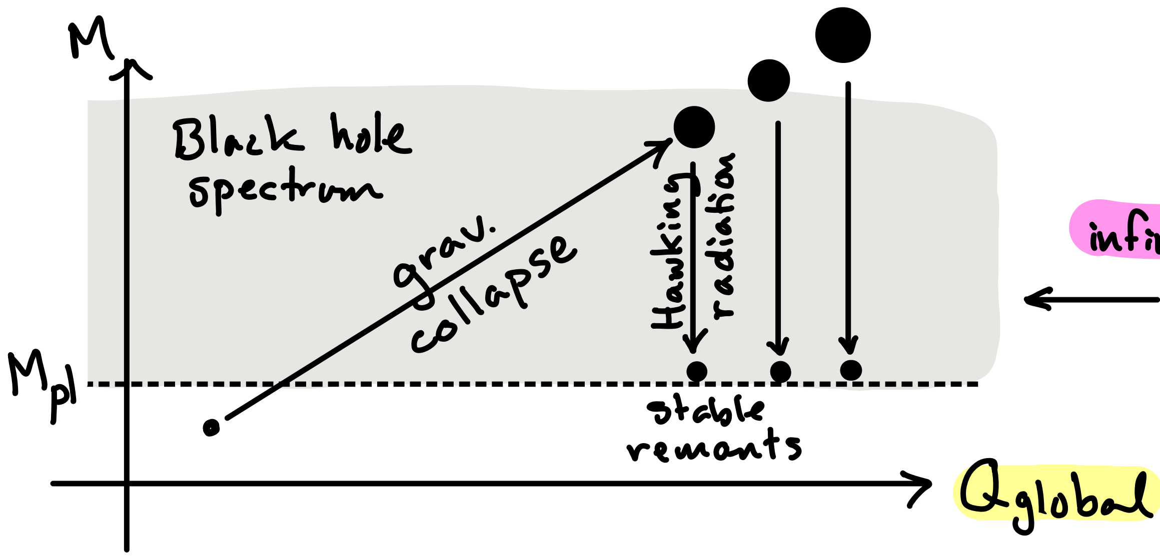

infinite density of states!
(BHs can have any global charge)

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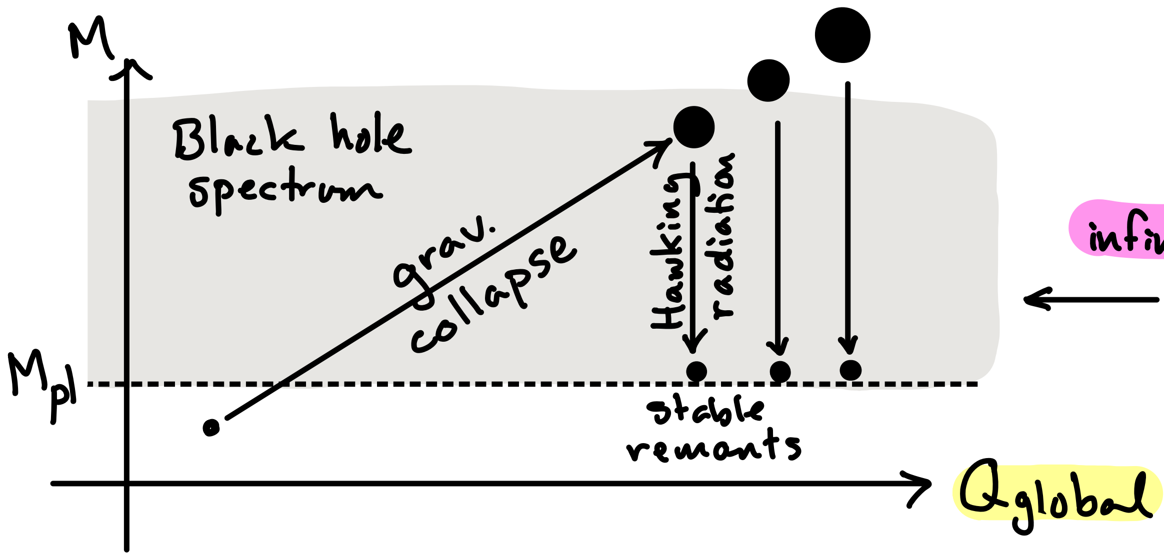

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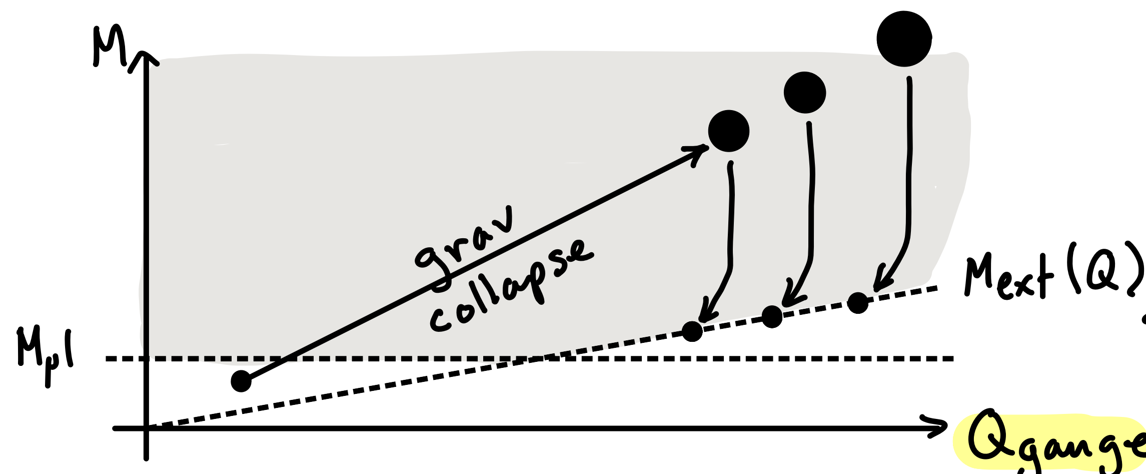
<u>VS.</u>



<u>finite</u> density of states

Top-down evidence: heterotic ST on T^k

$$\boxed{\frac{\alpha'}{4}m^2 = \frac{1}{2}Q_L^2 + N - 1 = \frac{1}{2}Q_R^2 + \tilde{N}, \qquad N, \tilde{N} \in \mathbb{Z}_{\geq 0}}$$

where $Q_A = \{Q_{La}, Q_{R\tilde{a}}\}$ $\quad a = 1, \ldots, k+16, \quad \tilde{a} = 1, \ldots, k$

lies on even, self-dual charge lattice Γ

(even: $\forall Q \in \Gamma,\ Q \circ Q \equiv Q_L^2 - Q_R^2 \in 2\mathbb{Z}$)

(self-dual: $\Gamma = \Gamma^* \equiv \{Q \mid \forall Q' \in \Gamma,\ Q \circ Q' \in \mathbb{Z}\}$)

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$$\left(\begin{array}{l} \underline{\text{even}}: \ \forall Q \in \Gamma, \ Q \circ Q \equiv Q_L^2 - Q_R^2 \in 2\mathbb{Z}) \\ \underline{\text{self-dual}}: \ \Gamma = \Gamma^* \equiv \{Q \mid \forall Q' \in \Gamma, \ Q \circ Q' \in \mathbb{Z}\} \end{array}\right)$$

$$\mathcal{M} \sim \frac{SO(16+k, k)}{SO(16+k) \times SO(k)}$$

<u>moduli</u>

size/shape of torus
+ Wilson lines
for A_1, B_2

At generic point $G = U(1)^{16+2k}$ (each $U(1)$ a <u>mix</u> of left & right)

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$$\frac{\alpha'}{4}M_{BH}^2 \geq \frac{1}{2}\max(Q_L^2, Q_R^2) \equiv \frac{\alpha'}{4}M_{ext}^2(Q)$$ Sen '94

VS. lightest string mode of charge Q:

$$\frac{\alpha'}{4}m^2 = \frac{1}{2}\max(Q_L^2 - 2, Q_R^2)$$

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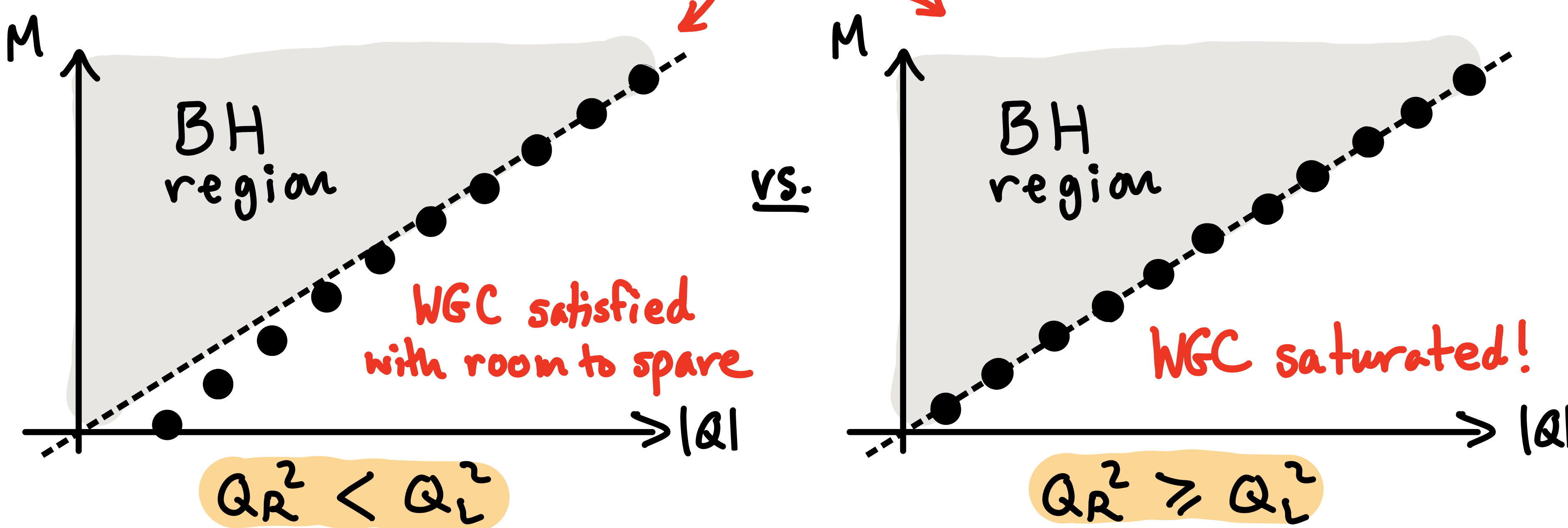

When can the WGC be saturated?

BPS bound is: $\frac{\alpha'}{4} m^2 \geq \frac{1}{2} Q_R^2$

$\Rightarrow$ lightest charge-Q mode BPS when $Q_R^2 \geq Q_L^2 - 2$

1) $Q_R^2 \geq Q_L^2$ string mode & extremal BH both BPS
WGC is saturated

2) $Q_R^2 = Q_L^2 - 2$ string mode is BPS, extremal BH is not
WGC satisfied, not saturated

3) $Q_R^2 < Q_L^2 - 2$ neither is BPS
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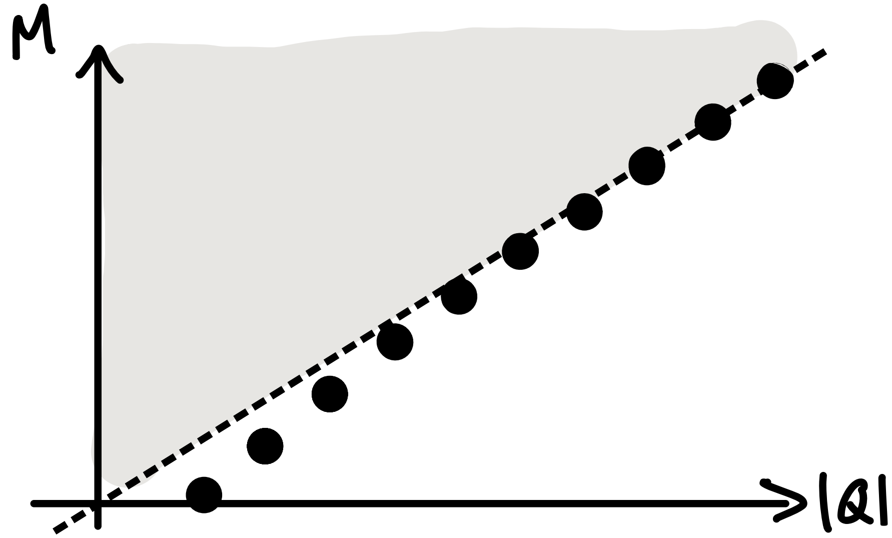
Ooguri-Vafa '16:	WGC can only be saturated if the superextremal particle is BPS

Careful: BPS particles do not always saturate ext. bound, as above!

The lattice WGC

(BH, Reece, Rudelius '15)



$\exists$ superextremal particles of <u>every</u> charge $Q \in \Gamma$!

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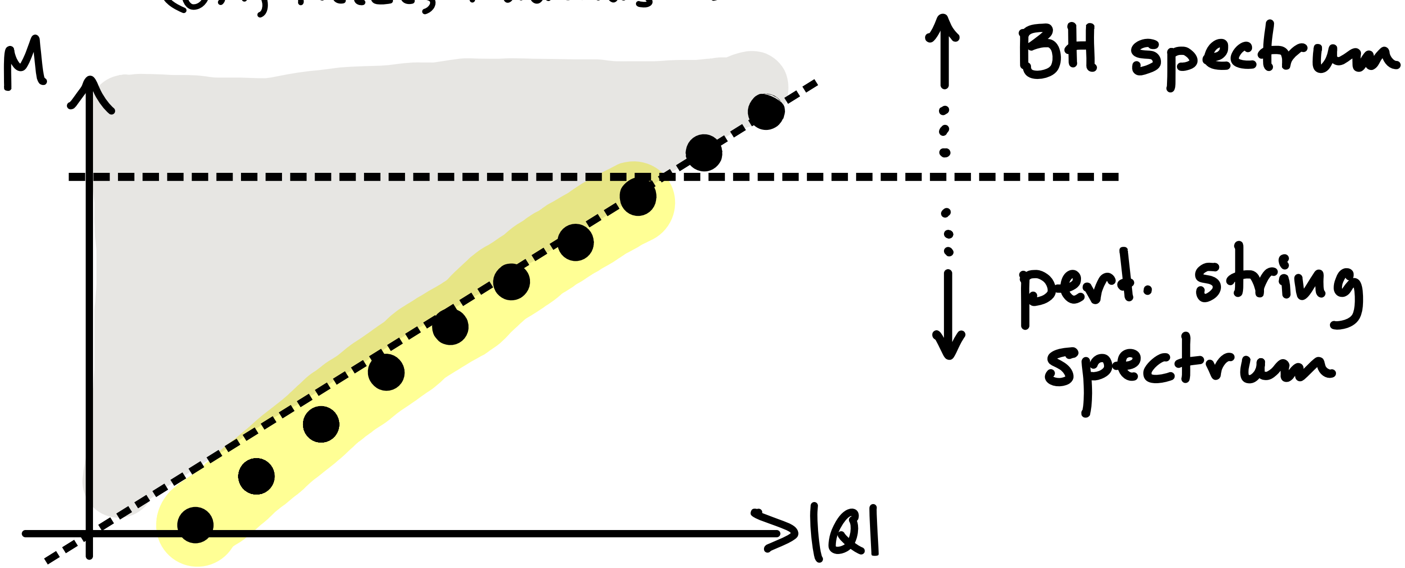

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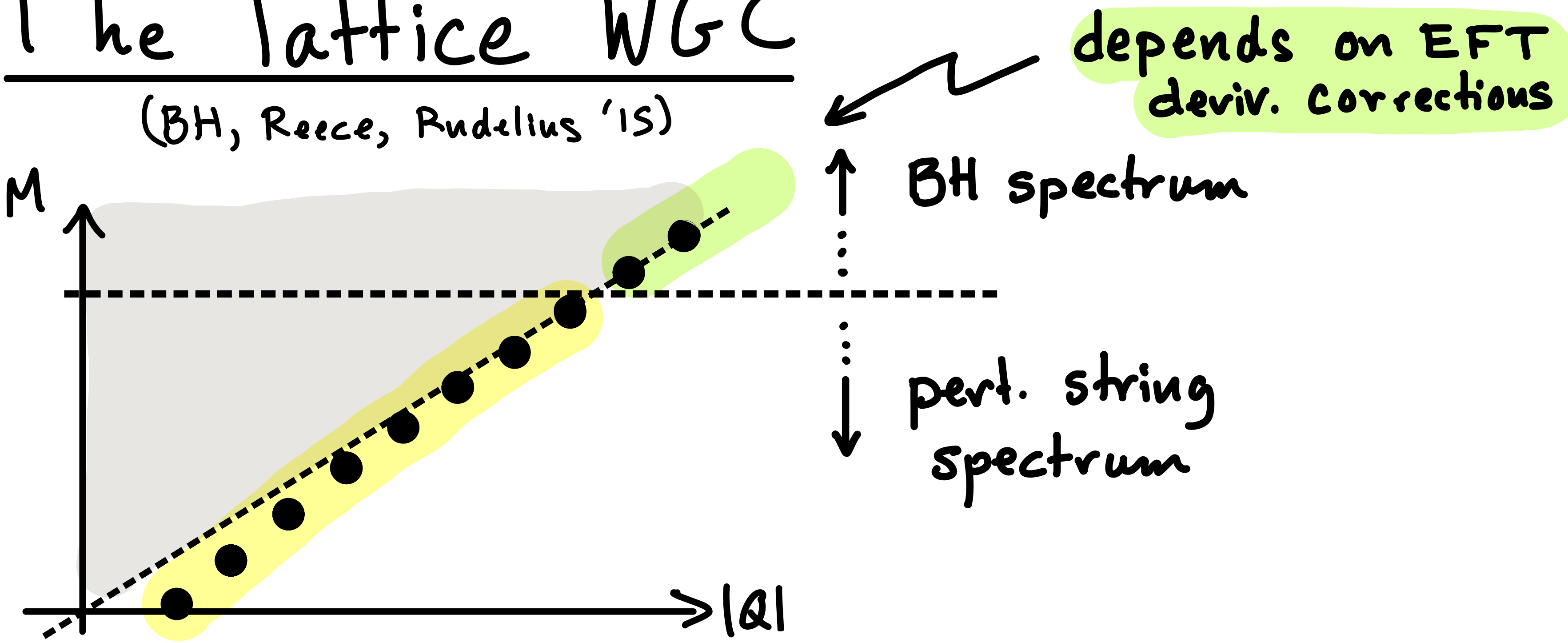

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The ~~lattice~~ sublattice WGC

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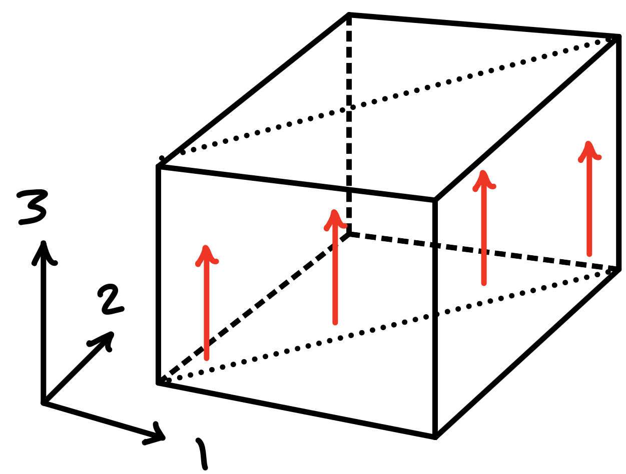
Lattice WGC true in many QGTs, but <u>not all</u>

E.g., $T^3/\mathbb{Z}_2$ orbifold



$\mathbb{Z}_2$: translation <u>&</u> reflection

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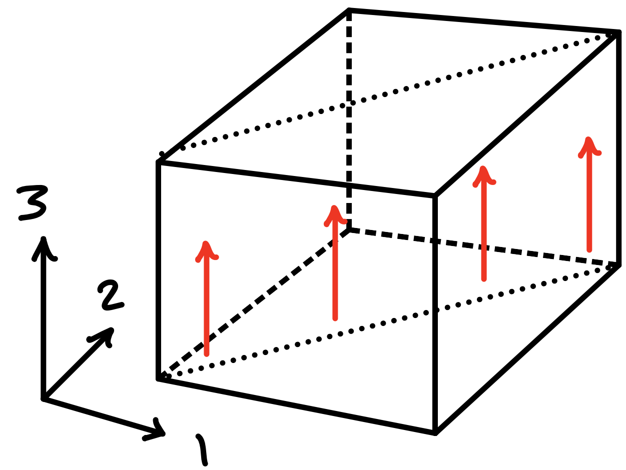

$\mathbb{Z}_2$: translation & reflection

$R_1 = R_2$ to preserve $\mathbb{Z}_2$

n_1, n_2 not individually conserved

KK modes: $m^2 = \frac{n_1^2}{R_1^2} + \frac{n_2^2}{R_2^2} + \frac{n_3^2}{R_3^2}$, charges: $Q_A = n_1 + n_2$, $Q_B = n_3$

v.s. $M_{ext}^2 = \frac{Q_A^2}{2R_1^2} + \frac{Q_B^2}{R_3^2} \Rightarrow m^2 - M_{ext}^2 = \frac{(n_1 - n_2)^2}{2R_1^2} \geq 0$

Modes w/ $n_1 = n_2$ are extremal

others are subextremal

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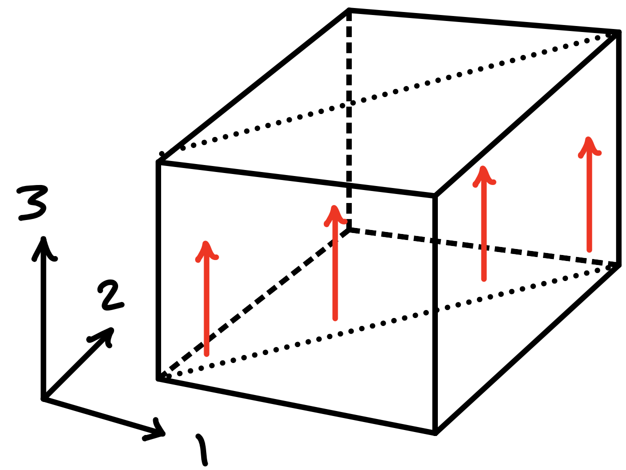

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$Q_A = n_1 + n_2 \in 2\mathbb{Z} + 1 \Rightarrow n_1 \neq n_2$
$\Rightarrow$ Lattice WGC violation!

The sublattice WGC

(BH, Reece, Rudelius '16; Montero, Shin, Soler '16)

Still true that:

$$\boxed{\begin{array}{l}\text{Sublattice WGC: } \exists \text{ finite index sublattice } \Gamma_0 \subseteq \Gamma_Q \\ \qquad \text{s.t. } \forall\, Q \in \Gamma_0,\ \exists \text{ charge } Q \text{ superext. particle}\end{array}}$$

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Weaker form:

Tower WGC: $\forall\, Q \in \Gamma_Q$, $\exists\, n \in \mathbb{Z}_{>0}$ s.t. $\exists$ charge nQ superext particle

(Andriolo, Junghaus, Noumi, Shin '18; BH, Reece, Rudelius '19)

A version of this follows from mild WGC for QGT on S^1 (BH, Reece, Rudelius '15 — see Timo's talk for caveats)

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Typically $n \sim \mathcal{O}(1) \implies$ tower of particles @ scale $g M_{Pl}^{\frac{D-2}{2}}$

... related to Distance Conjecture / QG resistance to $g \to 0$

The ~~lattice~~ sublattice WGC

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Existing evidence for sublattice WGC:

1. Proved in KK theory

2. Partial proof in tree level ST

   Thm: $\exists$ finite index sublattice $\Gamma_0 \subseteq \Gamma_Q$ ← electric NSNS charge lattice

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   $$\frac{\alpha'}{4} m^2 = \frac{1}{2} \max\left(Q_L^2, Q_R^2\right)$$

   Proof: We'll review it

   ... but unknown what $M_{ext}(Q)$ is in general } difficult PDE problem with general moduli

3. Checked case by case in many non-perturbative exs. (using BPS states, see Tom's talk)

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THIS TALK
Complete this argument for bosonic string

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Strategy

How to tell if a particle is superextremal?

Apparent digression: long-range forces

$$\vec{F}_{12} = \frac{\mathcal{F}_{12}}{V_{D-2}\, r^{D-2}}\, \hat{r}_{12}, \quad \mathcal{F}_{12} = f^{ab} q_{1a} q_{2b} - \kappa_N m_1 m_2 - G^{ij} \frac{\partial m_1}{\partial \phi^i} \frac{\partial m_2}{\partial \phi^j}$$

where

$$S_{EFT} \supseteq \int d^D x \sqrt{-g} \left[\frac{1}{2\kappa_D^2} R - \frac{1}{2} G_{ij}(\phi)\, \nabla\phi^i \cdot \nabla\phi^j - \frac{1}{2} f_{ab}(\phi)\, F^a \cdot F^b \right]$$

and f^{ab}, G^{ij} inverses of f_{ab}, G_{ij}, $\quad \kappa_N \equiv \frac{D-3}{D-2} \kappa_D^2$

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"self-repulsive" means $\mathcal{F}_{11} \geq 0$

Repulsive Force Conjecture (RFC): Replace superextremal $\hookrightarrow$ self-repulsive in WGC

Without moduli WGC = RFC (ext. BHs have zero self force)

But with moduli they are independent.

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However: A particle (mass $m(\phi)$, charge q) that exists & is self-repulsive everywhere in moduli space is superextremal.

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To prove, one writes:

$$M_{BH} = (\text{non-negative}) + \frac{1}{2}\int_{r_h}^{\infty} e^{2u}\,\frac{f^{ab}Q_aQ_b - k_N W^2(\phi) - G^{ij}W_{,i}W_{,j}}{V_{D-2}\,r^{D-2}}\,dr$$

$$+ W(\phi_\infty) \qquad \text{for any function } W(\phi)$$

where BH metric is $ds_D^2 = -e^{2u} f(r)\,dt^2 + e^{-\frac{2}{D-3}u}\left[\frac{dr^2}{f^2} + r^2 d\Omega^2\right]$

$$f(r) = 1 - \frac{r_h^{D-3}}{r^{D-3}}$$

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Then, picking $Q = Nq$, $W = Nm$ $(N \gg 1)$

$$M_{BH} \geq \text{non-negative} + W(\phi_\infty) \geq N m(\phi_\infty) = \frac{|Q|}{|q|} m(\phi_\infty)$$

$\Rightarrow$ particle is superextremal !

Proof Outline

(1) Prove $\exists$ tower of string modes with

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(2) Prove tower is self-repulsive

(3) $\Rrightarrow$ tower is superextremal, QED

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(3) $\Longrightarrow$ tower is superextremal, QED

BONUS: Prove Ooguri-Vafa WGC @ tree level
(no saturating bound unless BPS)
$\Longrightarrow$ safe from loop corrections for $g_s \ll 1$.

Bosonic string proof

First *define* what a (closed, oriented) bosonic string theory is. In a flat background, the worldsheet CFT *factors*:

$(X)^D \times \mathcal{C}_{int}$ where $\mathcal{C}_{int}$ is unitary, modular invariant, compact, with $c = \bar{c} = 26 - D$

$(X)^D$: free boson

$\mathcal{C}_{int}$: "internal CFT"

Bosonic string proof

First *define* what a (closed, oriented) bosonic string theory is. In a flat background, the worldsheet CFT *factors*:

$$(X)^D \times \mathcal{C}_{int}$$

$\uparrow$ free boson — $\mathcal{C}_{int}$: "internal CFT"

where $\mathcal{C}_{int}$ is unitary, modular invariant, compact, with $c = \tilde{c} = 26 - D$

Certain $\mathcal{C}_{int}$ primaries give rise to massless EFT fields:

Weight	Operator	EFT field(s)	
$(0,0)$	1	$g_{\mu\nu}, B_{\mu\nu}, \Phi^0 \leftarrow$ dilaton	(also tachyon)
$(1,0)$	$J^a(z)$	A^a_μ	worldsheet global symms become EFT gauge symms
$(0,1)$	$\tilde{J}^{\tilde{a}}(\bar{z})$	$A^{\tilde{a}}_\mu$	
$(1,1)$	$\varphi^i(z,\bar{z})$	Φ^i	marginal ops become massless EFT scalars; exactly marginal $\rightarrow$ modulus

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First *define* what a (closed, oriented) bosonic string theory is. In a flat background, the worldsheet CFT *factors*:

$$(X)^D \times \mathcal{C}_{int}$$

$(X)^D$: free boson; $\mathcal{C}_{int}$: "internal CFT"

where $\mathcal{C}_{int}$ is unitary, modular invariant, compact, with $c = \tilde{c} = 26 - D$

In general, for any weight $(h, \tilde{h})$ $\mathcal{C}_{int}$ primary, there are physical states of mass:

$$\frac{\alpha'}{4} m^2 = \max(h, \tilde{h}) + N - 1 \qquad \forall\ N = 0, 1, 2, \ldots$$

where the available spins depend on N, etc.

A) Modular invariance

(based on BH, Reece, Rudelius '16
Montero, Shiu, Soler '16)

Torus partition function

$$Z(\tau,\bar\tau) = \mathrm{Tr}\left[q^{L_0 - \frac{c}{24}} \bar q^{\tilde L_0 - \frac{\tilde c}{24}} \right]$$

$$= \underbrace{\sum_{(h,\tilde h)}}_{\text{sum over spectrum}} q^{h - \frac{c}{24}} \bar q^{\tilde h - \frac{\tilde c}{24}} \qquad q \equiv e^{2\pi i \tau}$$

Z must be modular invariant. With primaries inserted

$$Z[\mathcal{O}_1(w_1,\bar w_1)\cdots](\tau,\bar\tau) = Z[\mathcal{O}_1(w_1,\bar w_1)\cdots](\tau+1,\bar\tau+1)$$

$$= \frac{1}{\tau^{\sum h}\bar\tau^{\sum \tilde h}} Z\left[\mathcal{O}_1\left(\frac{w_1}{\tau},\frac{\bar w_1}{\bar\tau}\right)\cdots\right]\left(-\frac{1}{\tau},-\frac{1}{\bar\tau}\right)$$

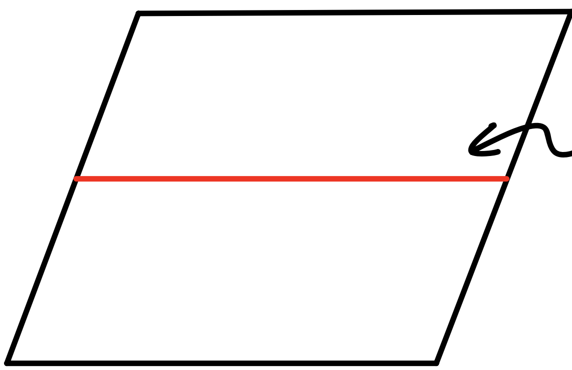
where we use cylindrical quantization $w \cong w+2\pi \cong w+2\pi\tau$.

To constrain charged spectrum, consider "flavored" partition function:

$$Z(\mu, \tau; \tilde{\mu}, \bar{\tau}) = \sum q^{h - \frac{c}{24}} y^{Q} \bar{q}^{\tilde{h} - \frac{\tilde{c}}{24}} \tilde{y}^{\tilde{Q}}$$

$$y^{Q} \equiv e^{2\pi i \mu_a Q^a}, \quad \tilde{y}^{\tilde{Q}} \equiv e^{-2\pi i \tilde{\mu}_{\tilde{a}} \tilde{Q}^{\tilde{a}}}$$

This corresponds to inserting a line operator on the A cycle:



$$e^{i\mu_a \oint J^a dw - i\tilde{\mu}_a \oint \bar{J}^a d\bar{w}}$$

... which is a symm. op for $\underbrace{\mu, \tilde{\mu} \in \mathbb{R}}_{\text{chemical potentials}}$.

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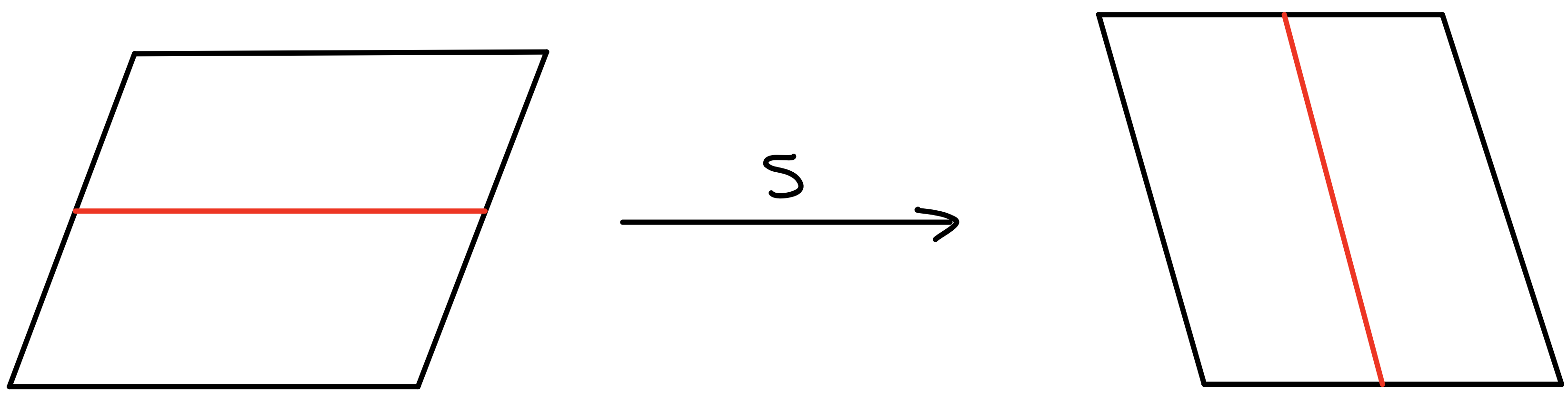

twisted sector!

$\Rightarrow$ flavored Z is <u>not</u> modular invariant!

<u>In fact</u>, w/ normalization

$$J^a(w)\,J^b(0) \sim -\frac{\delta^{ab}}{w^2}, \quad \tilde{J}^a(\bar{w})\,\tilde{J}^b(0) \sim -\frac{\delta^{ab}}{\bar{w}^2}$$

can argue that:

$$\boxed{Z(u,\tau;\tilde{u},\bar{\tau}) = e^{-\pi i u^2/\tau + \pi i \tilde{u}^2/\bar{\tau}}\, Z\left(\frac{u}{\tau},-\frac{1}{\tau};\frac{\tilde{u}}{\bar{\tau}},-\frac{1}{\bar{\tau}}\right)}$$

(Benjamin, Dyer, Fitzpatrick, Kachru '16)

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$$Z(\mu,\tau;\tilde{\mu},\bar{\tau}) = e^{-\pi i\mu^2/\tau + \pi i\tilde{\mu}^2/\bar{\tau}}\, Z\left(\frac{\mu}{\tau}, -\frac{1}{\tau}; \frac{\tilde{\mu}}{\bar{\tau}}, -\frac{1}{\bar{\tau}}\right)$$

(Benjamin, Dyer, Fitzpatrick, Kachru '16)

New argument (BH, Lotito '24):

$$:J^a(w_1)J^b(w_2):_\tau \equiv J^a(w_1)J^b(w_2) + \delta^{ab}\frac{1}{4\pi^2}\wp\left(\frac{w_1-w_2}{2\pi}\Big|\tau\right)$$

$$\mathring{\circ} J^a(w_1)J^b(w_2)\mathring{\circ}_\tau \equiv J^a(w_1)J^b(w_2) + \delta^{ab}\frac{1}{4\pi^2}\wp_0\left(\frac{w_1-w_2}{2\pi}\Big|\tau\right)$$

$\wp(z|\tau) \equiv$ Weierstrass $\wp$ func. (periods $1,\tau$)

$\wp_0(z|\tau) \equiv \wp(z|\tau) + \frac{\pi^2}{3}E_2(\tau)$ ← holomorphic Eisenstein series

$$\Rightarrow \int_0^1 \wp_0(z|\tau)\,dz = 0$$

$:(\ldots):_\tau$ is <u>modular</u> in that

$$Z[:J^{a_1}(w_1)\cdots J^{a_n}(w_n):_\tau] = Z[\underbrace{:J^{a_1}(0)\cdots J^{a_n}(0):_\tau}_{\text{entire function on } T^2}]$$

is a weight $(n,0)$ modular form.

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entire function on T^2

is a weight $(n,0)$ modular form.

... whereas ${}^{\circ}_{\circ}(\dots){}^{\circ}_{\circ\,\tau}$ is constructed to satisfy:

$$Z\left[\,{}^{\circ}_{\circ}J^{a_1}(w_1)\cdots J^{a_n}(w_n){}^{\circ}_{\circ\,\tau}\right] = Z\left[\,{}^{\circ}_{\circ}J^{a_1}(0)\cdots J^{a_n}(0){}^{\circ}_{\circ\,\tau}\right]$$

entire on T^2

integrate

$$\frac{1}{2\pi}\int_0^{2\pi} dw_i \longrightarrow \quad = Z\left[J_0^{a_1}\cdots J_0^{a_n}\right]$$

where $J_0^a \equiv \frac{1}{2\pi}\int_0^{2\pi} J^a(w)\,dw$ is the charge op.

$$\Rightarrow \quad Z(\mu,\tau) = Z\left[\,{}^{\circ}_{\circ}\, e^{2\pi i \mu_a J^a(0)}\, {}^{\circ}_{\circ\,\tau}\right]$$

is flavored partition function!

Compare:

$$Z(\mu,\tau) = Z\left[{}^{\circ}_{\circ}\, e^{2\pi i \mu_a J^a(0)}\, {}^{\circ}_{\circ}\, \tau\right]$$

with

$$\tilde{Z}(\mu,\tau) \equiv Z\left[:e^{2\pi i \mu_a J^a(0)}:\, \tau\right]$$ modular!

Compare:

$$Z(\mu,\tau) = Z\left[\,{}^{\circ}_{\circ}\, e^{2\pi i \mu_a J^a(0)}\, {}^{\circ}_{\circ}\,{}_{\tau}\right]$$

with

$$\tilde{Z}(\mu,\tau) \equiv Z\left[\,:e^{2\pi i \mu_a J^a(0)}:\,{}_{\tau}\right]$$ modular!

Relating these by "reordering" and using

$$E_2(\tau) = E_2(\tau+1) = \frac{1}{\tau^2} E_2(-1/\tau) + \frac{6i}{\pi\tau}$$

we get:

$$Z(\mu,\tau) = Z(\mu,\tau+1) = e^{-\frac{\pi i}{\tau}\mu^2} Z\left(\frac{\mu}{\tau}, -\frac{1}{\tau}\right)$$

due to quasimodular trans. of $E_2(\tau)$

QED.

Assuming symmetry is <u>compact</u>:

$$Z(\mu,\tau) = Z(\mu+\rho,\tau) \qquad \forall\ \rho \in \Gamma_Q^*$$

period lattice

Transforming by $S \in SL(2,\mathbb{Z})$ $\left(Z(\mu,\tau) = e^{-\frac{\pi i \mu^2}{\tau}} Z\left(\frac{\mu}{\tau}, -\frac{1}{\tau}\right) \right)$

$$\Rightarrow \quad Z(\mu+\tau\rho,\tau) = e^{-2\pi i \mu\rho - \pi i \rho^2 \tau} Z(\mu,\tau)$$

<u>quasiperiod cond.</u>

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<u>quasiperiod cnd.</u>

Define $\hat{h} \equiv h - \frac{1}{2}Q^2$, then

$$Z = \sum q^{\hat{h}-\frac{c}{24}} q^{\frac{1}{2}Q^2} y^Q = \sum q^{\hat{h}-\frac{c}{24}} q^{\frac{1}{2}(Q+\rho)^2} y^{Q+\rho}$$

quasiperiod

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Transforming by $S \in SL(2,\mathbb{Z})$ $\quad \left(Z(\mu,\tau) = e^{-\frac{\pi i \mu^2}{\tau}}\, Z\!\left(\frac{\mu}{\tau}, -\frac{1}{\tau}\right) \right)$

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$$Z = \sum q^{\hat{h} - \frac{c}{24}}\, q^{\frac{1}{2}Q^2}\, y^Q = \sum q^{\hat{h} - \frac{c}{24}}\, q^{\frac{1}{2}(Q+\rho)^2}\, y^{Q+\rho}$$

quasiperiod

So the spectrum is invariant under

$$(Q,\tilde{Q}) \to (Q,\tilde{Q}) + (\rho,\tilde{\rho}) \qquad \forall\, (\rho,\tilde{\rho}) \in \Gamma_Q^*$$

with $(\hat{h},\hat{\tilde{h}})$ <u>fixed</u>

<u>Spectral flow theorem</u>

BH, Reece, Rudelius '16

Montero, Shiu, Soler '16

Note: Charge lattice $\Gamma_Q \equiv \Gamma^*$ is:

$$\Gamma^* \equiv \{ (Q,\tilde{Q}) \mid \forall (p,\tilde{p}) \in \Gamma,\ Qp - \tilde{Q}\tilde{p} \in \mathbb{Z} \}$$

Quasiperiod condition requires $\Gamma \subseteq \Gamma^*$, i.e., period lattice is integral.

Likewise $T \in SL(2,\mathbb{Z})$ invariance requires $h - \tilde{h} \in \mathbb{Z}$. Under the quasiperiod, 1 maps to ops with weights:

$$(h,\tilde{h}) = \left(\tfrac{1}{2}p^2, \tfrac{1}{2}\tilde{p}^2\right) \Rightarrow p^2 - \tilde{p}^2 \in 2\mathbb{Z} \quad \forall (p,\tilde{p}) \in \Gamma$$

so Γ must also be even.

Weaker but related to the more-familiar even, self-dual case.

$\underbrace{K \equiv \Gamma^*/\Gamma}_{\text{finite abelian grp.}}$ characterizes "level" of the abelian currents.

Starting with 1 of $\mathcal{V}_{int}$, we get a tower of primaries with

$$(h, \tilde{h}) = \left(\tfrac{1}{2}Q^2, \tfrac{1}{2}\tilde{Q}^2\right) \qquad \forall \quad (Q, \tilde{Q}) \in \underset{\text{period lattice}}{\Gamma} \subseteq \underset{\text{charge lattice}}{\Gamma^*}$$

Thus, there's a target space tower of massive particles with

$$\boxed{\frac{\alpha'}{4} m^2 = \frac{1}{2}\max\left(Q^2, \tilde{Q}^2\right) - 1 \qquad \forall \ (Q, \tilde{Q}) \in \Gamma}$$

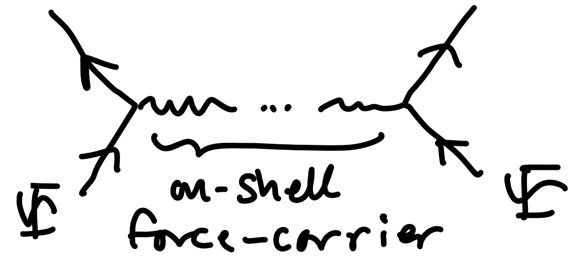
assuming a <u>compact</u> gauge group this covers a finite index sublattice of the charge lattice.

(Result of BH, Reece, Rudelius '16 with minor refinements)

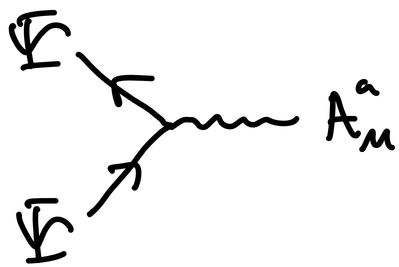
Are these particles superextremal?

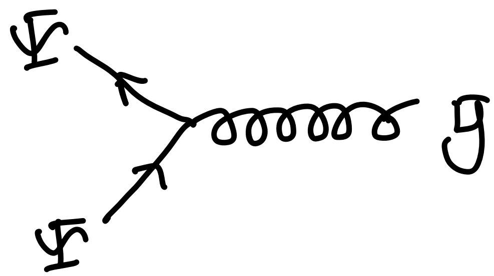
B) Long-range forces

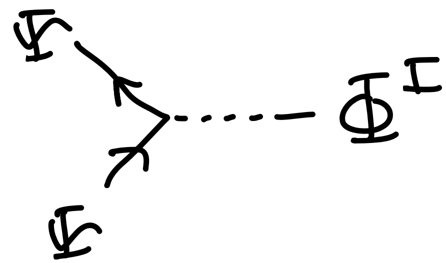
To answer, compute their long range forces. Controlled by the diagrams:



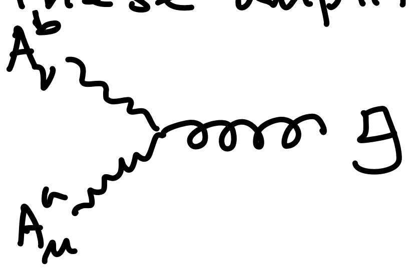
These factorize into three point amplitudes:

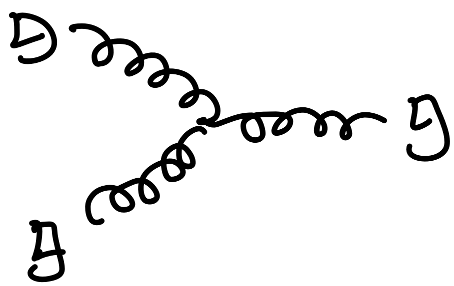


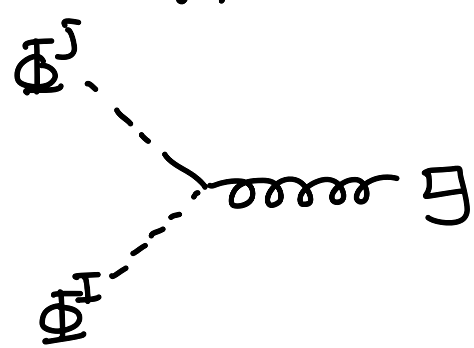




To normalize the vertex ops. correctly, we also consult these amplitudes:







Result is that $\vec{F} = \dfrac{\mathcal{F}}{V_{D-2}\, r^{D-2}}\,\hat{r} + (\text{subleading})$

where:

$$\mathcal{F} = k_N\, m m' \left[\frac{\langle \bar{\Psi}\Psi\gamma^A\rangle \langle \gamma\gamma g\rangle^{-1}_{AB} \langle \bar{\Psi}'\Psi'\gamma^B\rangle}{\langle \bar{\Psi}\Psi g\rangle \langle ggg\rangle^{-1} \langle \bar{\Psi}'\Psi' g\rangle} - 1 - \frac{\langle \bar{\Psi}\Psi\Phi^I\rangle \langle \Phi\Phi g\rangle^{-1}_{IJ} \langle \Phi^J \bar{\Psi}'\Psi'\rangle}{\langle \bar{\Psi}\Psi g\rangle \langle ggg\rangle^{-1} \langle \bar{\Psi}'\Psi' g\rangle} \right]$$

up to momentum/polarization-dependent factors.

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up to momentum/polarization-dependent factors.

We will <u>not</u> compute these factors. Rather, note that they depend only on $\mathcal{C}_{\text{ext}} = (X)^D$ free boson CFT, which is <u>universal</u> (indep. of our choice of $\mathcal{C}_{\text{int}}$).

Separate out all the $\mathcal{C}_{ext}$-dependent stuff and focus on the $\mathcal{C}_{int}$ portion:

$$g_{int} = \Phi^0_{int} = 1, \quad \Phi^i_{int} = \varphi^i(z,\bar{z})$$

$$\Rightarrow \langle g_{int}\, \Phi^0_{int}\, \Phi^i_{int} \rangle = \langle \varphi^i \rangle = 0$$

no kinetic mixing between dilaton & other scalars.

The graviton/dilaton contribution to $\mathcal{F}$ is completely universal. Can fix by computing one example, e.g., bosonic string on a torus

$$\Rightarrow \boxed{\mathcal{F}^{\mathcal{D}+\Phi^0} = -\kappa_D^2\, m m'}$$

Now consider

$$\mathcal{F}^{\text{gauge}} = N_J \frac{\langle \bar{\psi}\psi J^A \rangle \langle JJ \rangle^{-1}_{AB} \langle J^B \bar{\psi}'\psi' \rangle}{\langle \bar{\psi}\psi \rangle \langle 1 \rangle^{-1} \langle \bar{\psi}'\psi' \rangle}$$

N_J ↑ universal

where $J^A = (J^a, \tilde{J}^{\tilde{a}})$

Rewrite as matrix elements:

$$\mathcal{F}^{\text{gauge}} = N_J \frac{\langle \psi | J^A | \psi \rangle \langle J | J \rangle^{-1}_{AB} \langle \psi' | J^B | \psi' \rangle}{\langle \psi | \psi \rangle \langle 1 \rangle^{-1} \langle \psi' | \psi' \rangle}$$

$$\left.\begin{aligned} \langle J^a | J^b \rangle &= \delta^{ab} \langle 1 \rangle \\ \langle \tilde{J}^{\tilde{a}} | \tilde{J}^{\tilde{b}} \rangle &= \delta^{\tilde{a}\tilde{b}} \langle 1 \rangle \end{aligned}\right\}$$ follows from OPE normalization

$\langle\psi|J^a(z)|\psi\rangle \propto \frac{1}{z}$ by conformal inv., whereas

$$\oint \frac{dz}{2\pi} \langle\psi|J^a(z)|\psi\rangle = -\langle\psi|J_0^a|\psi\rangle = -Q^a\langle\psi|\psi\rangle$$

$$\underset{\text{Cauchy's int. formula}}{\Longrightarrow} \quad \langle\psi|J^a(z)|\psi\rangle = \frac{i}{z} Q^a \langle\psi|\psi\rangle$$

Likewise, $\langle\psi|\tilde{J}^{\tilde{a}}(\bar{z})|\psi\rangle = -\frac{i}{\bar{z}} \tilde{Q}^{\tilde{b}} \langle\psi|\psi\rangle$

$$\Rightarrow \mathcal{F}^{\text{gauge}} = \mathcal{N}_L \, \delta_{ab} Q^a Q^{b'} + \mathcal{N}_R \, \delta_{\tilde{a}\tilde{b}} \tilde{Q}^{\tilde{a}} \tilde{Q}^{\tilde{b}'}$$

universal

$\Longrightarrow$ comparing with example calc.

$$\boxed{\mathcal{F}^{\text{gauge}} = \frac{2\kappa_D^2}{\alpha'} \left(\delta_{ab} Q^a Q^{b'} + \delta_{\tilde{a}\tilde{b}} \tilde{Q}^{\tilde{a}} \tilde{Q}^{\tilde{b}'} \right)}$$

Finally, consider

$$\mathcal{F}^{\Phi^i} = -\mathcal{N}_\varphi \frac{\langle \psi | \varphi^i | \psi \rangle \langle \varphi | \varphi \rangle^{-1}_{ij} \langle \psi' | \varphi^j | \psi' \rangle}{\langle \psi | \psi \rangle \langle 1 \rangle^{-1} \langle \psi' | \psi' \rangle}$$

$\uparrow$ universal

A subset of the φ^i are $\lambda^{a\tilde{b}}(z,\bar{z}) = J^a(z)\tilde{J}^{\tilde{b}}(\bar{z})$

$\langle \lambda^{a\tilde{b}} | \lambda^{c\tilde{d}} \rangle = \delta^{ac}\delta^{\tilde{b}\tilde{d}}\langle 1 \rangle$ again from OPEs

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$\langle \lambda^{a\tilde{b}} | \lambda^{c\tilde{d}} \rangle = \delta^{ac} \delta^{\tilde{b}\tilde{d}} \langle 1 \rangle$ again from OPEs

Let $\chi(z,\bar{z})$ be any other <u>neutral</u> $(1,1)$ primary. WLOG we can require

$$\langle \lambda^{a\tilde{b}} | \chi \rangle = 0 \qquad \Longleftrightarrow \qquad J_1^a \tilde{J}_1^{\tilde{b}} | \chi \rangle = 0$$

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$$\xRightarrow[\text{steps}]{\text{few}} \quad J_1^a | \chi \rangle = 0 \quad \underline{\text{and}} \quad \tilde{J}_1^{\tilde{b}} | \chi \rangle = 0$$

$$\Rightarrow J_n^a | \chi \rangle = \tilde{J}_n^{\tilde{b}} | \chi \rangle = 0 \quad \forall\, n \geq 0$$

$\chi(z,\bar{z})$ is a neutral "current primary".

By similar arg. to above, we find:

$$\langle \psi | \lambda^{a\tilde{b}}(z,\bar{z}) | \psi \rangle = \frac{1}{|z|^2} Q^a \tilde{Q}^{\tilde{b}} \langle 1 \rangle$$

Still need to compute $\langle \psi | X(z,\bar{z}) | \psi \rangle$; this is the most non-trivial part.

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Still need to compute $\langle \psi | X(z,\bar{z}) | \psi \rangle$; this is the most non-trivial part.

Using <u>Sugawara construction</u>, stress tensor splits into two pieces:

$$T(z) = T^{j}(z) + \hat{T}(z) \quad \underline{\text{where}} \quad T^{j}(z) = -\frac{1}{2}\delta_{ab} \underbrace{:J^a J^b:}_{\text{"conformal normal order"}}(z)$$

Each piece has its own decoupled Virasoro algebra with central charges:

$$c^{j} = \underset{\substack{\uparrow \\ \#\text{ left-moving currents, etc.}}}{N_L}, \quad \tilde{c}^{j} = N_R, \quad \hat{c} = 26 - D - N_L, \quad \hat{\tilde{c}} = 26 - D - N_R$$

Then $L_0^j|\psi\rangle = \frac{1}{2}Q^2|\psi\rangle$

$\Rightarrow \quad (h^j, \tilde{h}^j) = (\frac{1}{2}Q^2, \frac{1}{2}\tilde{Q}^2), \quad \underbrace{(\hat{h}, \tilde{\hat{h}}) = (0,0)}_{\text{same as in modular invariance arg.}}$

For $X(z,\bar{z})$, we get instead:

$(h^j, \tilde{h}^j) = (0,0), \quad (\hat{h}, \tilde{\hat{h}}) = (1,1)$

since X is a neutral current primary.

(ψ is also a current primary, but <u>charged</u>.)

$\langle \psi(z_1) X(z_2) \psi(z_3) \rangle$ must be conformally invariant under <u>both</u> $\hat{T}$ and T^j conformal trans.

$(\hat{h}_\psi, \tilde{\hat{h}}_\psi) = (0,0)$

$\Rightarrow$ this is a <u>one-point function</u> according to $\hat{T}$

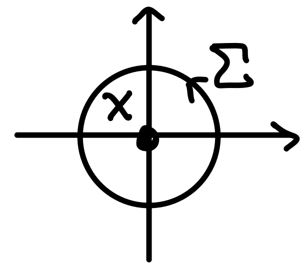
$$\Longrightarrow \quad \boxed{\langle \psi X \psi \rangle = 0}$$

($\nexists$ conformally inv. 1-pt func. on S^2)

Another way to derive this crucial fact is as follows:

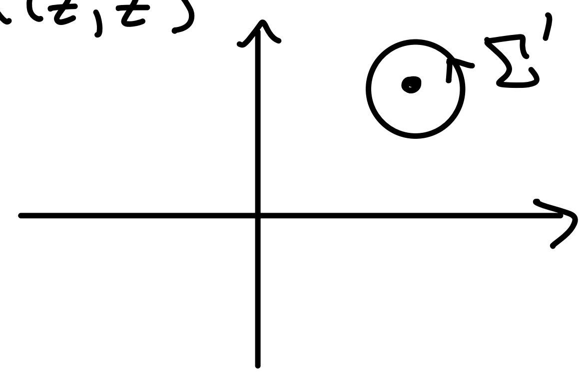
$$\hat{L}_0 |\chi\rangle = |\chi\rangle \xRightarrow[\text{integral}]{\text{path}} \chi(0) = \oint_{\Sigma} \frac{dz'}{2\pi i} z' \hat{T}(z') \chi(0)$$

(radial quantization)



Translating in z plane, we get:

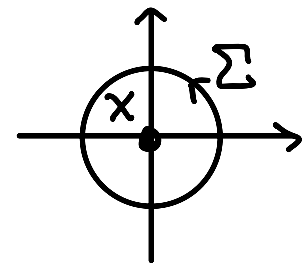
$$\chi(z,\bar{z}) = \oint_{\Sigma'} \frac{dz'}{2\pi i} (z'-z) \hat{T}(z') \chi(z,\bar{z})$$



Another way to derive this crucial fact is as follows:

$$\hat{L}_0|\chi\rangle = |\chi\rangle \xRightarrow[\text{integral}]{\text{path}} \chi(0) = \oint_{\Sigma} \frac{dz'}{2\pi i} z' \hat{T}(z')\, \chi(0)$$

(radial quantization)



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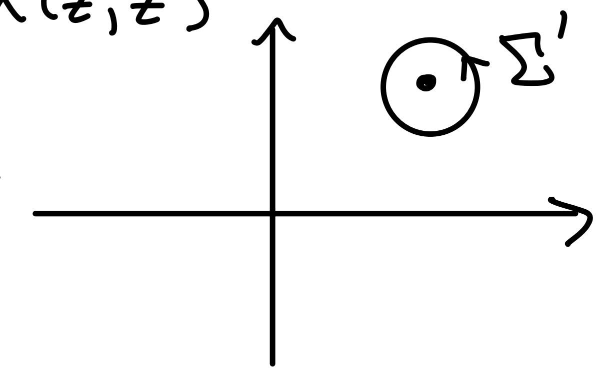

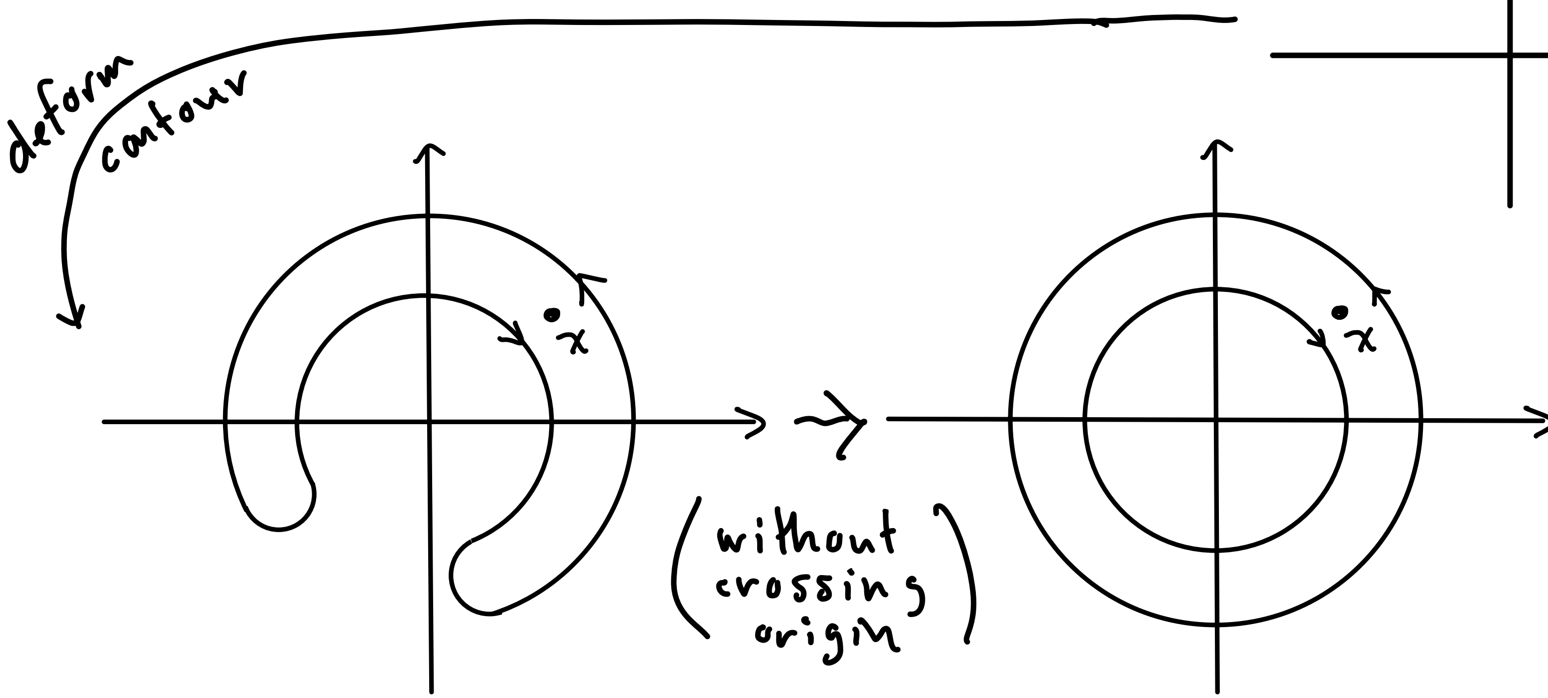

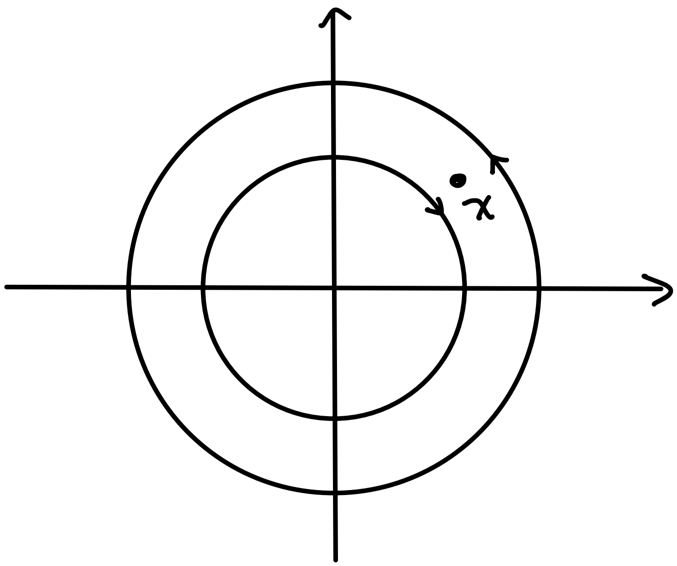

Here time is radially outwards, so this is now a commutator:

$$\chi(z,\bar{z}) = \left[\oint_{\Sigma''} \frac{dz'}{2\pi i}(z'-z)\hat{T}(z'),\ \chi(z,\bar{z}) \right]$$

$$= [\hat{L}_0 - z\hat{L}_{-1},\ \chi(z,\bar{z})]$$

$$\Rightarrow \langle\psi|\chi(z,\bar{z})|\psi\rangle = \langle\psi|[\hat{L}_0 - z\hat{L}_{-1},\ \chi(z,\bar{z})]|\psi\rangle$$

$$= 0 \qquad \underbrace{\text{b/c } \hat{L}_0, \hat{L}_{\pm 1} \text{ annihilate } |\psi\rangle}_{\text{follows from } h_\psi = 0}$$

... so finally we obtain:

$$\mathcal{F}^{\Phi^i} = N_\ell \, \delta_{ab} Q^a Q^{b'} \delta_{\tilde{a}\tilde{b}} \tilde{Q}^{\tilde{a}} \tilde{Q}^{\tilde{b}'}$$

Normalizing with an example, we get:

$$\mathcal{F}^{\Phi^i} = -\frac{4\kappa_D^2}{\alpha'} \frac{\delta_{ab} Q^a Q^{b'} \delta_{\tilde{a}\tilde{b}} \tilde{Q}^{\tilde{a}} \tilde{Q}^{\tilde{b}'}}{m m'}$$

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$$\mathcal{F}^{\Phi^i} = N_\psi \, \delta_{ab} Q^a Q^{b'} \delta_{\tilde{a}\tilde{b}} \tilde{Q}^{\tilde{a}} \tilde{Q}^{\tilde{b}'}$$

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Adding up all the pieces:

$$\boxed{\mathcal{F} = -\frac{4\kappa_D^2}{\alpha'^2 m m'} \left(\frac{\alpha'}{2} m m' - \delta_{ab} Q^a Q^{b'} \right) \left(\frac{\alpha'}{2} m m' - \delta_{\tilde{a}\tilde{b}} \tilde{Q}^{\tilde{a}} \tilde{Q}^{\tilde{b}'} \right)}$$

Phew!

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Phew!

$\mathcal{F}_{self} \geq 0$ when either

$$\frac{1}{2} Q^2 \leq \frac{\alpha'}{4} m^2 \leq \frac{1}{2} \tilde{Q}^2 \qquad \underline{\text{OR}} \qquad \frac{1}{2} \tilde{Q}^2 \leq \frac{\alpha'}{4} m^2 \leq \frac{1}{2} Q^2$$

Recall the ψ spectrum:

$$\frac{\alpha'}{4} m^2 = \frac{1}{2} \max(Q^2, \tilde{Q}^2) + N - 1, \qquad N = 0, 1, 2, \ldots$$

Thus,

$N = 0$: $\mathcal{F}_{self} > 0$ when $|Q^2 - \tilde{Q}^2| > 2$

$\mathcal{F}_{self} = 0$ when $|Q^2 - \tilde{Q}^2| = 2$

$\mathcal{F}_{self} < 0$ when $|Q^2 - \tilde{Q}^2| = 0$

Self-repulsive cases

$N = 1$: $\mathcal{F}_{self} = 0$ always

$N > 1$: $\mathcal{F}_{self} < 0$ always

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$N = 1$: $\mathcal{F}_{self} = 0$ always

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Self-repulsive cases

So there's at least one self-repulsive particle for every $(Q, \tilde{Q}) \in \Gamma$, e.g., the $N=1$ particles above.

$$\Rightarrow \boxed{\frac{\alpha'}{4} M_{BH}^2 \geq \frac{1}{2}\max(Q^2, \tilde{Q}^2)}$$

mass of everywhere self-repulsive particle

$$\boxed{\frac{\alpha'}{4} M_{BH}^2 \geqslant \frac{1}{2}\max(Q^2, \tilde{Q}^2)}$$

With this result, can finally conclude that our tower

$$\frac{\alpha'}{4} m^2 = \frac{1}{2}\max(Q^2, \tilde{Q}^2) - 1\,, \qquad \forall\ (Q, \tilde{Q}) \in \Gamma \subseteq \Gamma^*$$

is indeed (strictly) superextremal, which proves the strict (Ooguri-Vafa) sublattice WGC in pert. bosonic ST!

QED

Summary / Future Directions

* Proved WGC (in sublattice form) in perturbative closed bosonic ST
* Safe from loops when $g_s \ll 1$ *because* WGC not saturated
* Superstring generalization is W.I.P.
* Moving beyond electric NSNS sector would be very interesting / challenging