Kulikov Models and the Emergent String Conjecture

- 2111.xxxx w/ Seung-Joo Lee
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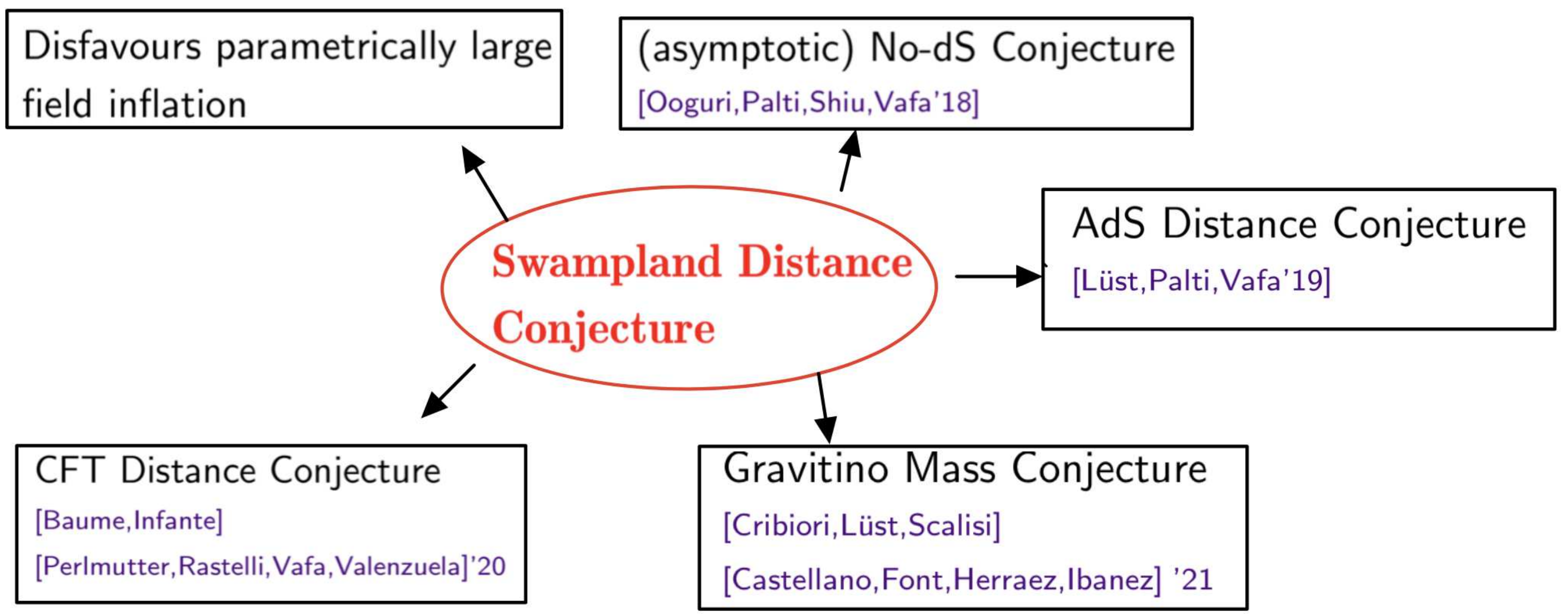
Swampland Distance Conjecture

Within the web of Swampland Conjectures, special role played by **Swampland Distance Conjecture** [Ooguri,Vafa'06]:

At infinite distance in the moduli space of a consistent Quantum Gravity, a tower of infinitely many states becomes asymptotically light:

$$m(\Phi) \to m(\Phi_0)\, e^{-c\frac{\Delta\Phi}{M_{\rm Pl}}}$$

with $c = \mathcal{O}(1)$ (Refined SDC [Baume,Palti][Kläwer,Palti]'16)



Swampland Distance Conjecture

Many successful tests in various corners of string landscape:

Example: Complex structure moduli of Type IIB compactifications

Origin of towers:
4d N=2 BPS states from D3-branes wrapped on vanishing 3-cycles

[Grimm,Palti,Valenzuela'18] [Grimm,Li,Palti'18] [Grimm,Li,Valenzuela'19] . . . [Grimm'20]

[Bastian,Grimm,van de Heisteeg'20/21]

[Blumenhagen,Kläwer,Schlechter,Wolf'18]

[Klemm,Joshi'19]

[Font,Herraez,Ibanez'19]

[Gendler,Valenzuela'20] [Palti'21]

[Kläwer'21]

What's the nature of the theory at infinite distance?

Emergent String Conjecture

Conjecture: [Lee,Lerche,TW'19]

If a quantum gravity theory admits an infinite distance limit, then

- *either it reduces to a weakly coupled string theory*
  $\Rightarrow$ *infinite tower of string excitations*
- *or it decompactifies*
  $\Rightarrow$ *infinite tower of Kaluza-Klein excitations*

Confirmed in non-trivial (non-perturbative) setups:

Existence and uniqueness of emergent critical string $\Longleftrightarrow$ (Quantum) geometry of string compactification

Kähler moduli F/M/IIA-theory in 6d/5d/4d	[Lee,Lerche,TW'18,'19,'20]
4d N=2 hypermultiplets	[Baume,Marchesano,Wiesner'19]
M-theory on G_2	[Xu'20]
4d N=1 F-theory	[Lee,Lerche,TW'19]
+ corrections	[Lee,Kläwer,TW,Wiesner'20]
cf. Distant axionic string conjecture	[Lanza,Marchesano,Martucci,Valenzuela'20/21]

This Talk

Aim: *Understand* physics on the boundary of complex structure (CS) moduli spaces

Method: Detailed analysis of degenerating geometry at infinite distance in CS moduli space

First step: Analyse infinite distance CS degenerations of elliptic K3 surfaces
$\Longleftrightarrow$ Physics of 8d F-theory on boundary on CS moduli space

Main results:

1) Math: Classification of CS infinite distance limits of elliptic K3
Refinement of Kulikov degenerations

- Type II: Type II.a and II.b [Clingher,Morgan'03] [Baily-Borel] [Mumford]
- Type III: Type III.a and Type III.b [Lee,TW'21] cf.[Alexeev,Brunyante,Engel'20]

2) Physics: Characterisation of infinite distance limits for F-theory on K3

This Talk

Main results:

1) Math: Classification of CS infinite distance limits of elliptic K3

Refinement of Kulikov degenerations

- Type II: Type II.a and II.b [Clingher,Morgan'03] [Baily-Borel] [Mumford]
- Type III: Type III.a and Type III.b [Lee,TW'21] cf.[Alexeev,Brunyante,Engel'20]

2) Physics: Characterisation of infinite distance limits for F-theory on K3

- Type II.a: Decompactification limit 8d $\to$ 10d cf [Morrison,Vafa'96]
  Type II.b: Weak coupling/emergent string limit in 8d
  cf [Aspinwall,Morrison'97]
- Type III.a: Decompactification limit 8d $\to$ 9d
  Type III.b: Decompactification limit 8d $\to$ 10d

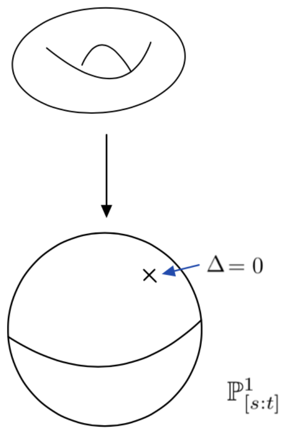
$\Longrightarrow$ **Agreement with Emergent String Conjecture**

F-theory on K3

K3: Elliptic fibration over $\mathbb{P}^1_{[s:t]}$

$$\begin{aligned} y^2 &= x^3 + f_8(s,t)xz^4 + g_{12}(s,t)z^6 \\ \Delta &= 4f^3 + 27g^2 \end{aligned}$$

7-branes $\leftrightarrow$ $\Delta = 0$



finite enhancements $\longleftrightarrow$ Kodaira-Néron classification

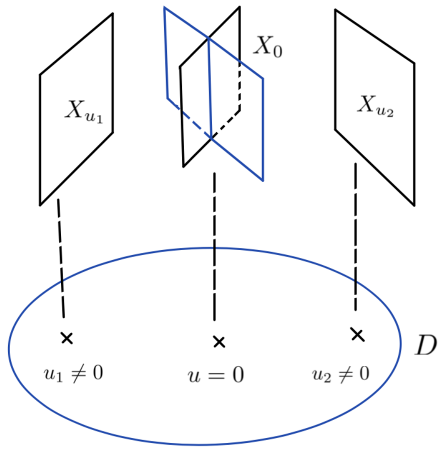
Branes	Algebra	Kodaira-type	ord(f)	ord(g)	ord(Δ)
A^{n+1}	A_n	I_{n+1}	0	0	n+1
$A^n BC$	D_n	I^*_{n-4}	2	3	n+2
A^5BC^2	E_6	IV*	≥ 3	4	8
A^6BC^2	E_7	III*	3	≥ 5	9
A^7BC^2	E_8	II*	≥ 4	5	10
	non-min	≥ 4	≥ 6	≥ 12	

Semi-Stable Degenerations

Consider 1-parameter family of K3 surfaces

$X_u \qquad u \in D = \{u \in \mathbb{C}, |u| < 1\}$,

- $X_{u \neq 0}$ smooth K3
- X_0 is degenerate



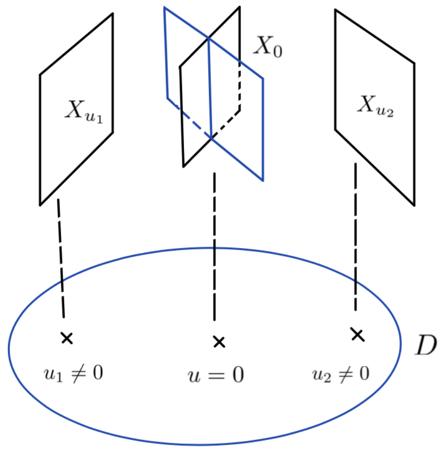
$\Longrightarrow$ 3-fold $\mathcal{X}$ fibered over D with smooth fiber $X_{u \neq 0}$ and degenerate central fiber X_0

Semi-stable reduction theorem: [Mumford]

Every such degeneration can be brought into semi-stable form.

- Semi-stable:
  $X_0 = \cup_i X^i$ with surface components X^i appearing with multiplicity one (reduced) and all singularities of X_0 are of local normal crossing type
- This may require birational transformations on $\mathcal{X}$ (leaving $X_{u \neq 0}$ invariant) or a base change $u \to u^n$, $n \in \mathbb{Z}$.

Semi-Stable Degenerations



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Semi-stable reduction theorem: [Mumford]
Every such degeneration can be brought into semi-stable form.

Theorem: [Kulikov'77] [Persson,Pinkham'81]
Up to birational transformations and base-change, a semi-stable K3 degeneration $\mathcal{X}$ can be arranged to have trivial canonical bundle.
$\Longrightarrow$ **Kulikov models**

Kulikov models

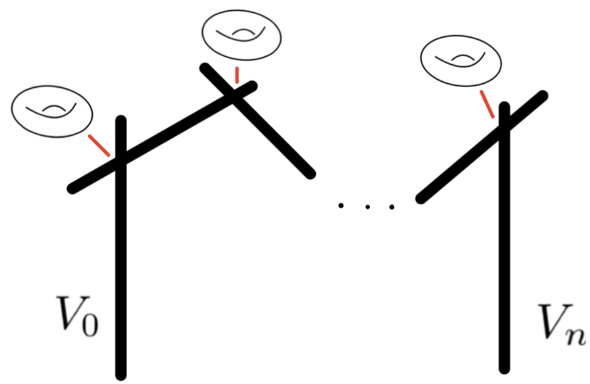
Theorem [Kulikov'77] [Persson'77] [Friedman,Morrison'81]

Kulikov models admit a classification as models of Type I, Type II, Type III.

Kulikov Type I: X_0 is single smooth reduced surface.
This occurs at finite distance in complex structure moduli space.

Infinite distance degenerations:

Kulikov Type II: degenerate K3 $X_0 = V_0 \cup V_1 \cup \ldots \cup V_n$



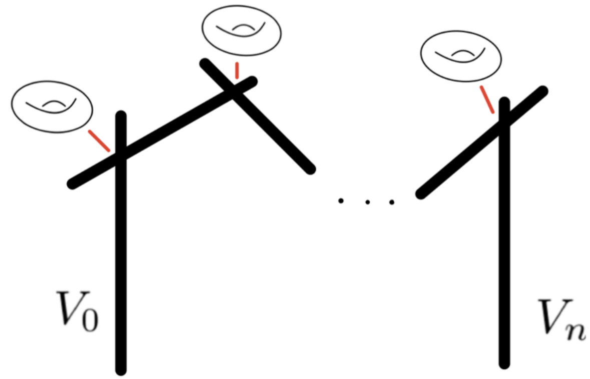
- V_0, V_n: rational surfaces
- $V_1, \ldots, V_{n-1}$: elliptic ruled surface
- $V_i \cap V_{i+1}$: elliptic curve

Kulikov Type III: degenerate $X_0 = \cup_i V_i$

- Each is V_i a rational surface
- $V_i \cap V_j$ is a rational curve or empty

Kulikov models

Kulikov Type II: degenerate K3 $X_0 = V_0 \cup V_1 \cup \ldots \cup V_n$



- $V_i \cap V_{i+1}$: elliptic curve
- 2 transcendental 2-tori $\gamma_j \in H_2(X_0, \mathbb{Z})$:
  $\int_{\gamma_j} \Omega = 0\,, \qquad j = 1, 2$

Kulikov Type III: degenerate $X_0 = \cup_i V_i$

- $V_i \cap V_j$ is a rational curve or empty
- 1 transcendental 2-torus $\gamma_1 \in H_2(X_0, \mathbb{Z})$: $\int_{\gamma_1} \Omega = 0$

M-theory on X_u in limit $u \to 0$:

- Obtain 2 or 1 towers of asymptotically massless BPS particles from M2-branes wrapped n-times on γ_j for $n \in \mathbb{Z}$.

  Similar arguments on CY3 and CY4:
  [Grimm,Palti,Valenzuela'18] [Grimm,Li,Palti'18] [Grimm,Li,Valenzuela'19] ...

- These in general form a *subset* of the asymptotically massless states.
  More details on degeneration required to extract asymptotic physics.

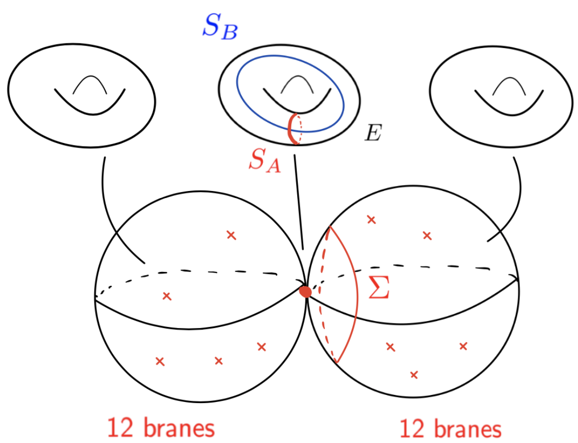
Type II Kulikov models

Theorem: [Clingher,Morgan'03] [Baily-Borel] [Mumford]

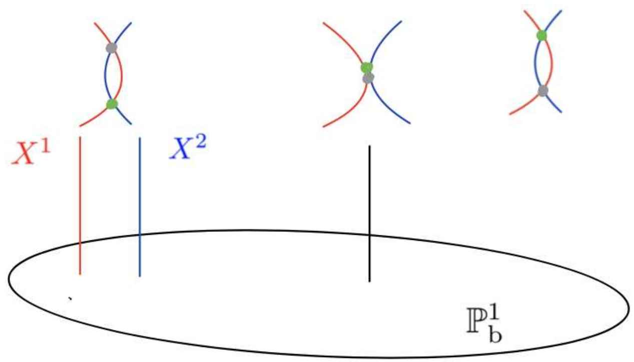
For elliptically fibered K3, the Type II degenerations are (birationally) of the form

$$X_0 = X^1 \cup X^2\,, \qquad X^1 \cap X^2 = E \qquad E : \text{elliptic curve}$$

- Type II.a:
  X^1 and X^2 are both dP_9 surfaces cf. [Morrison,Vafa'96]

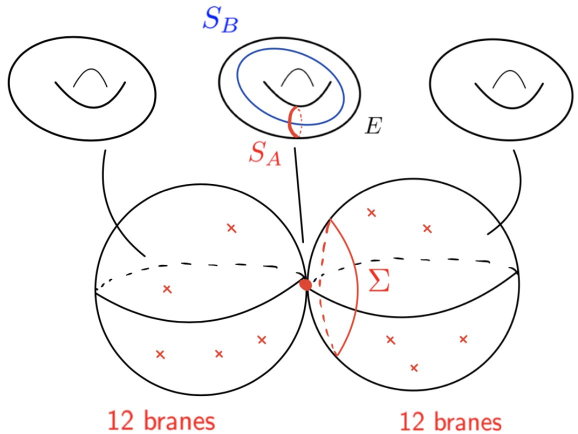


- Type II.b:
  X^1, X^2 are $\mathbb{P}^1$-fibrations over $\mathbb{P}^1_{\mathrm{b}}$ cf. [Aspinwall,Morrison'96]



Kulikov Type II.a models

'Stable degeneration limit' of F-theory - heterotic duality cf. [Morrison,Vafa'96]



$\gamma_1 = S_A \times \Sigma$, $\gamma_2 = S_B \times \Sigma$

Particle towers in M-theory:

δ_i: M2-brane on γ_i $\quad i = 1, 2$

F-theory: δ_1: $(1, 0)$ string on Σ $\quad$ δ_2: $(0, 1)$ string on Σ

$\Longrightarrow$ encircling configuration of 12 branes of total monodromy

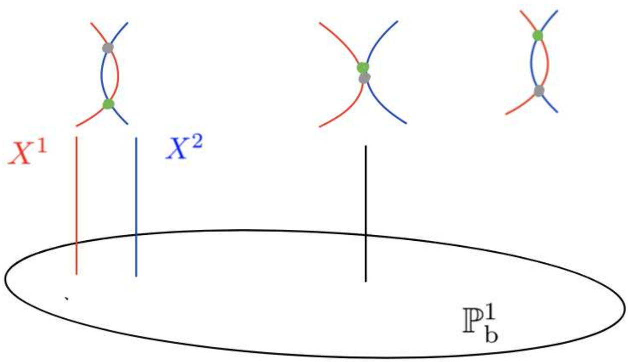
$$\prod_{i=1}^{12} M_{[p_i, q_i]} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = M_{I_0}^{-1} = M_{\hat{E}_9} \qquad \hat{E}_9 = A(A^7 BC^2) X_{[3,1]}$$

Hallmark: Component with I_0 fiber intersecting component in I_0 fiber

Physics: 8d $\rightarrow$ 10d decompactification limit of dual heterotic theory on $T^2_{\rm het} = S^1_{\rm het,1} \times S^1_{\rm het,2}$ $\qquad$ δ_i: KK tower associated with $S^1_{\rm het,1}$ $i = 1, 2$

Kulikov Type II.b models

Realises limits of form [Aspinwall,Morrison'96]



$X^1 \cap X^2 = E = \sigma_1 \times \sigma_2$:
bisection (double cover of $\mathbb{P}^1_b$)

In generic fiber over $\mathbb{P}^1_b$:
S^1_A is vanishing 1-cycle
(I_2 degeneration)

$\Longrightarrow$ 2 vanishing 2-tori $\quad \gamma_i = S^1_A \times \sigma_i,\ i = 1, 2$

Asymptotically massless particle towers in M-theory:

1. BPS particles from M2-branes on γ_i, $\qquad i = 1, 2$

$$\frac{M_{\mathrm{w},i}}{M_{11}} \sim \mathcal{V}_{11}(S_A) \times \mathcal{V}_{11}(\sigma_i) \sim \mathcal{V}_{11}(\gamma_i) \to 0$$

2. Tower of excitations of (non-BPS) tensionless string from M2 on S^1_A:

$$\frac{M^2_{\mathrm{str}}}{M^2_{11}} \sim \mathcal{V}_{11}(S_A) \to 0$$

Kulikov Type II.b models

F-theory interpretation:

Realised in Sen limit to weakly coupled Type IIB orientifold on E

[Aspinwall,Morrison'96] [Donagi,Wijnholt '12]

Asymptotically tensionless weakly coupled $(1,0)$ string with tower of winding states parametrically at same scale:

$$M_{\mathrm{w},i} \sim M_{\mathrm{str}}^2 \mathrm{Vol}(\sigma_i) \sim M_{\mathrm{str}} \mathcal{V}_{\mathrm{IIB}}(\sigma_i)$$

$\Longrightarrow$ Effective 8d theory rather than decompactification

Summary so far

Classification as Type II Kulikov not sufficient to distinguish between

- Decompactification limits 8d $\rightarrow$ 10d: Realisation of $\hat{E}_9$ loop algebra
- Emergent string limit in 8d: Realisation of Type IIB orientifold/Sen limit

Kulikov Type II/III - Systematics

Degenerate K3 X_0 has structure of fibration over B_0

Blow down all exceptional fibers $\Longrightarrow$ degenerate Weierstrass model

- $\pi : Y_0 = \cup_{i=0}^{P} Y^i \longrightarrow B_0 = \cup_{i=0}^{P} B^i$

  $y^2 = x^3 + f x z^4 + g z^6$

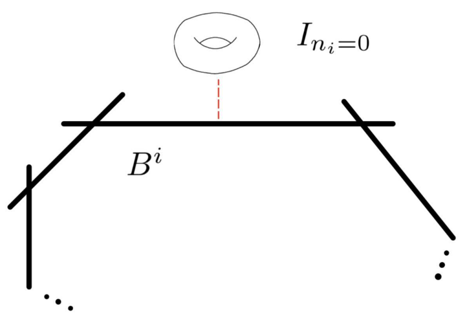
  f: degree 8 on B_0

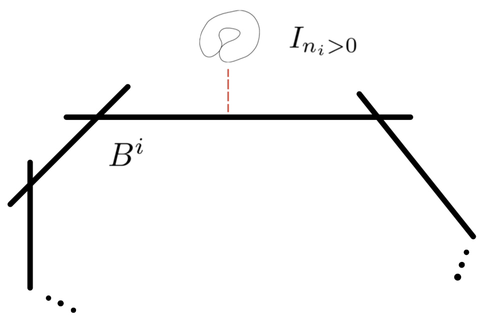
  g: degree 12 on B_0

  $\Delta = 4f^3 + 27g^2$: degree 24

- If write $B^i = \{e_i = 0\}$, then

  $$f = \prod_{i=0}^{P} e_i^{a_i} \tilde{f}\,, \quad g = \prod_{i=0}^{P} e_i^{b_i} \tilde{g}\,, \quad \Delta = \prod_{i=0}^{P} e_i^{n_i} \tilde{\Delta}$$





$n_i = 0$:
$Y^i =$ elliptic fibration over B^i

$n_i > 0$:
degenerate fiber over generic point of base component B^i

Systematics

Observation 1: [Lee,TW'21]

Fiber over generic point must be of Kodaira Type I_{n_i} for $n_i \geq 0$

i.e. $a_i \geq 0, b_i = 0$ or $a_i = 0, b_i \geq 0$

Reason: All others lead to not normal crossing or higher multiplicities
not semi-stable

Further degenerations over special points on B^i (codimension-one fibers):

- from intersection of 2 components
- in the interior of B^i: physical 7-branes in the theory

  Read off from vanishing orders of

  $f_i = f|_{e_i}, \quad g_i = g|_{e_i}, \quad \tilde{\Delta}_i = \tilde{\Delta}|_{e_i}$

  at special points:

  $\mathrm{ord}_{\mathrm{K3}}(f, g, \Delta)|_{\mathcal{P} \in B^i} = (\mathrm{ord}(f_i), \mathrm{ord}(g_i), \mathrm{ord}(\tilde{\Delta}_i))|_{\mathcal{P}}$

Systematics

Observation 2:

1. On a component with I_0-fibers: general Kodaira fibers are possible
2. On a component with generic $I_{n_i>0}$, the codimension-one fibers can only be of

   D-type: $\mathrm{ord}_{\mathrm{K3}}(f,g,\Delta)|_{\mathcal{P}\in B^i} = (2,3,2+k) \leftrightarrow$ O7-planes

   A-type: $\mathrm{ord}_{\mathrm{K3}}(f,g,\Delta)|_{\mathcal{P}\in B^i} = (0,0,k) \leftrightarrow$ perturbative 7-branes

If the $I_{n_i>0}$-component is

- an end component, then precisely 2 D-type singularities
- a middle component, then no D-type singularities

Reason: An $I_{n_i>0}$ component requires

$$f_i = -3h_i^2\,, \qquad g_i = 2h_i^3 \qquad \text{for } h_i \in H^0(B^i, L_i^2)\,.$$

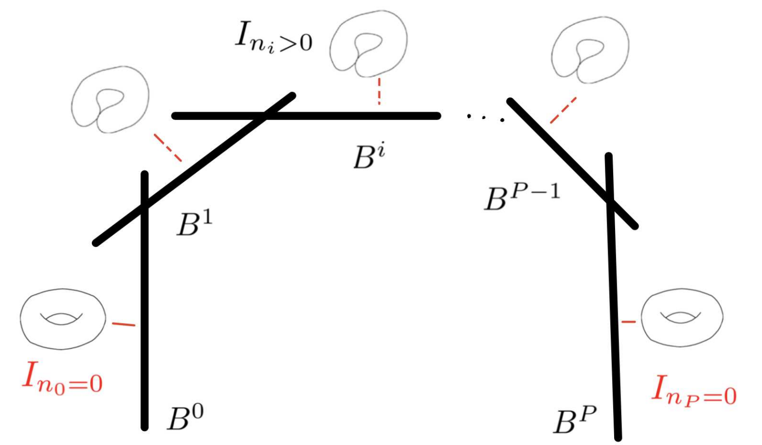
Explicit analysis of discriminant shows claim

Elliptic Type III - Classification

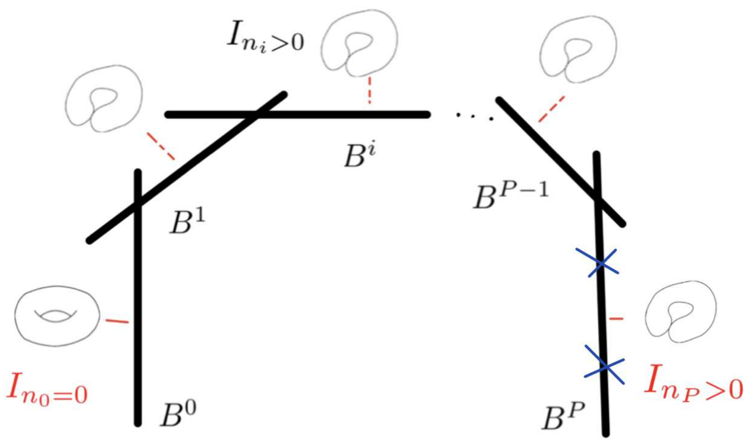
Theorem: [Lee,TW'21] see also: [Alexeev,Brunyante,Engel'20]

Every Type III Kulikov model must have a Weierstrass model with central element Y_0 *degenerating as a chain* $Y_0 = \cup_{i=0}^{P} Y^i$ *with* $P \geq 1$*:*

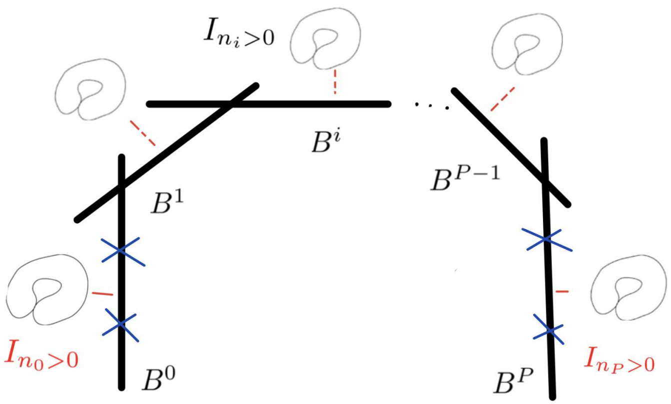
1. **Type III.a degenerations:**



or



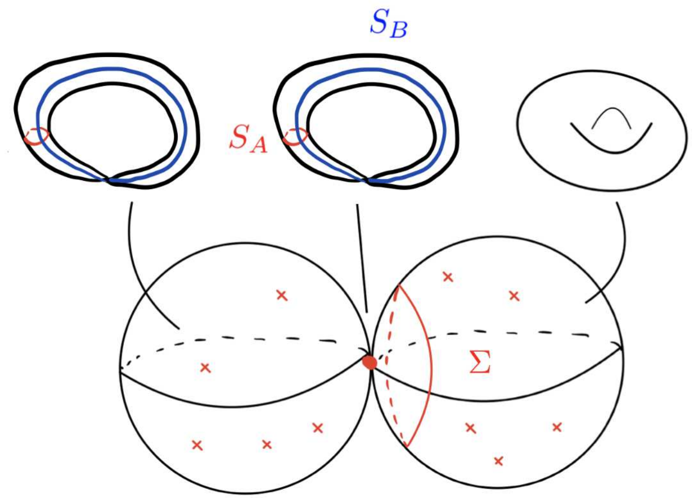
2. **Type III.b degenerations:**



Physics of Type III.a

Type III.a:

One or both ends are dP$_9$ surfaces intersecting $I_{n>0}$ component $\Longrightarrow$ decompactification to 9d



$Y_0 = \cup_{i=0}^{P} Y^i$

- Y^P: I_0 fiber over B^P (dP$_9$ surface)
- Y^{P-1}: $I_{n>0}$ fiber over B^{P-1}
- $B_P \cap B_{P-1} = 1$ point

$$\prod_{i=1}^{12-n} M_{[p_i,q_i]} = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = M_{I_n}^{-1} = M_{\hat{E}_{9-n}}$$

(12-n) branes

$\gamma_1 = S_A^1 \times \Sigma \Longleftrightarrow$ affine node δ_1 within $\hat{E}_{9-n}$

Physics: Partial decompactification of dual heterotic theory on

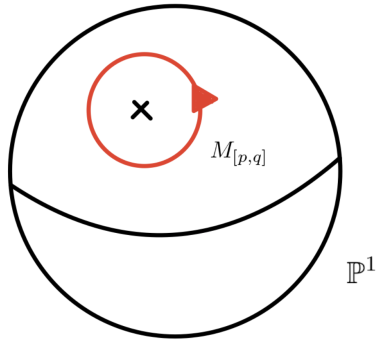
$S^1_{\text{het},1} \times S^1_{\text{het},2}$: 8d $\longrightarrow$ 9d

Digression: Affine Enhancements

$\begin{pmatrix} p \\ q \end{pmatrix}$ strings end on $[p,q]$-7-branes with $SL(2,\mathbb{Z})$ monodromy

$$\begin{pmatrix} r \\ s \end{pmatrix} \to M_{[p,q]} \begin{pmatrix} r \\ s \end{pmatrix}$$

$$M_{[p,q]} = \begin{pmatrix} 1+pq & -p^2 \\ q^2 & 1-pq \end{pmatrix}$$



ADE Lie algebras by collision of $[p,q]$-branes of type

$A = X_{[1,0]}$

$B = X_{[1,-1]}$

$C = X_{[1,1]}$

[Gaberdiel,Zwiebach'97]

[DeWolfe,Zwiebach'98]

G	branes	Monodromy M_G
A_N	A^{N+1}	$\begin{pmatrix} 1 & -N-1 \\ 0 & 1 \end{pmatrix}$
D_N	$A^N BC$	$\begin{pmatrix} -1 & -N-4 \\ 0 & -1 \end{pmatrix}$
E_N	$A^{N-1}BCC$	$\begin{pmatrix} -2 & 2N-9 \\ -1 & N-5 \end{pmatrix}$

Digression: Affine Enhancements

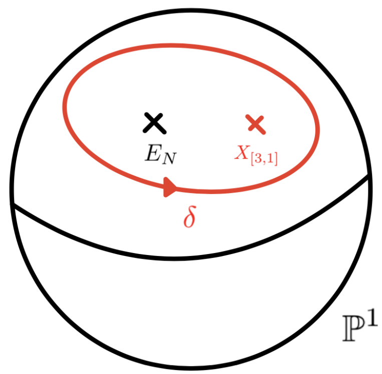
Analysis of monodromies shows: [DeWolfe,Hauer,Iqbal,Zwiebach'98]

$$\text{Affine enhancement} \qquad E_N \xrightarrow{+X_{[3,1]}} \hat{E}_N$$

$$\hat{E}_N = E_N X_{[3,1]} = (A^{N-1}BCC)\, X_{[3,1]} \qquad M_{\hat{E}_N} = \begin{pmatrix} 1 & 9-N \\ 0 & 1 \end{pmatrix}$$

- $M_{\hat{E}_N}\delta = \delta\,, \qquad \delta = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

  $\Longrightarrow$ $\delta =$ string encircling $\hat{E}_N$ gives BPS state, massless for coincident E_N and $X_{[3,1]}$



- $\hat{E}_N$ [a] is affine extension of finite Lie algebra E_N
  simple roots: $\{\alpha_i\}_{E_N}$, δ: imaginary root $\delta \cdot \delta = 0$, $\delta \cdot \alpha_i = 0$

[a] E_8 has two equivalent enhancements: 1) $E_8 \to \hat{E}_8 = E_8 X_{[3,1]}$ 2) $E_8 \to E_9 = AE_8$

Physics of Type III.a

Symmetry algebra (non-abelian part):

$$G_\infty = H \oplus (\hat{E}_{n_0} \oplus \hat{E}_{n_P})/\sim \qquad \text{for 2 } dP_9 \text{ ends}$$

Interpretation: Non-abelian gauge algebra in 9d:

$$G_{9\text{d}} = H \oplus E_{n_0} \oplus E_{n_P}$$

$$\hat{E}_8, \quad \hat{E}_7, \quad \hat{E}_6, \qquad \hat{E}_5 = \hat{D}_5,$$

$$\hat{E}_4 = \hat{A}_4, \quad \hat{E}_3 = \widehat{A_2 \oplus A_1}, \quad \hat{E}_2 = \widehat{A_1 \oplus u}(1), \quad \hat{E}_1 = \hat{A}_1, \quad \hat{\hat{E}}_0 = \hat{\emptyset}\,.$$

Application: Classification of maximal non-abelian gauge algebras in 9d

2 dP_9 ends:

$$G_\infty^{\max} = A_{17-n-m} \oplus (\hat{E}_n \oplus \hat{E}_m)/\sim \quad \Longrightarrow \quad G_{9\text{d}}^{\max} = A_{17-n-m} \oplus (E_n \oplus E_m)\,,$$

$$n, m \in \{0, 1, 3, \ldots, 8\}$$

1 dP_9 end:

$$G_\infty^{\max} = D_{17-k} \oplus \hat{E}_k \quad \Longrightarrow \quad G_{9\text{d}}^{\max} = D_{17-k} \oplus E_k\,, \qquad k \in \{0, 1, 3, \ldots, 8\}$$

Reproduces results of [Cachazo,Vafa'00] [Font,Fraiman,Grana,Nunez,Freitas'20]

Physics of Type III.b

Only $I_{n>0}$ fibers:

Weak coupling limit + large complex structure limit of torus $T^2_{\rm IIB}$ to 10d

- same $(1,0)$ cycle S_A in fiber degenerates over all components B^i
  $\Longrightarrow$ M2 on S_A: asymptotically tensionless fundamental IIB string
  $\Longrightarrow$ globally weak coupling limit
- In addition to weak coupling limit, 2 or more O-planes collide
  Vanishing orders: $(2,3,*) \longrightarrow (4,6,*) \Longrightarrow$ blowup
  $\Longrightarrow$ Degenerating complex structure of $T^2_{\rm IIB}$

Massless towers:

- $(1,0)$ string around intersection points: winding tower
- From above picture we know there must in addition be a SUGRA KK tower which we cannot see in this simple manner

$\Longrightarrow$ Decompactification to weakly coupled Type IIB in 10d

Realisation via non-minimal fibers

Theorem [Lee,TW'21]
All Type III Kulikov models are blowups of Weierstrass models with suitable non-minimal singularities:

- Start with Weierstrass over base $\mathbb{P}^1_{[s:t]}$
  $y^2 = x^3 + f_u(s,t)xz^4 + g_u(s,t)z^6$
- Consider non-minimal Kodaira singularity at $s = 0$ in the limit $u \to 0$:
  $\mathrm{ord}(f, g, \Delta)|_{u=0,s=0} = (4 + \alpha, 6 + \beta, 12 + \gamma)$

Then possibly upon base change, a chain of blowups in the base leads to a family of Weierstrass models without non-minimal singularities

- If $\gamma = 0$ (and hence $\alpha = 0$ or $\beta = 0$ or both):
  Blowup gives Type II model birational to Type II.a
- If $\gamma > 0$ and $\alpha = 0 = \beta$:
  blowup gives Type III.a (generically) or III.b model (non-generically)
- If $\gamma > 0$ and $\alpha > 0$ and $\beta > 0$: Type I (finite distance)

$E_7 \times E_8$ Weierstrass model

$$f = t^3 s^4 (a\, t + c\, s)\,, \qquad g = t^5 s^5 (d\, s^2 + b\, s\, t + e\, t^2)$$

- on base $\mathbb{P}^1_{[s:t]}$: $\quad E_7|_{t=0} \times E_8|_{s=0}$
- $\mathrm{ord}(f, g, \Delta)|_{t=0} \geq (4, 6, 12)$ for $c \to 0, \quad d \to 0$

$$4a^3 + 27b^2 \sim u^k\,, \quad c \sim u^n\,, \quad d \sim u^m \qquad u \to 0$$

  - $k = 0$: Type II limit: $\hat{E}_9 \times \hat{E}_9$
    Full decompactification to 10d with non-ab. gauge group $E_8 \times E_8$
  - $k \geq 1$: Type III limit:
    decompactification to 9d, with variety of further enhancements

towers from $\hat{G}_1$ and $\hat{G}_2$ are equivalent and identified

Loop algebra	non-ab part in 9d
$\hat{E}_7 \times \hat{E}_8$	$E_7 \times E_8$
$\hat{E}_7 \times \hat{E}_8 \times SU(2)$	$E_7 \times E_8 \times SU(2)$
$\hat{E}_7 \times \hat{E}_8 \times SU(3)$	$E_7 \times E_8 \times SU(3)$
$\hat{E}_8 \times \hat{E}_8$	$E_8 \times E_8$
$\hat{E}_8 \times \hat{E}_8 \times SU(2)$	$E_8 \times E_8 \times SU(2)$

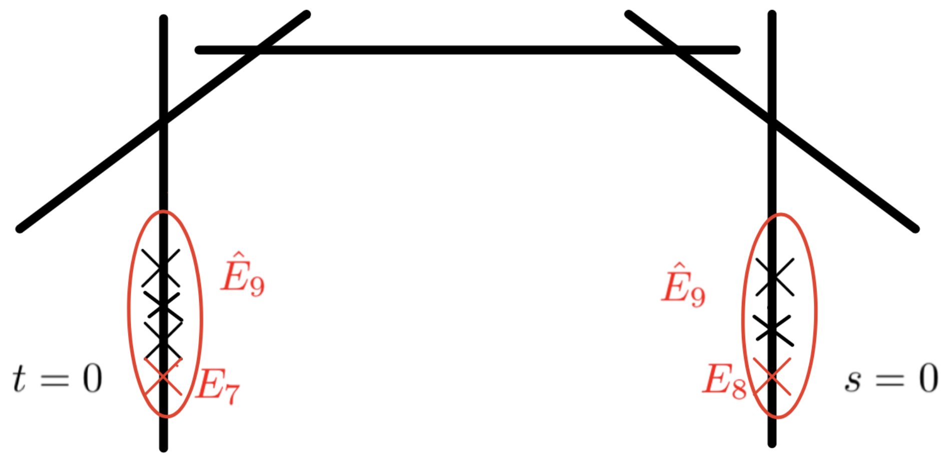
$E_7 \times E_8$ Weierstrass model

$$f_8 = t^3 s^4 (a\,t + c\,s)\,, \qquad g_{12} = t^5 s^5 (d\,s^2 + b\,s\,t + e\,t^2)$$

$$\text{Example :} \qquad 4a^3 + 27b^2 \sim u^k\,, \qquad c \sim u^4\,, \quad d \sim u^4 \qquad u \to 0$$

$k = 0$: Kulikov Type II

$K3 \to Y^0 \cup Y^1 \cup Y^2 \cup Y^3 \cup Y^4$

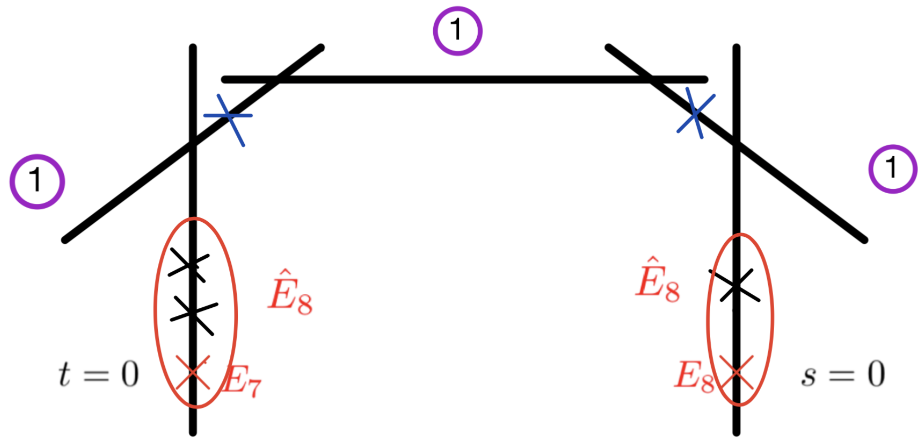


- Y^0 and Y^4 are dP_9, $\qquad Y^1$, Y^2, $Y^3 \simeq T^2 \times \mathbb{P}^1$,
  $Y^i \cap Y^{i+1} = T^2$
- Y^0 and Y^4 contain those branes which coalesce in inf. distance limit
  $\Longrightarrow \hat{E}_9 \times \hat{E}_9$
- Each $\hat{E}_9$ gives two towers, and towers from Y^0 and Y^4 are isomorphic
  $\Longrightarrow$ decompactification 8d $\to$ 10d

$E_7 \times E_8$ Weierstrass model

$$\text{Example}: \quad 4a^3 + 27b^2 \sim u^k\,, \qquad c \sim u^4\,, \quad d \sim u^4$$

$k = 1$, otherwise generic:
Kulikov Type III



- Y^0, Y^4: dP_9, $\quad Y^1, Y^2, Y^3$: rational fibration over $\mathbb{P}^1$,
  $Y^i \cap Y^{i+1} = \mathbb{P}^1$
- From Y^0, Y^4: $\hat{E}_{7/8} \times \hat{E}_8$ in infinite distance limit
- Interpretation:
  1 BPS tower from each $\hat{E}_n$, both identified
  $\Longrightarrow$ decompactification $8d \to 9d$ with $G^{9\mathrm{d}}_{\mathrm{non-ab}} = E_7 \times E_8$

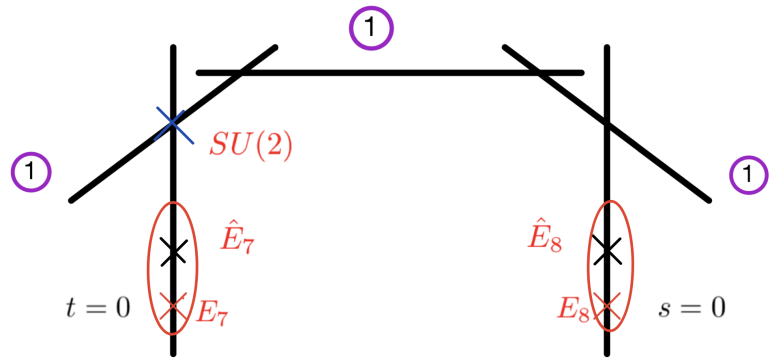
$E_7 \times E_8$ Weierstrass model

Type III degeneration

$4a^3 + 27b^2 \sim u$

Specialisation $c = -i\frac{\sqrt{3}}{\sqrt{a}}d$

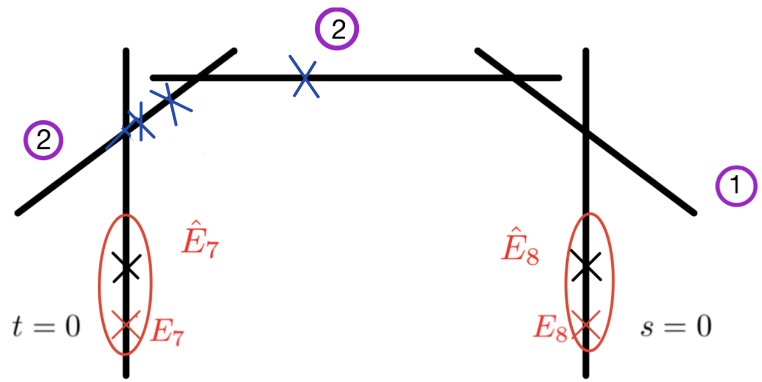
$E_7 \times E_8 \times SU(2)$ in 9d



$4a^3 + 27b^2 \sim u^2$

Specialisation $c = -i\frac{\sqrt{3}}{\sqrt{a}}d$

$E_7 \times E_8$ in 9d



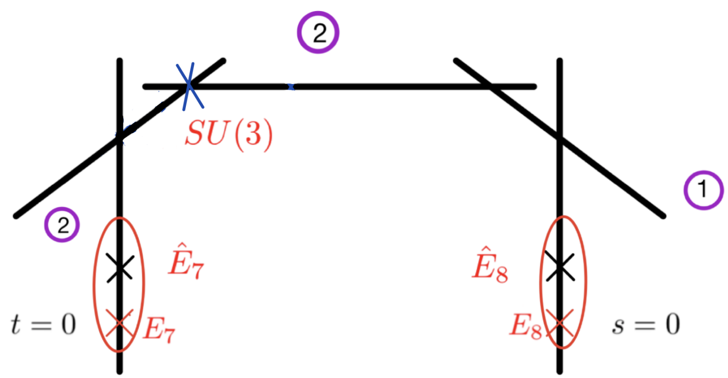
$4a^3 + 27b^2 \sim u^3$

Specialisation $c = -i\frac{\sqrt{3}}{\sqrt{a}}d$

$+$ 1 more tuning

Type III degeneration

$E_7 \times E_8 \times SU(3)$ in 9d



Heterotic dual

Match with dual heterotic on T^2 cf. [Malmendier,Morrison'14] [Jockers,Gu'15]

[Klemm, Poretschkin,Schimannek,Raum'15]

Map to Siegel modular forms [Font,Garcia-E.,Lüst,Massai,Mayrhofer'16]

$$a = -\frac{\psi_4(\underline{\tau})}{48}\,, \quad b = -\frac{\psi_6(\underline{\tau})}{864}\,, \quad c = -4\chi_{10}(\underline{\tau})\,, \quad d = \chi_{12}(\underline{\tau})\,, \quad e = 1\,.$$

$\underline{\tau} = \begin{pmatrix} \tau & z \\ z & \rho \end{pmatrix}$ τ: compl. struct. ρ: Kähler mod. , z: Wilson line

$$c \sim \chi_{10} \sim q_\tau q_\rho(-2 + \xi + \frac{1}{\xi}) + \ldots \sim u^4 \quad \to 0$$

$$d \sim \chi_{12} \sim q_\tau q_\rho(10 + \xi + \frac{1}{\xi}) + \ldots \sim u^4 \quad \to 0$$

$$4a^3 + 27b^2 \sim (\psi_4^3 - \psi_6^2) \sim q_\tau + q_\rho + \ldots \sim u^k$$

$$q_\tau = e^{2\pi i \tau}\,, \qquad q_\rho = e^{2\pi i \rho}\,, \qquad \xi = e^{2\pi i z}$$

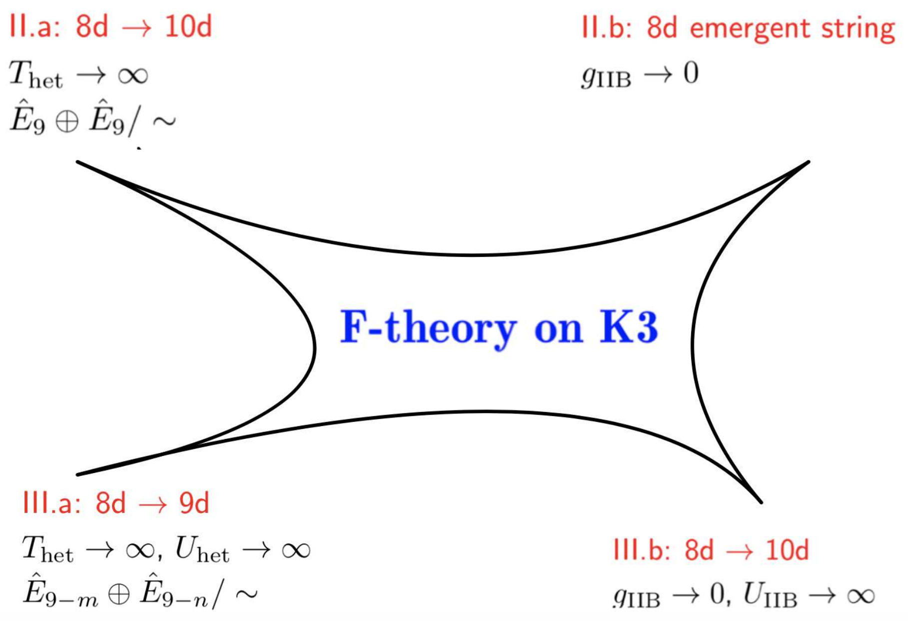
$k = 0$ (Type II): $\rho \to i\infty$, τ finite: $\longrightarrow$ 10d limit ✓

$k > 0$ (Type III): $\rho \to i\infty$, $\tau \to i\infty$, $\tau/\rho = \mathcal{O}(1)$: $\longrightarrow$ 9d limit ✓

Conclusions

Mathematics and physics of CS infinite distance limits for K3 surfaces

Refinement of Kulikov classification in agreement with physics:



✓ In agreement with idea of Emergent String Conjecture

✓ Reproduces classification of maximal 9d non-ab. gauge symmetries

[Cachazo,Vafa'00] [Font,Fraiman,Grana,Nunez,Freitas'20]

Next steps: Extension of this reasoning to CY_3 and CY_4