Black Hole Thermodynamics & the Weak Gravity Conjecture

Gary Shiu
University of Wisconsin-Madison

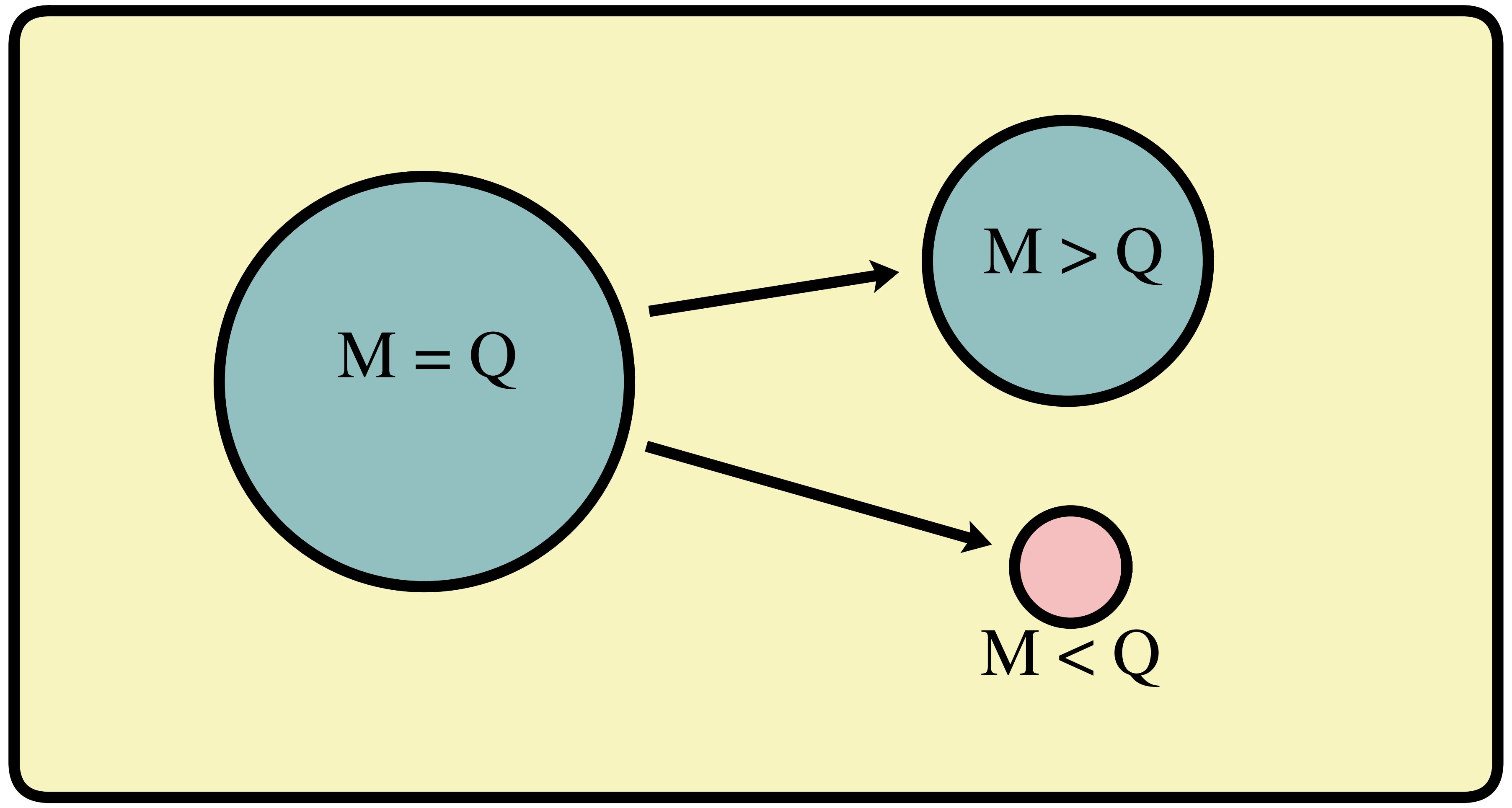
Unreasonable Effectiveness of Thermodynamics

- Thermodynamics offered insights into **quantum mechanics** before a microscopic treatment was available, e.g., UV catastrophe of blackbody radiation.
- Similarly, studies of thermodynamics of black holes have lent surprising insights into **quantum gravity**, e.g,
  - Hawking radiation: $$T_H = \frac{\hbar c^3}{8\pi G M k_B}$$
  - Bekenstein-Hawking area law: $$S = \frac{A}{4G}$$
  - Holographic principle
- It seems worthwhile to leverage this unreasonable effectiveness of thermodynamics for the **swampland program**.

WGC and Black Holes

[Arkani-Hamed, Motl, Nicolis, Vafa ’06]

- Extremal black holes are kinematically unstable:



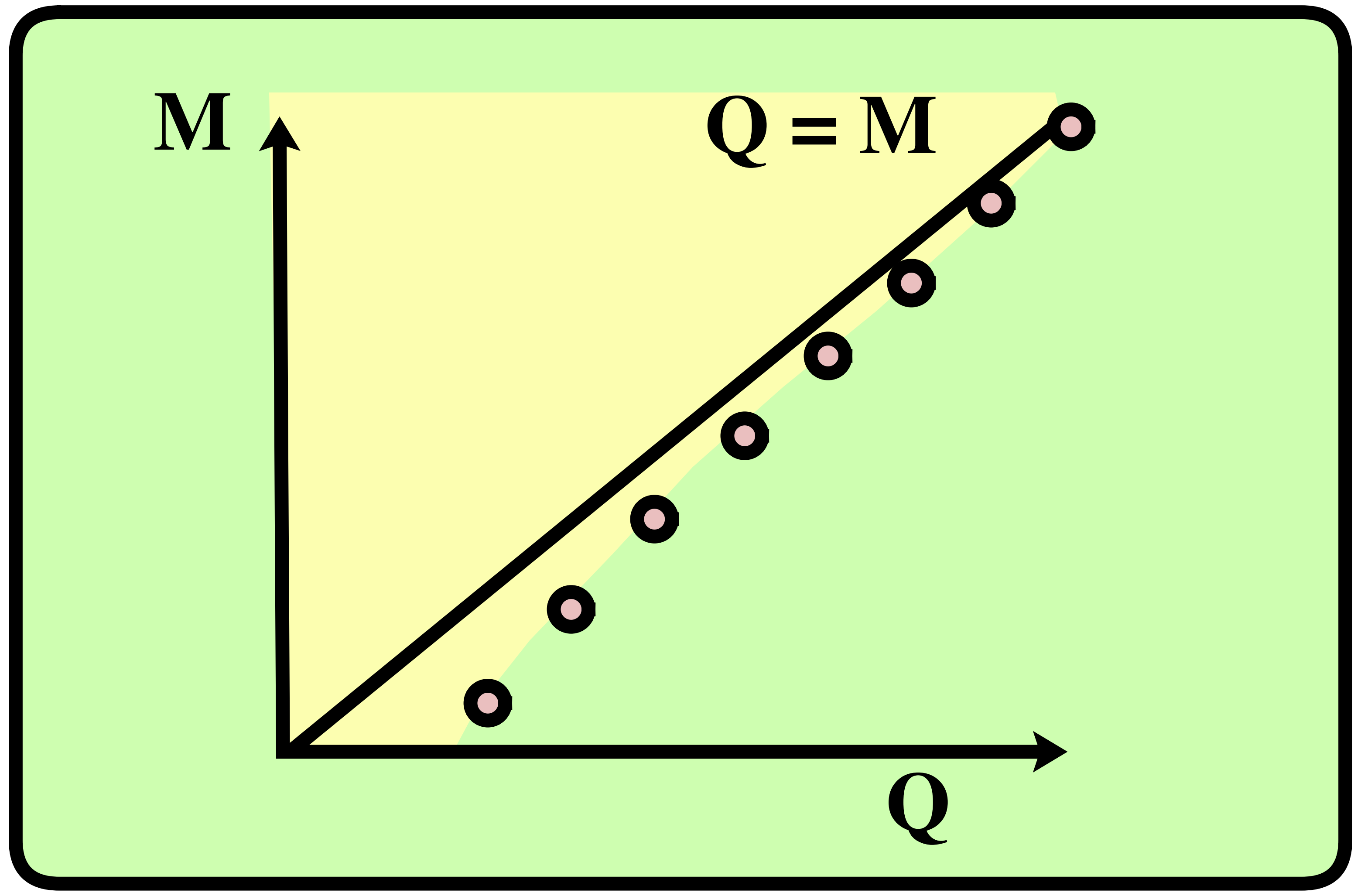
$$\exists \text{ state with } \frac{q}{m} \geq \lim_{M\to\infty} \frac{Q}{M}\Big|_{\text{ext}}$$

Mild form: WGC satisfied by large but finite mass BHs.

- Consistency with dimensional reduction, modular invariance, unitarity/causality suggests stronger version e.g. Tower WGC [Andriolo, Junghans, Noumi, GS, ‘18] and sub-Lattice WGC [Heidenreich, Reece, Rudelius, ‘16];[Montero, GS, Soler, 16].
- Tower WGC is also motivated by the distance conjecture [Ooguri, Vafa, ’06] and the emergence proposal [Heidenreich, Reece, Rudelius, ‘18];[Grimm, Palti, Valenzuela, ‘18].

Evidence from String Theory

- Evidence from string theory (e.g., heterotic string spectrum [Arkani-Hamed, Motl, Nicolis, Vafa '06]):

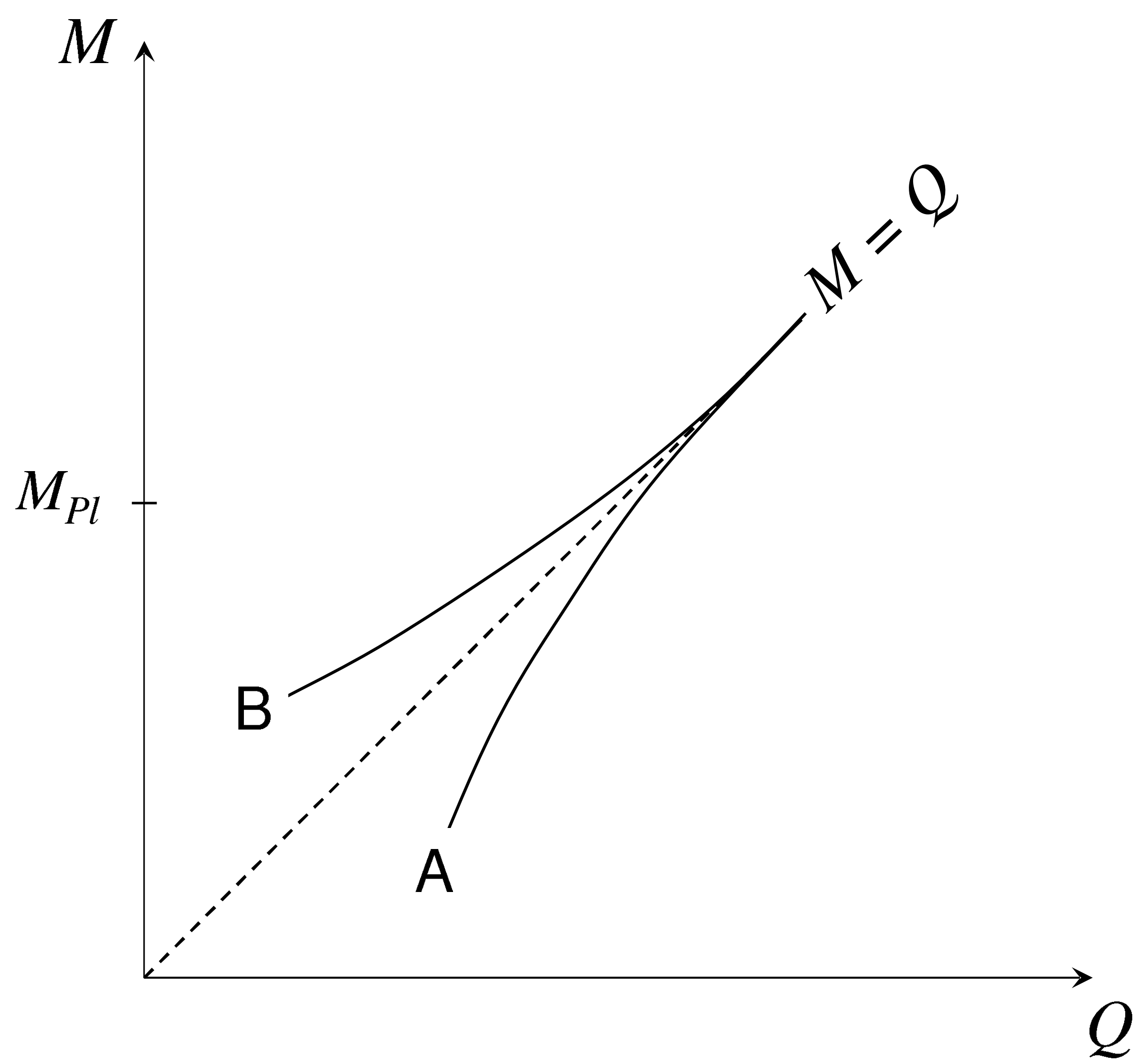


Further checks in string and F-theory constructions [Lee, Lerche, Weigand]; …

- While this provides evidence for stronger forms of the WGC, we need detailed knowledge of the UV complete theory (e.g. spectrum, modular invariance, …).
- Connection to BH instability is less clear.
- What *physically* goes wrong if the spectrum does not follow this pattern?

Extremal BHs

- Leading corrections to the extremality bound, e.g. in Einstein-Maxwell theory:



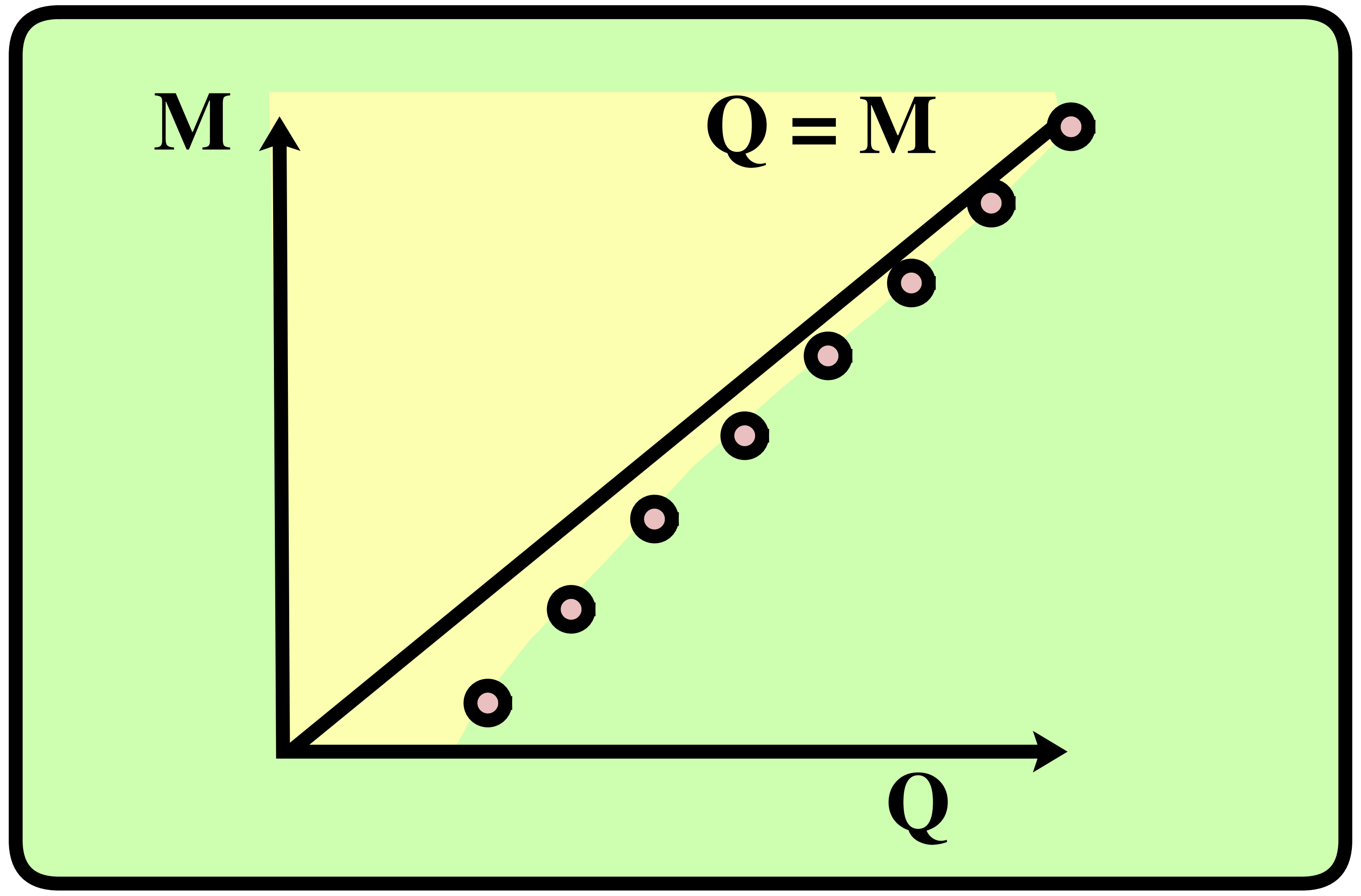
$$\mathscr{L} = \frac{1}{2}R - \frac{1}{4}F^2 + \frac{\alpha_1}{4}(F^2)^2 + \frac{\alpha_2}{4}(F\tilde{F})^2 + \frac{\alpha_3}{2}FFW$$

$$\frac{\sqrt{2(Q^2+P^2)}}{M} \leq 1 + \frac{32\pi^2}{5(Q^2+P^2)^3}\left[2\alpha_1(Q^2-P^2)^2 + 2\alpha_2(2QP)^2 - \alpha_3(Q^4-P^4)\right]$$

- The leading corrections increase Q/M for extremal electric heterotic BHs [Yats, Motl, Padi, '06].
- This behavior (A) was shown [Hamada, Noumi, GS, '18] to follow from gravitational positivity bounds (unitarity, causality) under the assumption of Regge boundedness & gravity subdominance.
- RG running to deep IR [Charles, '19];[Arkani-Hamed et al, '21]: log running from massless fields eventually dominates if the BH is exponentially large, $r_H \gtrsim 10^{4000} H_{\text{today}}^{-1}$.

From Black Holes to Strings

- String theory suggests a tower of superextremal states hugging extremality from below:



Turn on a small string coupling

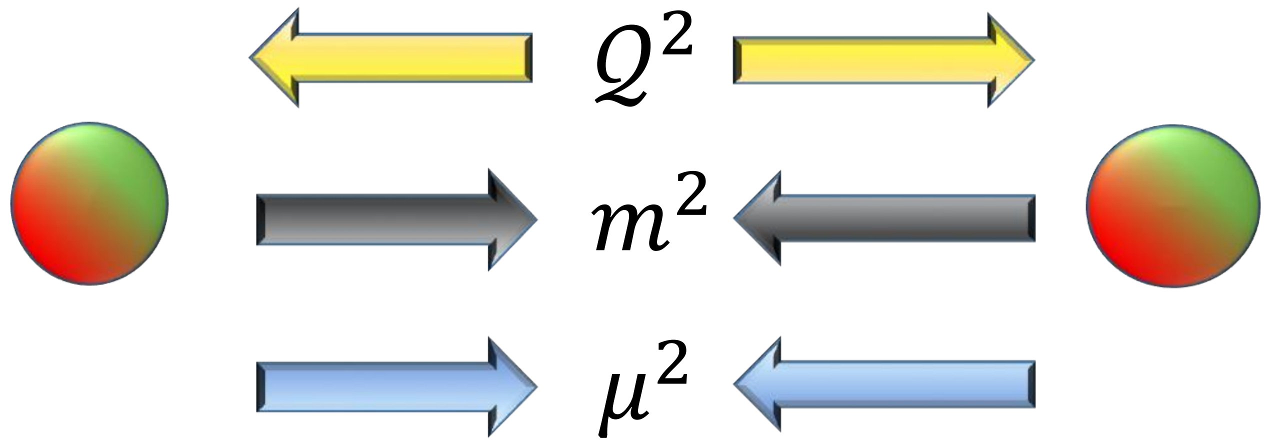
$$g_c \sim N^{-1/4} \ll 1$$

The excited string states turned into BHs

- The correspondence principle [Susskind, '93];[Horowitz, Polchinski '96]: $S_{\text{String}} = \mathcal{O}(1)\ S_{BH}$.
- Extremal BHs with near horizon BTZ geometry: matching of anomalies ensures $S_{\text{String}} = S_{BH}$, superextremal states stay superextremal upon turning on g_s [Aalsma, Cole, GS, '19].

Charge Convexity Conjecture

- Another way to formulate the WGC is the existence of a self-repulsive state [Arkani-Hamed, Motl, Nicolis, Vafa '06]. It was emphasized by [Palti, '17] that the formulations differ in the presence of scalars, and further studied in [Lust, Palti, '17]; [Lee, Lerche, Weigand, '18]; [Heidenreich, Reece, Rudelius, '19], ….



- This weakly coupled gravitational picture has motivated the **Charge Convexity Conjecture**:

[Aharony, Palti, '21]

Abelian Convex Charge Conjecture: *Consider any CFT with a $U(1)$ global symmetry. Denote by $\Delta(q)$ the dimension of the lowest dimension operator of charge q. Then this must satisfy a convex-like constraint*

$$\Delta(n_1 q_0 + n_2 q_0) \geq \Delta(n_1 q_0) + \Delta(n_2 q_0) \ , \qquad (1.1)$$

for any positive integers n_1, n_2, for some q_0 of order one.

Conjectured to hold for **all** CFTs, not only holographic ones.

Charge Convexity Conjecture

- For holographic CFTs, evidence of the CCC came mostly from studying the large q limit:

$$\Delta(q) = Aq^{d/(d-1)} + \dots$$

which coincides with the (classical) extremality bound for large AdS BHs: $M_{ext} \sim Q_{ext}^{d/(d-1)}$

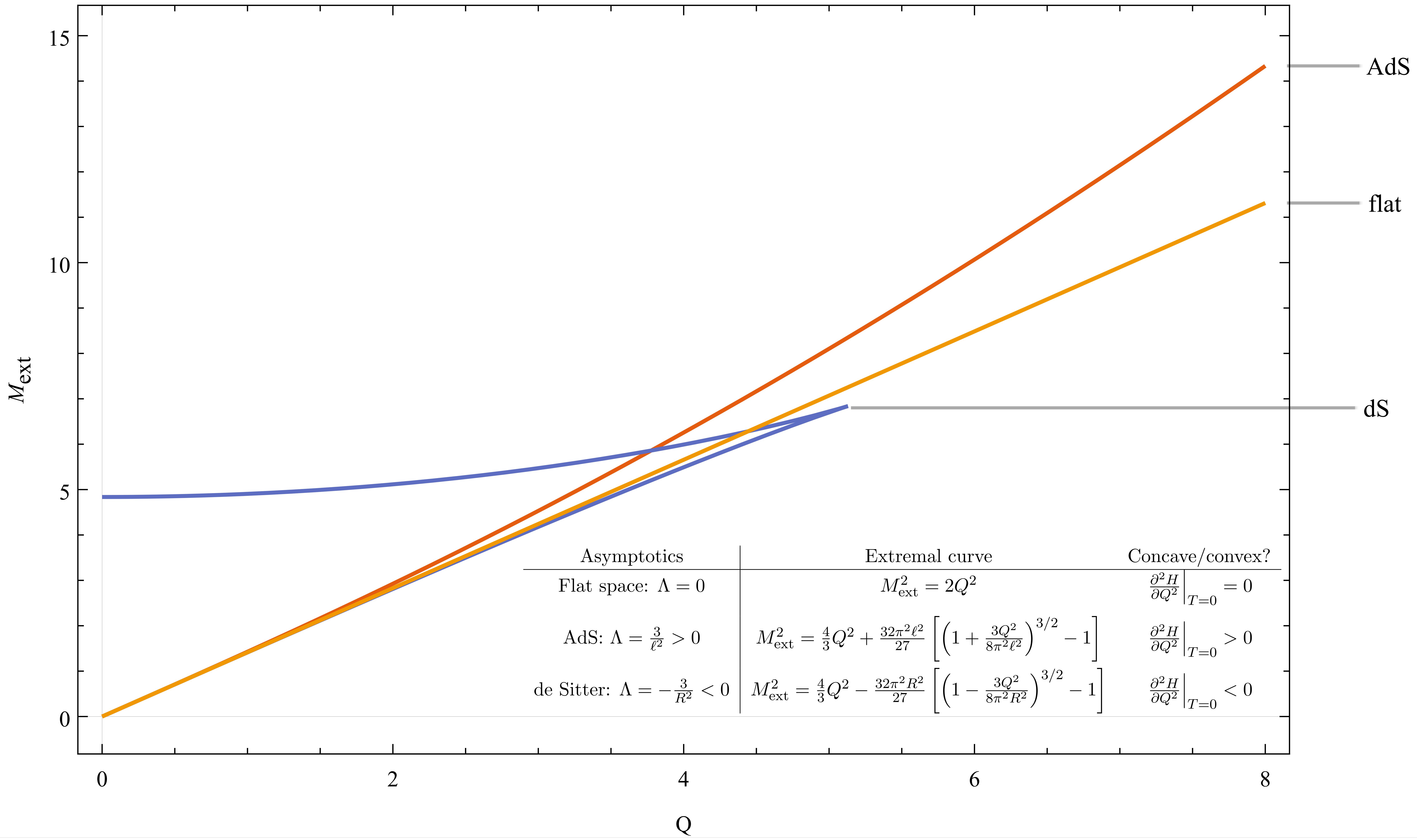
- However, there is a regime of q where q is large compared with the central charge c but not as large as that correspond to AdS black holes: flat space BHs have classically $M_{ext} \sim Q_{ext}$.

$$\Delta_{AdS\ BH} \gg \Delta_{\text{flat space BH}} \sim \mathcal{O}(c)$$

- *Higher derivative corrections turn the extremality curve* ***concave***. If $\exists$ a multi-particle state with the same charge but smaller mass, the CCC is a priori not violated.

- Why should the CCC impose an opposite condition on such a multi-particle state as that on a healthy BH?

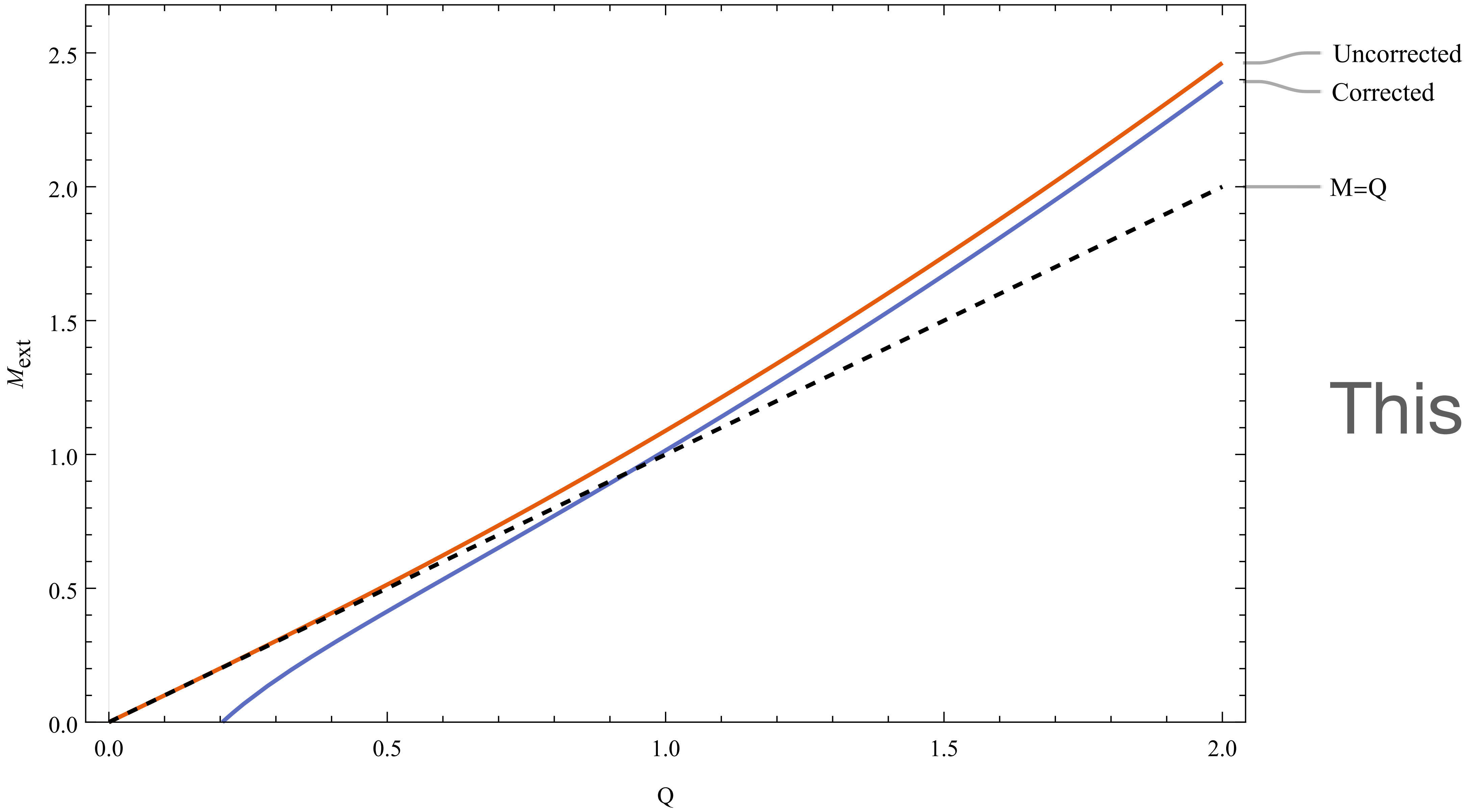
Black Hole Extremality



Asymptotics	Extremal curve	Concave/convex?
Flat space: $\Lambda = 0$	$M^2_{\mathrm{ext}} = 2Q^2$	$\left.\frac{\partial^2 H}{\partial Q^2}\right\vert_{T=0} = 0$
AdS: $\Lambda = \frac{3}{\ell^2} > 0$	$M^2_{\mathrm{ext}} = \frac{4}{3}Q^2 + \frac{32\pi^2\ell^2}{27}\left[\left(1 + \frac{3Q^2}{8\pi^2\ell^2}\right)^{3/2} - 1\right]$	$\left.\frac{\partial^2 H}{\partial Q^2}\right\vert_{T=0} > 0$
de Sitter: $\Lambda = -\frac{3}{R^2} < 0$	$M^2_{\mathrm{ext}} = \frac{4}{3}Q^2 - \frac{32\pi^2 R^2}{27}\left[\left(1 - \frac{3Q^2}{8\pi^2 R^2}\right)^{3/2} - 1\right]$	$\left.\frac{\partial^2 H}{\partial Q^2}\right\vert_{T=0} < 0$

Monotonicity Continues

- We have evidence that this monotonicity continues from flat space BHs to large AdS BHs [Loges, Noumi, GS, work in progress]. This result if universal has consequences on $1/q$ corrections to $\Delta(q) \geq \mathcal{O}(c)$ operators in the dual CFT (to be formulated more precisely).



This monotonicity implies a ***Tower*** WGC

- Here, we present evidence for this monotonic behavior (on the gravity side) based on earlier works on the thermodynamics of (axio)-dilatonic Einstein-Maxwell BHs [Loges, Noumi, GS, '19]; [Loges, Noumi, GS, '20] and BTZ black holes [Aalsma, Cole, Loges, GS, '20].

BH Thermodynamics and WGC

Corrections to Extremality

- Higher order corrections change the BH solution and the extremality bound. Finding corrected solutions is often intractable (especially with additional scalars).
- Thermodynamic approach sidesteps this difficulty:

$$Z = \mathrm{Tr}\left(e^{-\beta(H-\Phi Q)}\right) = \sum_{\text{saddles}} e^{-I_E} \equiv e^{-\beta G(T,\Phi,P)}$$

$$G = M - TS - Q\Phi \qquad dG = -SdT - Qd\Phi + \Psi dP$$

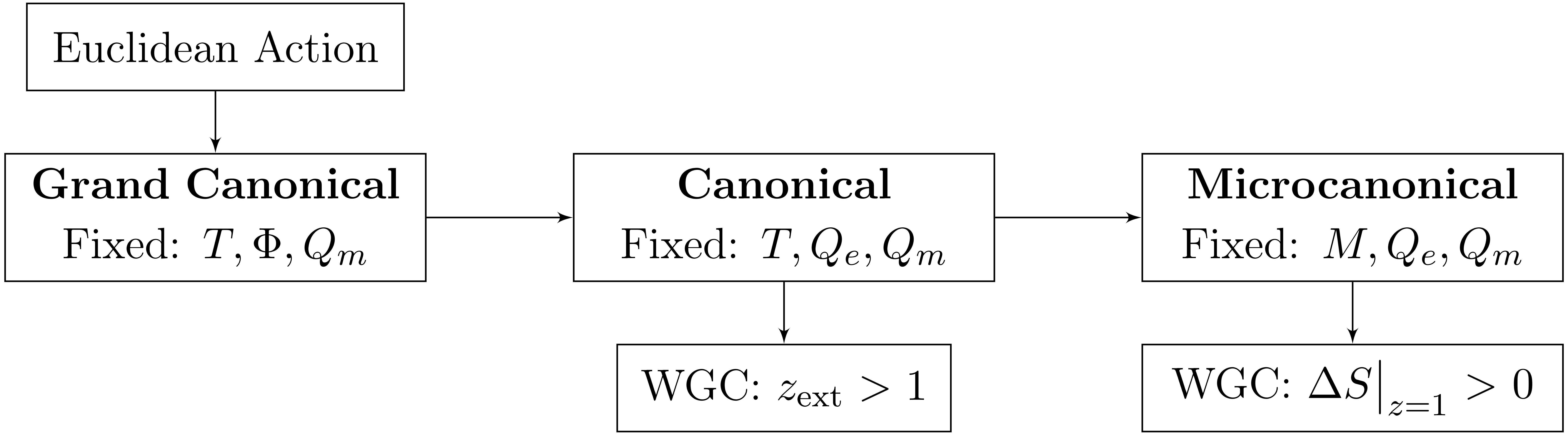
- Evaluating free energy G to first order requires only the **uncorrected solution**:

$$I_E = I_E^{(0)} + \alpha_i I_E^{(i)} + \mathcal{O}(\alpha^2)$$

$$I_E[\phi^{(0)} + \alpha\delta\phi] = I_E^{(0)}[\phi^{(0)} + \alpha\delta\phi] + \alpha_i I_E^{(i)}[\phi^{(0)} + \alpha\delta\phi] + \mathcal{O}(\alpha^2) = I_E^{(0)}[\phi^{(0)}] + \alpha_i I_E^{(i)}[\phi^{(0)}] + \mathcal{O}(\alpha^2)$$

- Subtleties for boundary terms & counter-terms [Reall Santos '19]

Ensembles



- WGC naturally phrased at fixed Q, P and $T \to 0$.
- Correlated with $\Delta S\,\big|_{z=1} > 0$ (microcanonical ensemble) [Hamada, Noumi, GS, '18];[Goon, Penco, '19]

Einstein-Maxwell-Dilaton

- Given the importance of scalars in formulating the WGC, consider:

$$\mathcal{L} = \frac{1}{2}R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{-2\lambda\phi}F^2 + \mathcal{O}(\alpha)\ \text{corrections}$$

- Zeroth-order dyonic BH solutions ($\lambda^2 = 1/2$):

$$ds^2 = -\frac{r(r-\xi)}{(r+P_e)(r+P_m)}dt^2 + \frac{(r+P_e)(r+P_m)}{r(r-\xi)}dr^2 + (r+P_e)(r+P_m)d\Omega_2^2$$

$$F = \frac{Q}{4\pi(r+P_e)^2}dt \wedge dr + \frac{P}{4\pi}d(\cos\theta) \wedge d\varphi$$

$$e^{-2\lambda\phi} = \frac{r+P_e}{r+P_m} \qquad Q^2 = (4\pi)^2 P_e(P_e+\xi), \quad P^2 = (4\pi)^2 P_m(P_m+\xi)$$

- Horizons: $r = 0,\ \xi$ Extremality: $\xi \to 0^+$

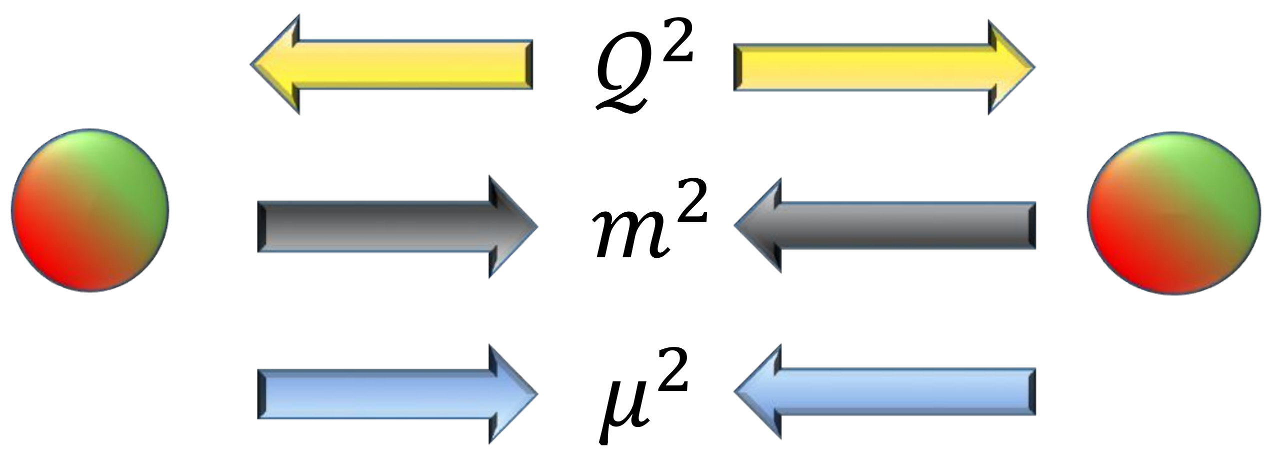
Scalar WGC

- Uncorrected solution satisfies the extremality bound:

$$M^2 = (4\pi\xi)^2 + 2\left(Q^2 + P^2 - Q_\phi^2\right) \geq 2\left(Q^2 + P^2 - Q_\phi^2\right)$$

- Long range scalar force: $e^{-2\lambda\phi} \sim 1 + \dfrac{Q_\phi}{r} + \dots$
- $Q_\phi \propto Q - P$ is not an independent parameter (no scalar hair), in particular $M^2 \geq 0$
- Scalar WGC is satisfied by BHs if the corrections make:

$$M^2 + 2Q_\phi^2 < 2\left(Q^2 + P^2\right)$$



Thermodynamics (Grand Canonical Ensemble)

- From I_E, it is most direct to obtain the thermodynamic functions in the GCE:

$$G(T,\Phi,P) = \frac{1-\Phi^2}{2T} + \frac{P^2 T}{2(1-\Phi^2)} + \mathcal{O}(\alpha) \text{ corrections}$$

$$S(T,\Phi,P) = \frac{1-\Phi^2}{2T^2} - \frac{P^2}{2(1-\Phi^2)} + \mathcal{O}(\alpha) \text{ corrections}$$

$$Q(T,\Phi,P) = \frac{\Phi}{T} - \frac{P^2 \Phi T}{2(1-\Phi^2)^2} + \mathcal{O}(\alpha) \text{ corrections}$$

$$\Psi(T,\Phi,P) = \frac{PT}{1-\Phi^2} + \mathcal{O}(\alpha) \text{ corrections}$$

$$M(T,\Phi,P) = \frac{1}{T} - \frac{P^2 \Phi^2 T}{(1-\Phi^2)^2} + \mathcal{O}(\alpha) \text{ corrections}$$

Extremality:

$$T \to 0, \quad \Phi^2 \to 1, \quad \frac{T}{1-\Phi^2} \text{ fixed}$$

Thermodynamics (Canonical Ensemble)

- The WGC is most naturally phrased at fixed Q, P and $T \to 0$ (hence in CE):

$$G(T,Q,P) = P\left[1 + \frac{1}{8}Q^2T^2 + \ldots\right] + \mathcal{O}(\alpha) \text{ corrections}$$

$$S(T,Q,P) = \frac{QP}{2}\left[1 + \frac{1}{2}(Q+P)T + \ldots\right] + \mathcal{O}(\alpha) \text{ corrections}$$

$$\Phi(T,Q,P) = \left[1 - \frac{1}{2}PT + \ldots\right] + \mathcal{O}(\alpha) \text{ corrections}$$

$$\Psi(T,\Phi,P) = \left[1 - \frac{1}{2}QT + \ldots\right] + \mathcal{O}(\alpha) \text{ corrections}$$

$$M(T,Q,P) = (Q+P)\left[1 + \frac{1}{8}QPT^2 + \ldots\right] + \mathcal{O}(\alpha) \text{ corrections}$$

Extremality:

$T \to 0$

Leading Corrections

- Seven independent 4-derivative operators:

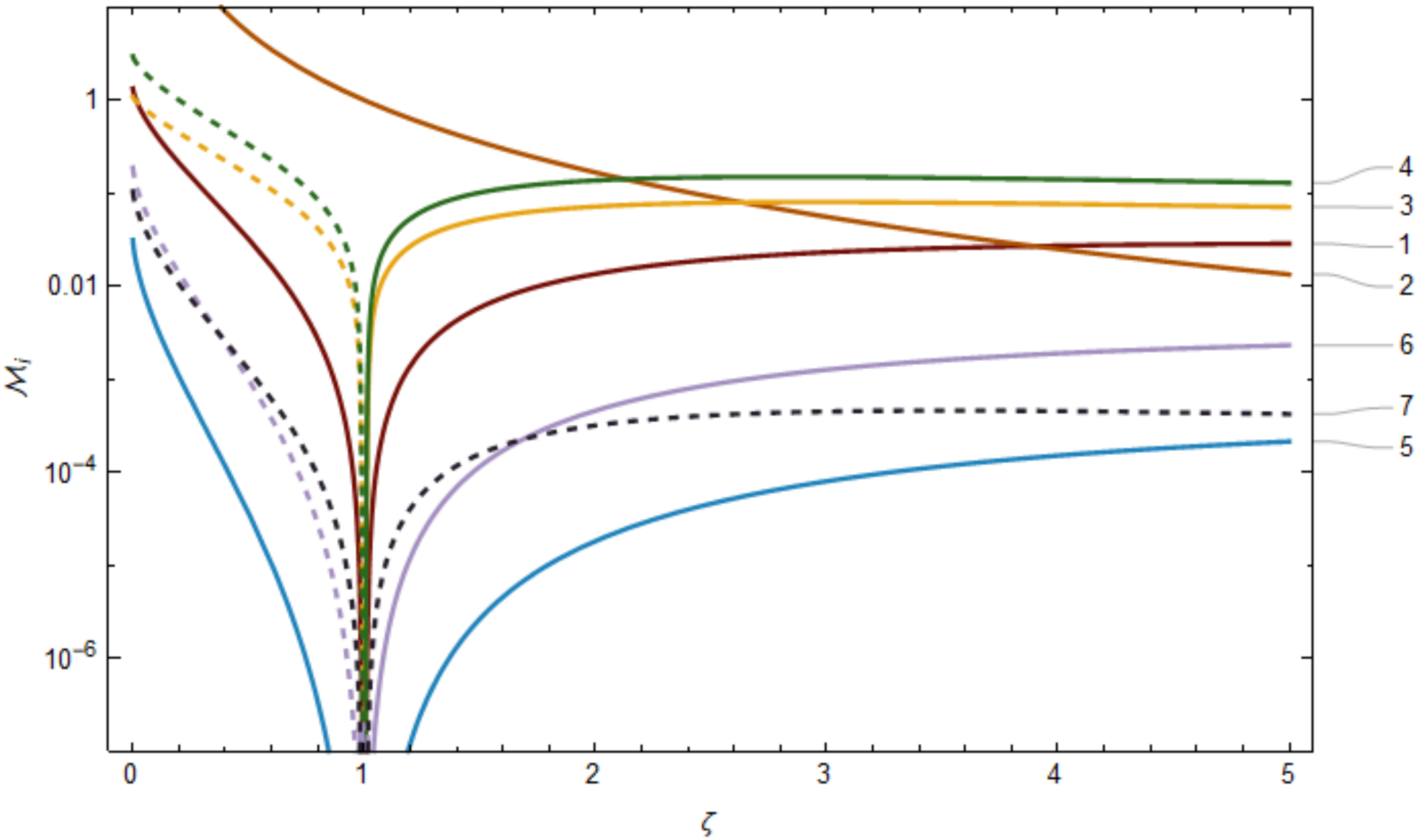
$$\alpha_i I_i \equiv \int \mathrm{d}^4x\,\sqrt{-g}\,\Big[\frac{\alpha_1}{4}e^{-6\lambda\phi}\big(F^2\big)^2 + \frac{\alpha_2}{4}e^{-6\lambda\phi}\big(F\widetilde{F}\big)^2 + \frac{\alpha_3}{2}e^{-4\lambda\phi}\big(FFW\big) + \frac{\alpha_4}{2}e^{-2\lambda\phi}(R_{\mathrm{GB}})$$
$$+ \frac{\alpha_5}{4}e^{-2\lambda\phi}(\partial\phi)^4 + \frac{\alpha_6}{4}e^{-4\lambda\phi}(\partial\phi)^2\big(F^2\big) + \frac{\alpha_7}{4}e^{-4\lambda\phi}(\partial\phi\partial\phi FF)\Big]$$

- We have computed all thermodynamic quantities to linear order in α_i [Loges, Noumi, GS, '19], e.g. for the F^4 operator:

$$M(T,Q,P) = (Q+P) - \frac{32\pi^2\alpha_1}{5QP}\;\frac{(1-\zeta)(8+103\zeta-137\zeta^2-37\zeta^3+3\zeta^4)+60\zeta(1-2\zeta^2)\log\zeta}{6(1+\zeta)(1-\zeta)^5}$$

with $\zeta = P/Q$.

A Positivity Puzzle



$$z_{ext} = 1 + \frac{32\pi^2}{5QP}\alpha_i M_i(\zeta)$$

Gravitational positivity bound : $\alpha_{1,2,5,7} > 0$

Staying Positive

- Under the same assumption of Regge boundedness and gravity subdominance (e.g. integrating out scalars or UV completions with open string Regge tower), $z_{ext} > 1$.
- Further non-trivial checks [Loges, Noumi, GS, '20]: including axion but with symmetries in action
  - $SL(2,\mathbb{R})$ (broken to $SL(2,\mathbb{Z})$ by non-perturbative effects): positivity bounds $\Rightarrow z_{ext} > 1$
  - $O(d, d, \mathbb{R})$: gravitational 4-derivative terms are not subdominant, but WGC follows from NEC.
  - $N \geq 2$ SUSY: puzzling term which give $\Delta z_{ext} < 0$ are necessary to ensure that the extremality bound is uncorrected for BPS states.
- For healthy theories, the leading corrections shift the extremality bound positively, making BHs a WGC state.

Covariant Formulation of WGC

A New Spin on the WGC

[Aalsma, Cole, Loges, GS, ’20]

- We reformulate the WGC as a **covariant integrated condition**:

$$\int_\Sigma \mathrm{d}^{d-1}x\,\sqrt{h}\,\delta T^{\text{eff}}_{ab}\,\xi^a n^b \leq 0$$

derived using the covariant phase formalism of Iyer-Wald.

- Construct a Hamiltonian generating a diffeomorphism parametrized by ξ and gauge transformation $A \to A + d\lambda$.
- The Hamiltonian obeys a conservation law on-shell. Off-shell variation gives:

$$\Bigg(\underbrace{\int_{S^{d-2}_\infty}}_{\text{Asymptotic charges}} - \underbrace{\int_{S^{d-2}_{\text{hor}}}}_{\text{correction to the horizon}}\Bigg)\,\delta\mathbf{H} = \int_\Sigma \mathrm{d}^{d-1}x\,\sqrt{h}\,\delta T^{\text{eff}}_{ab}\,n^a\xi^b$$

A New Spin on the WGC

[Aalsma, Cole, Loges, GS, '20]

- **Advantages:** No need to solve the corrected Einstein equations to derive corrections to the extremality bound; Valid for any corrections, not just higher derivative corrections

$$\left(\int_{S^{d-2}_\infty} - \int_{S^{d-2}_{\text{hor}}}\right)\delta\mathbf{H} = \int_\Sigma \mathrm{d}^{d-1}x\,\sqrt{h}\,\delta T^{\text{eff}}_{ab}n^a\xi^b$$

- For charged 4d BH and BTZ BHs respectively, the asymptotic charges are:

$$\partial_t \leftrightarrow M, \quad \partial_\phi \leftrightarrow J, \quad \lambda \leftrightarrow Q$$

- Our covariant energy condition amounts to evaluating:

$$\delta M - \Phi\delta Q + \frac{r_+}{2G_4}\delta f(r_+) = \int_\Sigma d^3x\sqrt{h}\delta T^{\text{eff}}_{ab}n^a\xi^b\ , \quad \xi = \partial_t$$

$$\delta M - \Omega\delta J + \frac{1}{8G_3}\delta N(r_+)^2 = \int_\Sigma d^2x\sqrt{h}\delta T^{\text{eff}}_{ab}n^a\xi^b\ , \quad \xi = \partial_t - \Omega\partial_\phi$$

Spinning WGC

Spinning WGC

[Aalsma, Cole, Loges, GS, '20]

- WGC is a statement that charged extremal BHs are unstable.
- Q: Is there an analogous statement for rotating BHs?
- A: Probably not - heuristic motivation is gone (Penrose process).
- However, for BTZ BHs, there is a **spinning WG Theorem:**

$$\boxed{\text{Extremal BTZ BHs satisfy } \frac{J}{M} \geq \lim_{M\to\infty} \frac{J}{M}\bigg|_{ext}}$$

- Even though gravity is not dynamical in 3d, BTZ geometry is distinguished as it appears as near-horizon limit of many stringy BHs.

Corrections to BTZ

[Aalsma, Cole, Loges, GS, '20]

- Consider the 3d action on an AdS_3 background perturbed by the leading 4-derivative operators (purely gravitational):

$$\int d^3x\sqrt{-g}\left[\frac{1}{16\pi G_3}\left(R+\frac{2}{\ell^2}\right)+\alpha_1\ell R^2+\alpha_2\ell R_{ab}R^{ab}\right]$$

- Directly compute the shift in BH horizon (covariant formulation):

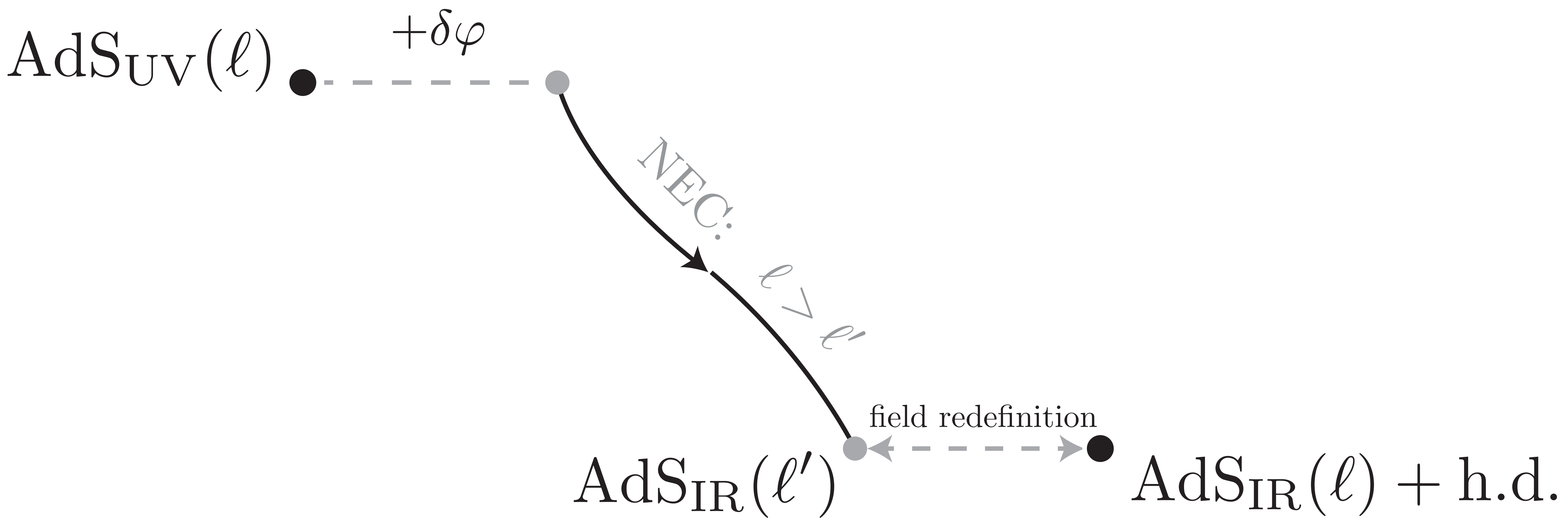
$$\delta T_{ab}=-\frac{4(3\alpha_1+\alpha_2)}{\ell^3}g_{ab}+\mathcal{O}(\alpha^2)\ ; \qquad \int_\Sigma d^2x\sqrt{h}\,\delta T_{ab}n^a\xi^b=-\frac{4\pi r_+^2}{\ell^3}\left(3\alpha_1+\alpha_2\right)$$

- Redefinition of the AdS length: $\ell'=\ell-42\pi G_3\left(3\alpha_1+\alpha_2\right)$

$$c'=\frac{3\ell'}{2G_3}=\frac{3\ell}{2G_3}\left[1-\frac{48\pi G_3(3\alpha_1+\alpha_2)}{\ell}\right] \quad \text{and} \quad \frac{|J_3|}{\ell M_3}\leq 1+\frac{48\pi G_3(3\alpha_1+\alpha_2)}{\ell}$$

Holographic RG Arugment

[Aalsma, Cole, Loges, GS, '20]

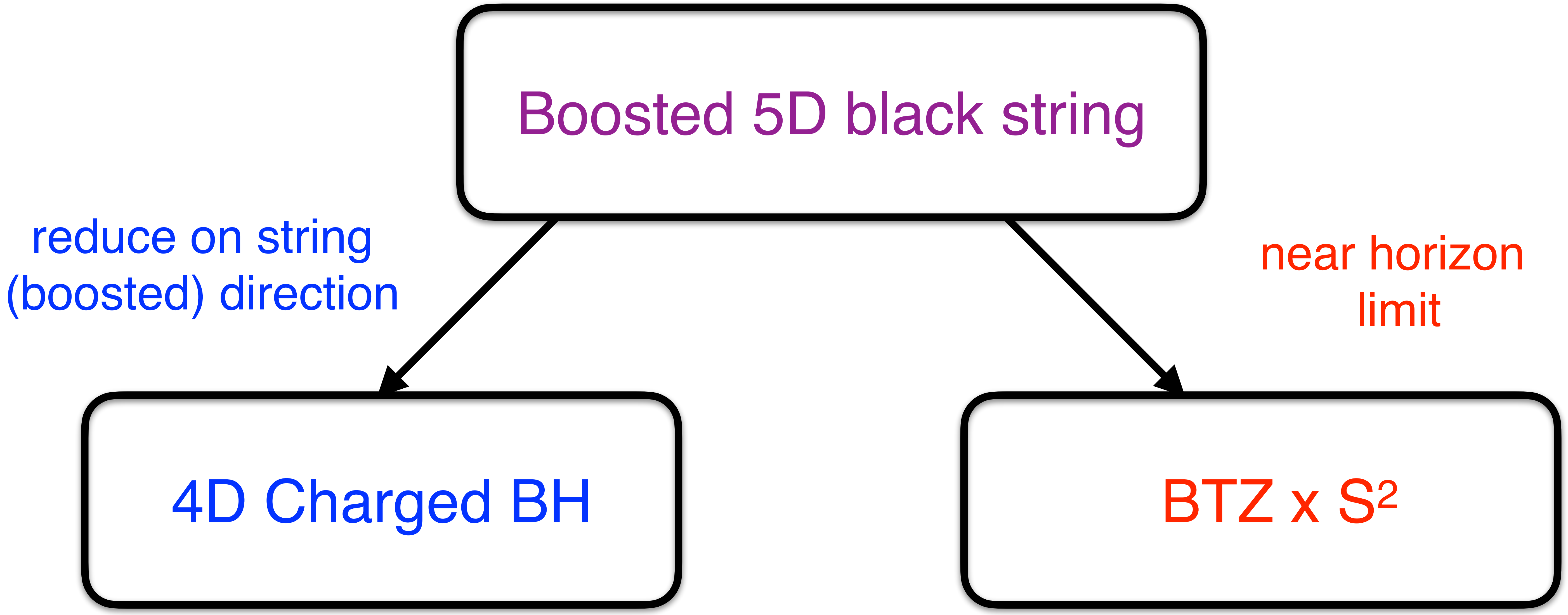


- Not necessary to assume that the UV CFT is dual to pure Einstein gravity (though convenient because the central charge takes the Brown-Henneaux form).
- As long as the NEC is satisfied in the bulk, the c-theorem implies a decrease in central charge in the IR and an increase in extremality bound.

Charged WGC

[Aalsma, Cole, Loges, GS, '20]

- Many charged extremal solutions have a near horizon BTZ geometry; in fact one often uses this to compute their entropy.
- Given the **entropy-extremality relation** [Hamada, Noumi, GS '18], can we use the spinning WGC to infer the charged WGC?
- An example is the boosted 5D black string which has an M-theory origin as the intersection of three M5-branes:



Extremality and Entropy

[Aalsma, Cole, Loges, GS, '20]

- The boosted 5D black string is described by the 5D action:

$$I = \frac{1}{16\pi G_5}\int d^5x\sqrt{-g}\left(R - \frac{3}{4}F_{MN}F^{MN} + \alpha_1 Q^2 F_{MN}F^{MN}F_{OP}F^{OP} + \alpha_2 Q^2 F_{MN}F_{OP}W^{MNOP} + \alpha_3 Q^2 E_5\right)$$

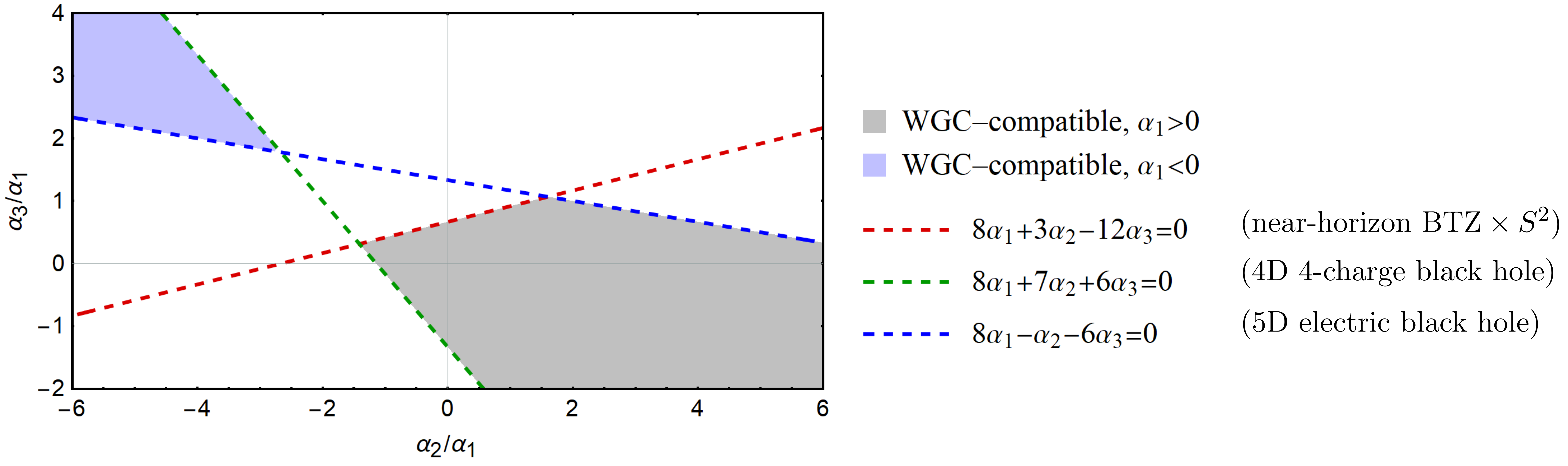
W_{MNOP} is the Weyl tensor and $E_5 = R_{MNOP}R^{MNOP} - 4R_{MN}R^{MN} + R^2$

$\mathrm{BTZ}\times S^2$	$T=0$	$z = 1 + \frac{8\alpha_1+3\alpha_2-12\alpha_3}{2}$	
		$S = 2\pi Q\sqrt{\frac{M_3}{G_3}}\left(1 - \frac{8\alpha_1+3\alpha_2-12\alpha_3}{2}\right)$	$= \frac{\pi Q^2}{G_4}\left(1 - \frac{8\alpha_1+3\alpha_2-12\alpha_3}{2}\right)$
	$z=1$	$T = \sqrt{\frac{G_3 J_3(8\alpha_1+3\alpha_2-12\alpha_3)}{\pi Q^3}}$	
		$S = 2\pi Q\sqrt{\frac{M_3}{G_3}}\left(1 + \sqrt{\frac{8\alpha_1+3\alpha_2-12\alpha_3}{2}}\right)$	$= \frac{\pi Q^2}{G_4}\left(1 + \sqrt{\frac{8\alpha_1+3\alpha_2-12\alpha_3}{2}}\right)$
4D	$T=0$	$z = 1 + \frac{2a_1+a_2}{10}$	$= 1 + \frac{8\alpha_1+7\alpha_2+6\alpha_3}{40}$
		$S = \frac{\pi Q^2}{G_4}(1 - 4a_1 + 4a_3)$	$= \frac{\pi Q^2}{G_4}\left(1 - \frac{8\alpha_1+3\alpha_2-12\alpha_3}{2}\right)$
	$z=1$	$T = \frac{\pi}{Q}\sqrt{\frac{2(2a_1+a_2)}{5}}$	$= \frac{\pi}{Q}\sqrt{\frac{8\alpha_1+7\alpha_2+6\alpha_3}{10}}$
		$S = \frac{\pi Q^2}{G_4}\left(1 + \sqrt{\frac{2(2a_1+a_2)}{5}}\right)$	$= \frac{\pi Q^2}{G_4}\left(1 + \sqrt{\frac{8\alpha_1+7\alpha_2+6\alpha_3}{10}}\right)$

Total Landscaping

[Aalsma, Cole, Loges, GS, '20]

- The entropy of the BTZ and 4d charged BH agree at zero temperature, but the extremality bounds do not.
- The **entropy-extremality relation** [Hamada, Noumi, GS '18] is between the extremality bound (fixed $Q/J, T$) & the **microcanonical** entropy (fixed $Q/J, M$).
- The extremality bounds for the spinning WGC and charged WGC do not line up; rather they together strengthen the WGC in 5D:



Summary

Summary

- Thermodynamics of BHs has lent insights into quantum gravity. Worthwhile to leverage this effectiveness for swampland criteria such as WGC, RFC, CCC, …

- We provided evidence that the extremality curve (M vs Q) for BHs approaches the classical extremality bound monotonically from below: (axio)-dilatonic Einstein-Maxwell BHs, BTZ BHs.

- Evidence that this monotonicity behavior continues from the flat space BH regime to the large AdS BH regime [Loges, Noumi, GS, work in progress]. If universal, this behavior has interesting consequences for $1/q$ corrections to $\Delta(q) \geq \mathcal{O}(c)$ operators in dual CFT.

- Covariant formulation of WGC in terms of effective stress tensor (Iyer-Wald).

- Spinning WGC via c-theorem and Total Landscaping Principle.