Generalized Clausen identities and the refined swampland distance conjecture

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Refined Distance Conjecture

At distances beyond $\mathcal{O}(M_{Pl})$ an infinite tower of states appears which becomes exponentially light with the distance:

$$m(\phi) = m(\phi_0) e^{-\alpha \frac{\Delta(\phi,\phi_0)}{M_{Pl}}} \tag{1}$$

- Metric on moduli space

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K \tag{2}$$

$$K = -\log(-i\, \bar{\Pi} \cdot \Sigma \cdot \Pi) - \log(S + \bar{S}) \tag{3}$$

- Requires *global* expressions for the periods
- This includes different phases and regions close to the boundaries.
- Numerically doable, but bad convergences for $h^{2,1} > 1$.
- Modular expressions can help. [Kläwer 21]

What is a global closed form?

For this talk: Any function that

- Can be evaluated for any value where it is defined
- Derivatives known
- Can be integrated

This includes all usual functions (log, sin, arctan, polynoms, etc.) as well as some more uncommon (polylogs, complete elliptic integrals K(x), ${}_2F_1$)

Periods

Let X be a CY hypersurface or CICY in $\mathbb{WCP}$. (This assumption will be dropped later)

Choose symplectic basis $\gamma^\alpha \in H_3(X, \mathbb{Z}) \qquad \alpha = 0, \ldots, 2h^{2,1} + 1.$

$$\Pi^\alpha(x) = \int_{\gamma^\alpha} \Omega(x) \tag{4}$$

The x denote the moduli of the CY, Ω is the unique holomorphic 3-form.

How to compute the periods?

The periods fulfill a system of differential equations, the Picard-Fuchs equations.

$$\mathcal{D}_l = \prod_{l_i>0} \left(\frac{\partial}{\partial_{a_i}}\right)^{l_i} - \prod_{l_i<0} \left(\frac{\partial}{\partial_{a_i}}\right)^{-l_i}, \qquad l \in \{l_i\}\,, \tag{5}$$

$$x_k = (-1)^{l_0^{(k)}} a_0^{l_0^{(k)}} \dots a_s^{l_s^{(k)}} \tag{6}$$

$$\mathcal{D}_l\,\omega = 0 \tag{7}$$

Local Solutions at LCS

Around the LCS point a 'closed' form is given by

[Hosono, Klemm, Theisen, Yau 93']

$$\omega_0 = \sum_{n_i=0;i=1,\ldots,h^{2,1}}^{\infty} \left(\prod_{i=1}^{h^{2,1}} x_i^{n_i+\rho_i} \right) \frac{\Gamma\left[1 - \sum_{k=1}^{h^{2,1}} l_k^{(0)}(n_k+\rho_k)\right]}{\Gamma\left[1 - \sum_{k=1}^{h^{2,1}} l_k^{(0)}\rho_k\right]} \cdot \prod_{j=1}^{p} \frac{\Gamma\left[1 - \sum_{k=1}^{h^{2,1}} l_k^{(j)}\rho_k\right]}{\Gamma\left[1 - \sum_{k=1}^{h^{2,1}} l_k^{(j)}(n_k+\rho_k)\right]} .$$

Depends on the charge vectors l, the moduli x_i and the indices ρ_i

Local Solutions at LCS

$$
\begin{aligned}
D_{1,i} &= \frac{1}{2\pi i}\partial_{\rho_i} \, , \\
D_{2,i} &= \frac{1}{2}\frac{K_{ijk}}{(2\pi i)^2}\partial_{\rho_j}\partial_{\rho_k} \, , \\
D_3 &= -\frac{1}{6}\frac{K_{ijk}}{(2\pi i)^3}\partial_{\rho_i}\partial_{\rho_j}\partial_{\rho_k} \, ,
\end{aligned} \tag{8}
$$

$$
\omega = \left.\begin{pmatrix} \omega_0 \\ D_{1,i}\,\omega_0 \\ D_{2,i}\,\omega_0 \\ D_3\,\omega_0 \end{pmatrix}\right|_{\rho_i=0} \tag{9}
$$

ϵ expansion of hypergeometric functions

One can rewrite the fundamental period at the LCS as a hypergeometric function, e.g. for 3 parameters:

$$\omega_0 = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \overline{x}^{n_1+\rho_1} \overline{y}^{n_2+\rho_2} \overline{z}^{\rho_z} f(n_1, n_2, \rho_1, \rho_2, \rho_3) \, {}_pF_q(\vec{a}, \vec{b}, \overline{z}) \, .$$

The periods are given by up to third derivatives with respect to the indices ρ. These appear in the parameters of the hypergeometric function.

$\rightarrow$ Need to expand the ${}_pF_q$ around its parameters (to order 3).

ϵ expansion of hypergeometric functions

- Well studied in the amplitudes community (2003-2013)

  [Weinzierl 04', Kalmykov, Kniehl 10', Greynat, Sesma 13'. . .]

- Recently much progress in the math community (2014-2020)

  [Wan, Zucker 14', Aiblinger 15', Campbell, D'Aurizio, Sondow 17', Cantarini, D'Aurizio 18', Zhao 19',Zhao 20'. . .]

- Still a hard computation!

The program

- Compute the elliptic curve case
- Reduce the computation of certain K3s to this case using Clausen's identity
- Generalize to any geometric 2-dimensional PF operator
- Use generalizations of Clausens's identity to compute the periods of any 1-parameter K3/Fano 3-fold in closed form

Elliptic curves

There are exactly 4 complete intersection elliptic curves.

$$\mathbb{P}_{1,1,1,1}[2\ 2] \qquad \mathbb{P}_{1,1,2}[4] \qquad \mathbb{P}_{1,1,1}[3] \qquad \mathbb{P}_{1,2,3}[6]$$

These correspond to the fundamental periods

$$ {}_2F_1(\frac{1}{2}, \frac{1}{2}, 1, x) \qquad {}_2F_1(\frac{1}{4}, \frac{3}{4}, 1, x) \qquad {}_2F_1(\frac{1}{3}, \frac{2}{3}, 1, x) \qquad {}_2F_1(\frac{1}{6}, \frac{5}{6}, 1, x)$$

Exactly matching Ramanujan's theory of elliptic functions of alternative bases.
Special cases of the Legendre family of hypergeometric functions:

$$ {}_2F_1\left(a, 1-a, 1, x\right) \tag{10}$$

Choosing other values for a allows for easy violations of the distance conjecture. But the possible values are restricted by the monodromy representations of the 3-punctured sphere!

The simplest case

$a = 1/2$

$$\omega_0 = x^{\rho}\, {}_3F_2(1, \frac{1}{2}+\rho_1, \frac{1}{2}+\rho_1; 1+\rho_1, 1+\rho_1; x) \,. \qquad (11)$$

We need the parameter derivatives of

$$ {}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x) = \frac{\pi}{2} K(x) \,. \qquad (12)$$

Can be rewritten as

$$\sum_{n=0}^{\infty} \frac{\Gamma[n+1/2]\Gamma[n+1/2]}{\Gamma[n+1]\Gamma[n+1]} x^n \left(\Psi(n+1/2) + \Psi(n+1/2) - 2\Psi(n+1)\right)$$

Or equivalently using the harmonic number representation of the polygamma functions:

$$\sum_{n=0}^{\infty} \frac{\Gamma[n+1/2]\Gamma[n+1/2]}{\Gamma[n+1]\Gamma[n+1]} x^n \left(H_{n-1/2} + H_{n-1/2} - 2H_n\right)$$

Expansion for Elliptic curves

$$\sum_{n=0}^{\infty} \frac{\Gamma[n+1/2]\Gamma[n+1/2]}{\Gamma[n+1]\Gamma[n+1]} x^n \left(H_{n-1/2} + H_{n-1/2} - 2H_n\right)$$

$$H_{xn} = \frac{1}{x}\left(H_n + H_{n-\frac{1}{x}} + H_{n-\frac{2}{x}} + \ldots + H_{n-\frac{x-1}{x}}\right) + \log(x)$$

$$H_{2n} = \frac{1}{2}\left(H_n + H_{n-\frac{1}{2}}\right) + \log(2)$$

Allows rewriting of the sum into combinations of harmonic numbers with integer coefficients.

Expansion for Elliptic curves

Allows rewriting of the sum into combinations of harmonic numbers with integer coefficients.

$$\partial_\rho \omega_0|_{\rho=0} = \sum_{n=0}^{\infty} \frac{\Gamma[n+1/2]\Gamma[n+1/2]}{\Gamma[n+1]\Gamma[n+1]} x^n \left(2H_{2n} - 2H_n\right)$$

Generating functions actually known!

$$\sum_{n=0}^{\infty} \binom{2n}{n}^2 H_n \frac{x^n}{4^{2n}} = K(1-x) + \frac{1}{\pi} K(x) \log\left(\frac{x^2}{16(1-x)}\right) \tag{13}$$

$$\sum_{n=0}^{\infty} \binom{2n}{n}^2 H_{2n} \frac{x^n}{4^{2n}} = \frac{1}{2} K(1-x) + \frac{1}{\pi} K(x) \log\left(\frac{x}{4(1-x)}\right) \tag{14}$$

Global closed forms for parameter derivatives

Even Better:

$$\partial_c \, {}_2F_1(a, 1-a, c, x)|_{c=1} = -\frac{\pi}{2\sin(\pi a)} \, {}_2F_1(a, 1-a, 1, 1-x) -$$

$$\frac{1}{2}\left(\Psi(1-\frac{a}{2}) + \Psi(\frac{a+1}{2}) - \Psi(1) - \Psi(\frac{1}{2}) - \frac{\pi}{\sin(\pi a)} - \log(\frac{1-x}{x})\right) \, {}_2F_1(a, 1-a, 1, x)$$

[Nicholson 18']

$$\partial_\epsilon \, {}_2F_1(a+\epsilon, b+\epsilon, a+b, x)|_{\epsilon=0} = \log\left(\frac{1}{1-x}\right) \, {}_2F_1(a, b, a+b, x)$$

[Blaschke 18']

- Combining these gives a closed form for the periods of all 4 elliptic curves.
- The expressions simplify for explicit values of a!

Combining Everything

$$t = \frac{\omega_1}{\omega_0} = \frac{1}{2\pi i} \frac{\partial_\rho \omega_0}{\omega_0}|_{\rho=0} = \frac{i}{2} \frac{K(1-x)}{K(x)} - \frac{i}{\pi} \log(4)$$

- Exactly of the form expected for the inverse of a triangle map.
- The only closed form for an ϵ-expansion needed!
- Rest of this talk: How to use this result to compute the periods of K3s and Fano 3-folds.

Clausen's identity

The Legendre family enjoys a classical identity

$$
\begin{aligned}
{}_3F_2(\{a, 1-a, \frac{1}{2}\}, \{1,1\}, x) &= {}_2F_1(\{\frac{a}{2}, \frac{1-a}{2}\}, \{1\}, x)^2 \\
&= {}_2F_1(\{a, 1-a\}, \{1\}, 4x(1-x))^2
\end{aligned}
$$

[Clausen 1828']

- relates the fundamental period of certain K3s to the periods of elliptic curves!
- The structure is such that the first ϵ-derivative can also be computed!
- The identity does not help with the second derivative.

Period identities

- Compactification on K3 manifolds results in an $\mathcal{N} = 4$ supersymmetric theory

$\rightarrow$ no instanton corrections to the prepotential
$\rightarrow$ condition on the periods:

$$\omega_2 = \frac{\omega_1^2}{\omega_0} \, . \tag{15}$$

$\rightarrow$ knowledge of the first two periods is sufficient!
Note: Both relations, Clausen's identity as well as the period relation follow from the fact, that the PF operator of K3s are the symmetric squares of second order operators of the elliptic curves.

Example: $\mathbb{P}_{11222}[8]$

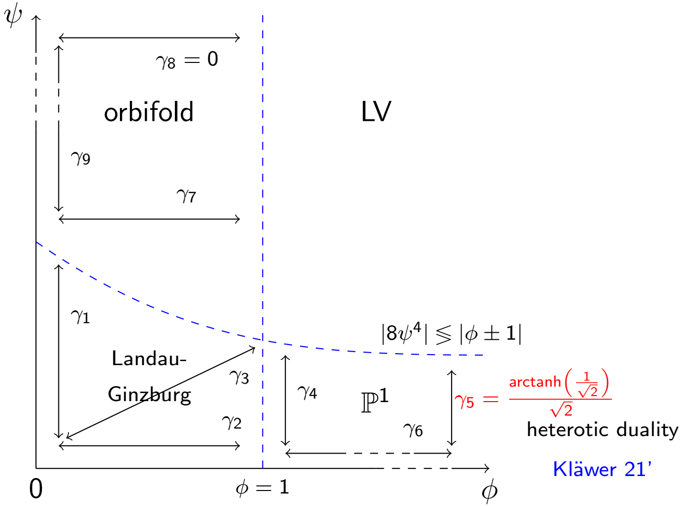

Figure: Definitions of curves in the moduli space of $\mathbb{P}_{11222}[8]$. The dotted blue lines represent a sketch of the phase boundaries.

Computing the distance

$$\mathcal{F} = -\frac{1}{6}K_{ijk}t_i t_j t_k + \frac{1}{2}a_{ij}t_i t_j + b_i t_i - \frac{1}{2}\chi\frac{\zeta(3)}{(2\pi i)^3}\,. \tag{16}$$

$$K_{111} = 8\,, \quad K_{112} = K_{121} = K_{211} = 4\,, \quad b_1 = \frac{7}{3}\,, \quad b_2 = 1\,, \quad \chi = -168$$

$$\to g_{t_1,\bar{t}_1}\big|_{|t_2|=\infty} = \frac{1}{2\mathrm{Im}(t_1)^2} \tag{17}$$

$$t_1(x) = \frac{i}{2}\frac{\sqrt{4+2\sqrt{2-2\sqrt{1-\overline{x}_1}}}}{\sqrt{2+\sqrt{2}\sqrt{1+\sqrt{1-\overline{x}_1}}}}\frac{K\left(\frac{2\sqrt{2}\sqrt{1+\sqrt{1-\overline{x}_1}}}{2+\sqrt{2}\sqrt{1+\sqrt{1-\overline{x}_1}}}\right)}{K\left(\frac{2\sqrt{2-2\sqrt{1-\overline{x}_1}}}{2+\sqrt{2-2\sqrt{1-\overline{x}_1}}}\right)}\,, \tag{18}$$

$$\to \Delta(\mathrm{LG},\mathrm{Conifold}) = \frac{\mathrm{arctanh}\left(\frac{1}{\sqrt{2}}\right)}{\sqrt{2}}$$

Some comments

- No logarithms appear in the mirror map!
- The usual log combines with the infinite instanton series.
- Underlying all of this is modularity, but it is neither used nor needed!

$$\overline{x}_1(t^1) = \frac{256}{j_2^+(t^1)} \,. \tag{19}$$

$$j_2^+(t^1) = 2^4 \frac{(\theta_3(q_{t^1})^4 + \theta_4(q_{t^1})^4)^4}{\theta_2(q_{t^1})^8 \theta_3(q_{t^1})^4 \theta_4(q_{t^1})^4} \,, \tag{20}$$

Inverting this relation is hard! (but extremely good approximations are possible, see Kläwer 21')
The hypergeometric treatment gives directly the inverse relation.

An aside: Apéry-like sequences

Apéry used the sequence

$$u_n = 1,\ 3,\ 19,\ 147,\ 1251,\ 11253,\ 104959\ldots$$

in his proof of the irrationality of $\zeta(2)$. It follows the recursion relation

$$n^2 u_n - (A(n-1)^2 + A(n-1) + B) u_{n-1} + C(n-1)^2 u_{n-2} = 0\,, \quad (21)$$

for A=11, B=3, C=-1.

$$\left[\theta^2 - x(A\theta^2 + A\theta + B) + x^2 C(\theta+1)^2\right] \tilde{f}(x) = 0\,, \quad (22)$$

This inspired Zagier to search for more integer sequences of this type.

A	B	C	Name	Generating Function
7	2	-8	A	$\frac{1}{1-2x}\ {}_2F_1(\frac{1}{3},\frac{2}{3},1,27x^2/(1-2x)^3)$
9	3	27	B	$\frac{1}{1-3x}\ {}_2F_1(\frac{1}{3},\frac{2}{3},1,-27x^3/(1-3x)^3)$ $=\ {}_2F_1(\frac{1}{3},\frac{1}{3},1,27x(1-9x+27x^2))$
12	4	32	E	$\frac{1}{1-4x}\ {}_2F_1(\frac{1}{2},\frac{1}{2},1,16x^2/(1-4x)^2)$ $=\ {}_2F_1(\frac{1}{2},\frac{1}{2},1,16x(1-4x))$
17	6	72	F	$\frac{1}{1-6x}\ {}_2F_1(\frac{1}{3},\frac{2}{3},1,-27x^3(8x-1)/(1-6x)^3)$
10	3	9	C	$\frac{\sqrt{2}}{p_1^{1/2}}\ {}_2F_1(\frac{1}{2},\frac{1}{2},1,64x^3/p_1^2)$
11	3	-1	D	$\frac{1}{p_2^{1/4}}\ {}_2F_1(\frac{1}{12},\frac{5}{12},1,1728x^5(1-11x-x^2)/(p_2)^3))$

Apéry-like sequences

- Periods of the 6 curves with 4 singularities. Beukers 02',Zagier 09'
- Closed forms for all sequences know!

$$\omega_0 = f(x)\, {}_2F_1(a, 1-a, 1, g(x)) \tag{23}$$

Exactly of the type whose ϵ expansion is known. They form a nice "basis" of functions. The (twisted) symmetric squares of them span all K3s and Fano 3-folds with PF operators of order 2!

(twisted) symmetric squares

Almkvist, van Straten, Zudilin 11'

$$\left[\theta^2 - x(A\theta^2 + A\theta + B) + x^2 C(\theta+1)^2\right] f(x) = 0\,,$$

$$\left[\theta^3 - 2x(2\theta+1)(A\theta^2 + A\theta + B) + 4Cx^2(\theta+1)(2\theta+1)(2\theta+3)\right]$$

$$f(x)^2 = \frac{1}{1-Cx^2}\tilde{f}\left(\frac{1-Ax+Cx^2}{(1-Cx^2)^2}\right)\,.$$

$$[\theta^3 - 2x(2\theta+1)(A\theta^2 + A\theta + B) + 4x^2 C(\theta+1)^3]\tilde{f}(x) = 0\,,$$

$$f(x)^2 = \frac{1}{1-Ax+Cx^2}\tilde{f}\left(\frac{-x}{1-Ax+Cx^2}\right)\,.$$

Rewriting these identities for $\tilde{f}(x)$ leads to identities of the form

$$\tilde{f}(x) = g(x)\,{}_2F_1(a, 1-a, 1, h(x))^2\,.$$

Compare to Clausen's identity:

$$\tilde{f}(x) = {}_2F_1(a, 1-a, 1, 4x(1-x))^2\,.$$

K3 manifolds Lian, Yau 95'

degree matrices $\mathbf{d}$	diff. operators L	potential $Q(x)$	genus 0 groups
(4)	$\Theta_x^3 - 8x(1+2\Theta_x)(1+4\Theta_x)(3+4\Theta_x)$	$\frac{1-304x+61440x^2}{4(1-256x)^2x^2}$	$\Gamma_0(2)+$
$(2 \quad 3)$	$\Theta_x^3 - 6x(1+2\Theta_x)(1+3\Theta_x)(2+3\Theta_x)$	$\frac{1-132x+11340x^2}{4(1-108x)^2x^2}$	$\Gamma_0(3)+$
$(2 \quad 2 \quad 2)$	$\Theta_x^3 - 8x(1+2\Theta_x)^3$	$\frac{1-80x+4096x^2}{4(1-64x)^2x^2}$	$\Gamma_0(4)+$
$\begin{pmatrix} 2 \\ 3 \end{pmatrix}$	$\Theta_x^3 - 8x(1+2\Theta_x)^3$	$\frac{1-80x+4096x^2}{4(1-64x)^2x^2}$	$\Gamma_0(4)+$
$\begin{pmatrix} 0 & 2 \\ 2 & 2 \end{pmatrix}$	$\Theta_x^3+36x^2(1+\Theta_x)(1+2\Theta_x)(3+2\Theta_x)$ $-2x(1+2\Theta_x)(3+10\Theta_x+10\Theta_x^2)$	$\frac{1-52x+1500x^2-6048x^3+15552x^4}{4(1-36x)^2(1-4x)^2x^2}$	$\Gamma_0(6)+$
$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$	$\Theta_x^3+x^2(1+\Theta_x)^3$ $-x(1+2\Theta_x)(5+17\Theta_x+17\Theta_x^2)$	$\frac{1-44x+1206x^2-44x^3+x^4}{4x^2(1-34x+x^2)^2}$	$\Gamma_0(6)+6$
$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$	$\Theta_x^3-32x^2(1+\Theta_x)(1+2\Theta_x)(3+2\Theta_x)$ $-2x(1+2\Theta_x)(2+7\Theta_x+7\Theta_x^2)$	$\frac{1-36x+972x^2+3712x^3+12288x^4}{4(-1-4x)^2x^2(-1+32x)^2}$	$\Gamma_0(6)+$
$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$	$\Theta_x^3-3x^2(1+\Theta_x)(2+3\Theta_x)(4+3\Theta_x)$ $-x(1+2\Theta_x)(4+13\Theta_x+13\Theta_x^2)$	$\frac{1-34x+745x^2+840x^3+648x^4}{4(-1-x)^2x^2(-1+27x)^2}$	$\Gamma_0(7)+$
$\begin{pmatrix} 0 & 1 & 1 & 2 \\ 2 & 1 & 1 & 0 \end{pmatrix}$	$\Theta_x^3+64x^2(1+\Theta_x)^3$ $-2x(1+2\Theta_x)(2+5\Theta_x+5\Theta_x^2)$	$\frac{1-28x+396x^2-1792x^3+4096x^4}{4(1-16x)^2(1-4x)^2x^2}$	$\Gamma_0(12)+$
$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$	$\Theta_x^3-4x^2(1+\Theta_x)(3+4\Theta_x)(5+4\Theta_x)$ $-2x(1+2\Theta_x)(1+3\Theta_x+3\Theta_x^2)$	$\frac{1-16x+224x^2+976x^3+3840x^4}{4(-1-4x)^2x^2(-1+16x)^2}$	$\Gamma_0(10)+$

Fano 3-folds, Golyshev 05'

$d = 1$

1	$D^3 - 24t(1+2D)(6D+5)(6D+1)$
2	$D^3 - 8t(1+2D)(4D+3)(4D+1)$
3	$D^3 - 6t(1+2D)(3D+2)(3D+1)$
4	$D^3 - 8t(1+2D)^3$
5	$D^3 - 2t(1+2D)\left(11D^2+11D+3\right) - 4t^2(D+1)(2D+3)(1+2D)$
6	$D^3 - t(1+2D)\left(17D^2+17D+5\right) + t^2(D+1)^3$
7	$D^3 - t(1+2D)\left(13D^2+13D+4\right) - 3t^2(D+1)(3D+4)(3D+2)$
8	$D^3 - 4t(1+2D)\left(3D^2+3D+1\right) + 16t^2(D+1)^3$
9	$D^3 - 3t(1+2D)\left(3D^2+3D+1\right) - 27t^2(D+1)^3$
11	$D^3 - 2/5\,t(2D+1)\left(17D^2+17D+6\right) - \frac{56}{25}t^2(D+1)\left(11D^2+22D+12\right) - -\frac{126}{125}t^3(2D+3)(D+2)(D+1) - \frac{1504}{625}t^4(D+3)(D+2)(D+1)$

$d = 2$

1	$D^3 - 192t^2(3D+5)(3D+1)(D+1)$
2	$D^3 - 64t^2(2D+3)(2D+1)(D+1)$
3	$D^3 - 12t^2(3D+2)(3D+4)(D+1)$
4	$D^3 - 64t^2(D+1)^3$
5	$D^3 - 4t^2(D+1)\left(11D^2+22D+12\right) - 16t^4(D+3)(D+2)(D+1)$

multi parameter models

For more than one modulus there exist analogues of the period relations: Hosono 00'

$$D_2\,\omega_0 = \frac{1}{2}\frac{\sum_{a,b} K_{a,b}\; D_{1,a}\,\omega_0\;\; D_{1,b}\,\omega_0}{\omega_0} - 1\,, \tag{24}$$

Moreover, in examples the solutions are expressible in terms of products of ${}_2F_1$ functions, e.g for the fibre of $\mathbb{P}^4_{112812}[24]$:

$$\mathcal{L}_1 = \theta_x\,(\theta_x - 2\theta_z) - x\left(\theta_x + \frac{1}{6}\right)\left(\theta_x + \frac{5}{6}\right)\,, \tag{25}$$

$$\mathcal{L}_2 = \theta_z^2 - z\,(2\theta_z - \theta_x + 1)\,(2\theta_z - \theta_x)\;. \tag{26}$$

$$\omega_0(x) = \;{}_2F_1(\frac{1}{6},\frac{5}{6},1,x)\,,$$

$$\omega_1(x) = -2\pi\;\;{}_2F_1(\frac{1}{6},\frac{5}{6},1,1-x) + \log(432)\;{}_2F_1(\frac{1}{6},\frac{5}{6},1,x)\,.$$

The combinations $\omega_0(S)\,\omega_0(R)$, $\omega_1(S)\,\omega_0(R)$,$\omega_1(R)\,\omega_0(S)$ and $\omega_1(S)\,\omega_1(R)$ are all annihilated by $\mathcal{L}_1$ and $\mathcal{L}_2$, where S and R are algebraic functions given by the solutions to the system

$$R + S - 864RS - x = 0\,, \tag{27}$$

$$RS - (1 - 432R)(1 - 432S) - x^2 z = 0\,. \tag{28}$$

CY 3-folds

Method can not work for CY 3-folds. Obvious obstruction: Instantons break the period relations.

- Remnants of period relations visible at special points in moduli space (e.g. Legendre relation of periods at the conifold).
- PF operator of more complicated, i.e. non-hypergeometric, operators often match the structure as expected from twisted cubes!

E.g AESZ 16:

$$\theta^4 - 4x(2\theta+1)^2(5\theta^2+5\theta+2) + 2^8x^2(\theta+1)^2(2\theta+1)(2\theta+3)$$

For comparison the K3 operator

$$\theta^3 - 2x(2\theta+1)(A\theta^2+A\theta+B) + 4Cx^2(\theta+1)(2\theta+1)(2\theta+3)$$

The order x^2 or higher is necessary! Easy to show that no factorization of cubes can exist for the hypergeometric case.

Summary

- We can compute closed forms for periods of K3s and Fano 3-folds.
- Requires solutions of a 2-d operator as well as a generalized Clausen identity/ twisted symmetric square.
- Distance computations agree with results from heterotic duality.
- No known CY 3-fold examples in the literature, but so far only non-twisted symmetric cubes have been studied.
- Higher orders in the quadratic equations?

The End

Thank You