The tadpole conjecture at large complex-structure

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This talk is based on ::

- *The tadpole conjecture at large complex-structure*
  E. Plauschinn  
arXiv:2109.00029

- *Moduli Stabilization in Asymptotic Flux Compactifications*
  T. Grimm, D. van de Heisteeg, E. Plauschinn  
arXiv:2110.05511
Some motivation.
motivation — the landscape

Compactifications of string theory give rise to an abundance of lower-dimensional theories — the **string theory landscape**.

Famous estimates for its **size** are $10^{500}, 10^{930}, 10^{1500}, 10^{272000}$.

Bousso, Polchinski — 2000
Schellekens — 2016
Lerche, Lüst, Schellekens — 1987
Taylor, Wang — 2015

But one is often only interested in four-dimensional theories with no or few **massless scalar fields**.
motivation — moduli stabilization

In type IIB orientifold compactifications on Calabi-Yau three-folds, fluxes generate a mass-term for the complex-structure and axio-dilaton moduli.

An underlying assumption of the KKLT and Large-Volume scenarios is that all of these moduli can be stabilized in a suitable regime.

This assumption can fail.

Bena, Dudas, Graña, Lüst — 2018
Betzler, EP — 2019
Braun, Valandro — 2020
The **tadpole conjecture** states that for a large number of complex-structure moduli, not all of them can be stabilized by fluxes.

**Goal** for this talk ::

- Explain why it is difficult to prove this conjecture,
- show that in the large complex-structure limit generically the conjecture is satisfied, and
- discuss moduli stabilization in asymptotic regions.
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3. boundary behaviour
4. large complex-structure
5. summary
6. asymptotic hodge theory
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1. motivation

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Explanation of the tadpole conjecture.
D-branes and O-planes are charged under Ramond-Ramond gauge potentials and therefore contribute to Bianchi identities. The integrated expressions are the \textit{tadpole cancellation conditions}.

In \textbf{F-theory}, the D3-brane tadpole equation reads (with $\chi$ the Euler number of the four-fold)

\[ N_{D3} + \frac{N_{\text{flux}}}{2} = \frac{\chi}{24}. \]
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$$N_{D3} + \frac{N_{\text{flux}}}{2} = \frac{\chi}{24} \quad \xrightarrow{h^{3,1} \gg 1} \quad \frac{N_{\text{flux}}}{2} \leq \frac{h^{3,1}}{4}.$$
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The tadpole conjecture states that in the large $h^{3,1}$-limit, the flux number satisfies

$$\frac{N_{\text{flux}}}{2} > \frac{h^{3,1}}{3}.$$
Implications ::
- If true, the tadpole conjecture implies that the landscape of theories with no massless scalar fields is smaller than expected.

Comments ::
- The conjecture also applies to type IIB orientifold compactifications.
- In a number of examples the tadpole conjecture is satisfied, even for
  \[
  \frac{N_{\text{flux}}}{2} > 0.44 \times n_{\text{mod}}.
  \]
  Bena, Blåbäck, Graña, Lüst — 2020 & 2021
- A scenario which violates the tadpole conjecture has been proposed by Marchesano, Prieto & Wiesner.
  Marchesano, Prieto, Wiesner — 2021
  see also :: Lüst — 2021
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Estimating the behaviour of $N_{\text{flux}}$ when approaching a boundary.
Consider type IIB orientifold compactification on Calabi-Yau three-folds with NS-NS and R-R fluxes ::

- The global minimum of the scalar potential corresponds to the self-duality condition

\[ \star G_3 = i G_3, \quad G_3 = F_3 - \tau H_3, \quad \tau = c + i s. \]

- The Hodge-star matrix $M$ depends on the complex-structure moduli, and the above relation reads in matrix notation

\[ F_3 = [\eta M s + 1 c] H_3, \quad \eta = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}. \]

- The flux number is defined and satisfies

\[ N_{\text{flux}} = F_3^T \eta H_3 \geq 0. \]
The Hodge-star matrix $M$ is a real, symmetric, symplectic matrix. It can therefore be decomposed as

$$M = U^T \Sigma U,$$

$$\Sigma = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},$$

$$\lambda = \text{diag}(\text{eigenvalues } M),$$

$$U \in \text{Sp}(2h^{2,1} + 2, \mathbb{R}) \cap \text{O}(2h^{2,1} + 2, \mathbb{R}).$$

The self-duality condition and the flux number (at the minimum) can then be expressed as

$$\tilde{F}_3 = \begin{pmatrix} 1 & c \\ -\lambda s & 1 \\ -\lambda s & \lambda^{-1} s \end{pmatrix} \tilde{H}_3,$$

$$\tilde{H}_3 = U H_3,$$

$$\tilde{F}_3 = U F_3,$$

$$N_{\text{flux}} = s \sum_I \left[\lambda_I (\tilde{h}_I)^2 + \frac{1}{\lambda_I} (\tilde{h}_I)^2\right],$$

$$\tilde{H}_3 = \begin{pmatrix} \tilde{h}_I \\ \tilde{h}_I \end{pmatrix}.$$
The flux number (at the minimum) can be expressed in the following way

\[ N_{\text{flux}} = s \sum_I \left[ \lambda_I \left( \tilde{h}^I \right)^2 + \frac{1}{\lambda_I} \left( \tilde{h}_I \right)^2 \right]. \]

Boundary behaviour of the flux number (for \( \tilde{h}^I \neq 0 \)) ::

- At a boundary in complex-structure moduli space, the matrix \( M \) is expected to degenerate. This implies

  \[ \lambda_I \rightarrow \infty \quad \longrightarrow \quad N_{\text{flux}} \rightarrow \infty. \]

- For the boundary in axio-dilaton moduli space one finds

  \[ s \rightarrow \infty \quad \longrightarrow \quad N_{\text{flux}} \rightarrow \infty. \]
boundary behaviour — growth of flux number

For simple examples and generic configurations, the flux number grows significantly near a boundary ::

\[ N_{\text{flux}} \]

\[ h^{2,1}=2 \text{ (LCS)} \]

\[ h^{2,1}=3 \text{ (LCS)} \]
Summary ::
- Near a boundary in moduli space, the flux number $N_{\text{flux}}$ generically diverges.
  (The behaviour in non-generic situations is not derived here.)

Implications ::
- The smallest values for $N_{\text{flux}}$ will be found in the interior of moduli space.
- It can thus be challenging to prove the tadpole conjecture or to find computationally-controlled counter-examples.

Comment ::
- The same behaviour is found using asymptotic Hodge theory.

Grimm — 2020
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The tadpole conjecture in the large complex-structure regime.
Consider the large complex-structure regime, for which the moduli space is described by the prepotential

$$\mathcal{F} = -\frac{1}{3!} \kappa_{ijk} \frac{X^i X^j X^k}{X^0},$$

where

$$z^i = \frac{X^i}{X^0} = u^i + i v^i, \quad i, j, k = 1, \ldots, h^{2,1}.$$

Using the self-duality condition, the flux number (at the minimum) can be expressed as

$$N_{\text{flux}} = s \left[ \frac{\kappa}{6} (h^0)^2 + \frac{2}{3} \kappa (h^i - u^i h^0) G_{ij} (h^j - u^j h^0) \right]$$

$$+ \frac{1}{s} \left[ \frac{\kappa}{6} (f^0)^2 + \frac{2}{3} \kappa (f^i - u^i f^0) G_{ij} (f^j - u^j f^0) \right],$$

with $G_{ij} = $ Kähler metric,

$$\kappa = \kappa_{ijk} v^i v^j v^K.$$

Using statistical results by Demirtas et al. for $\kappa_{ijk}$, the scaling of the flux number can be estimated as

$$N_{\text{flux}} \sim \begin{cases} 
(h^{2,1})^{1/2} \|v\| & \text{for } h^0 = f^0 = 0, \\
(h^{2,1})^{-1/2} \|v\|^3 & \text{else}.
\end{cases}$$
At large complex-structure, the complex-structure moduli space is **mirror-dual** to the Kähler moduli space.

Statistical data on the Kähler cone has been determined by Demirtas et al. for the Kreuzer-Skarke list.

Demirtas, Long, McAllister, Stillman — 2018
The **Kähler cone** contains all Kähler forms such that curves, divisors and the CY\textsubscript{3} itself have positive volume.
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The **stretched Kähler cone** contains all Kähler forms such that all volumes are larger than a constant $c$. 

![Wide Kähler cone](image1)

![Narrow Kähler cone](image2)
For the Kreuzer-Skarke list, one finds that the Kähler cone becomes narrower for larger dimensions $h^{1,1}$.

The **narrowness** can be encoded in the distance between the ordinary and stretched Kähler cones

$$d_{\min}[c] \approx 10^{-1.4} (h^{1,1})^{2.5} c.$$
For the Kreuzer-Skarke list, one finds that the Kähler cone becomes narrower for larger dimensions $h^{1,1}$.

The \textbf{narrowness} can be encoded in the distance between the ordinary and stretched Kähler cones

$$d_{\text{min}}[c] \simeq 10^{-1.4} (h^{1,1})^{2.5} c .$$

A \textbf{lower bound} can be estimated from the data as

$$d_{\text{min}}[c] \gtrsim 10^{-2.6} (h^{1,1})^{2.7} c .$$
large complex-structure — scaling of the flux number

The result from above can now be combined in the following way (here $\gamma = 1, 3$):

\[
N_{\text{flux}} \gtrsim (\hbar^{2,1})^{1-\gamma/2} \|v\|^{\gamma}
\]

\[
\|v\| \geq d_{\text{min}} \gtrsim 10^{-2.6} (\hbar^{2,1})^{2.7} c
\]

\[
N_{\text{flux}}[c] \gtrsim (\hbar^{2,1})^{1-\gamma/2} \left[ 10^{-2.6} (\hbar^{2,1})^{2.7} c \right]^{\gamma}
\]

The constant $c$ parametrizes minimal volumes of cycles on the mirror-dual side. For a well-controlled large complex-structure limit, $c$ should not be much smaller than one.
Even for small values of $c$, the flux number in the large complex-structure regime (orange curves) quickly exceeds the linear bound of the tadpole conjecture (blue curves).
Summary ::

- For generic flux choices and in the large complex-structure limit, the tadpole conjecture is satisfied.

- A smaller flux number, violating the conjecture, may be found outside of this regime or for non-generic flux choices.
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Summary ::

- The **tadpole conjecture** states that for large $h^{2,1}$, not all moduli can be stabilized.
- When approaching a **boundary** in moduli space, the flux number diverges.
- In the **large complex-structure** regime, the tadpole conjecture is satisfied for generic flux choices.

Implications ::

- If true, the tadpole conjecture implies that the **landscape** of theories with no or few massless scalar fields is smaller than expected.
- A common assumption in the **KKLT** and **Large Volume** scenarios may not be satisfied.