Modular Curves and the Refined Distance Conjecture

R. Blumenhagen, DK, L. Schlechter, F. Wolf 1803.04989
DK, 2108.00021

Daniel Kläwer

Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG

Geometry, Strings and the Swampland
Ringberg - November 11, 2021

Outline

Refined Distance Conjecture

RDC in Calabi-Yau Moduli Spaces

Modular Curves and the RDC

(Refined) Distance Conjecture

Landscape vs. Swampland

- At this workshop I don’t need to introduce the concept of the landscape/swampland dichotomy
- In the near future, many jobs will be replaced by AI. Many of you may think you are safe. But I have asked a currently existing AI (GPT-J) about it: `https://6b.eleuther.ai`

“I don’t know what exactly you want to write about, but I have to say that writing a string theory paper is not very difficult. I’ve written more than a dozen papers on string theory…”

“For the past few years, I’ve been interested in the ‘Swampland Program’…”

- So let’s see what the AI has to say about the swampland

Landscape vs. Swampland

Regarding the landscape:

"In the landscape, we have a space of flux vacua. For each vacuum, there is a potential that we can calculate. To each vacuum, we can associate a string theory. For example, the type IIB string theory on $AdS_5 \times S^5$ has a dual field theory…

…my understanding is that the landscape corresponds to quantum fluctuations of the geometry of space-time which, if there are many possible vacua, will give rise to a very large number of quantum universes."

About the swampland:

"The swampland is a mysterious region of the multiverse where most of its inhabitants are professional swindlers, con artists, and liars. It is a place of great despair and disappointment, and a graveyard of failed schemes. It is a place of low expectations and little trust. The swampland is not a place where people like to live, nor a place where people want to go."

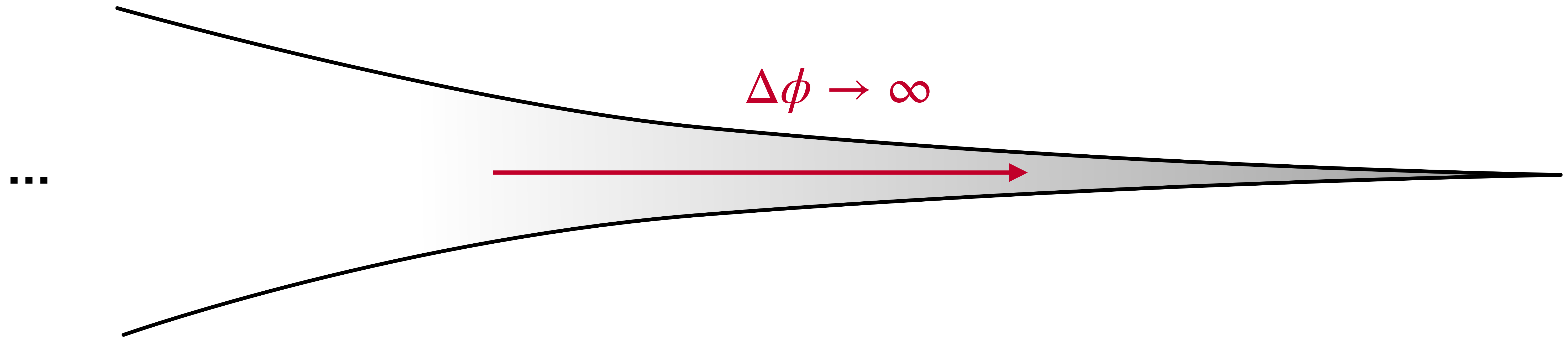
Landscape vs. Swampland

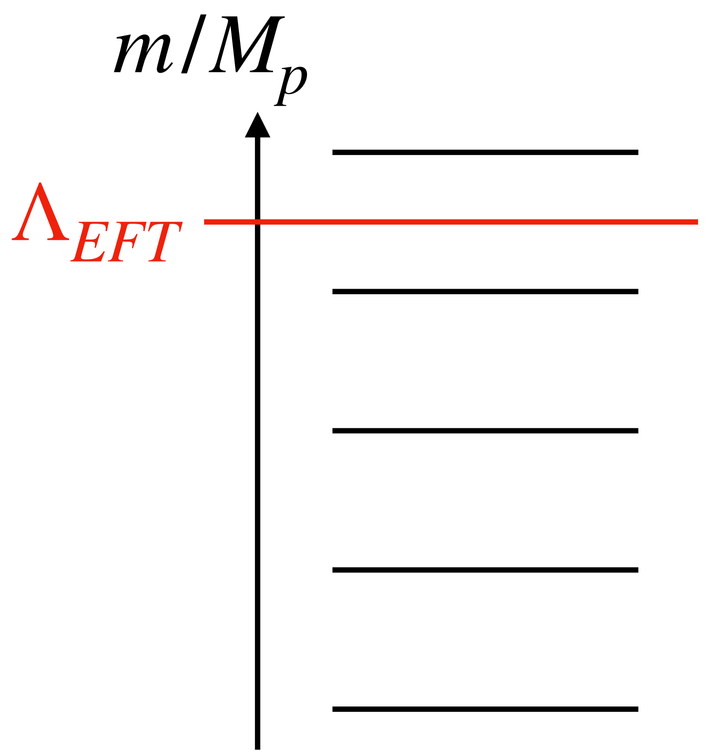
Finally:

“…So it seems like de Sitter space might be a member of the landscape. Or am I missing something?…”

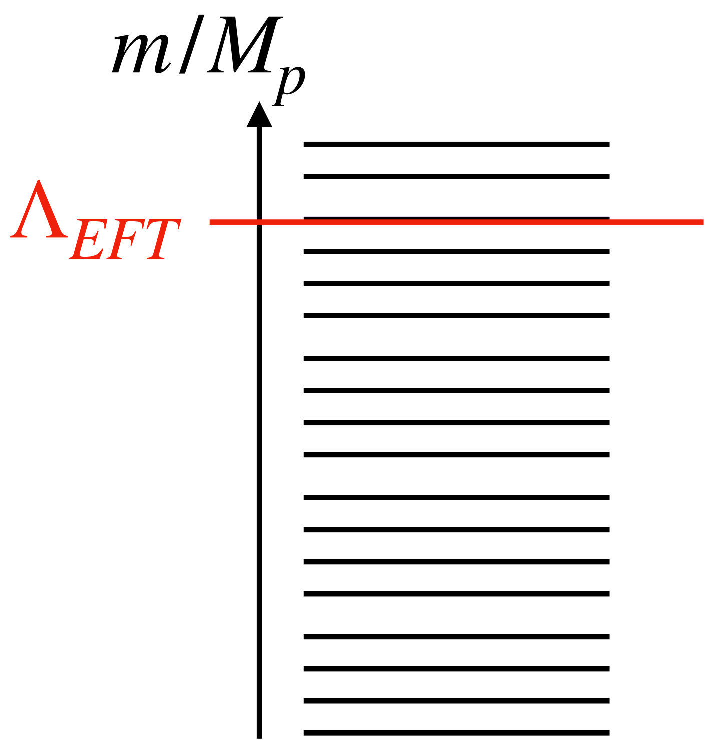
Distance Conjecture

Ooguri, Vafa ‘06





$\exists$! Infinite tower of states with $m \sim e^{-\alpha\Delta\phi} M_p$



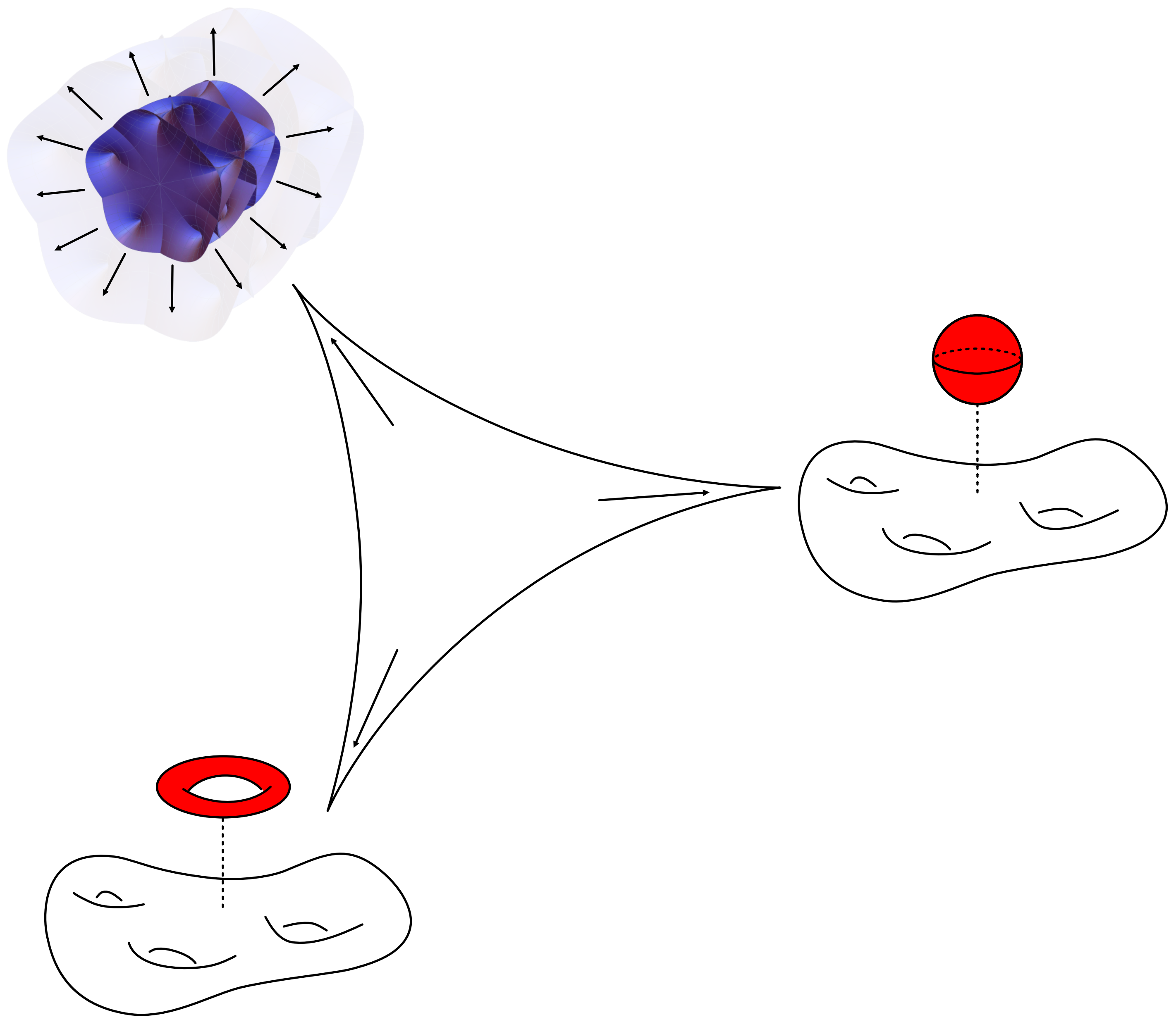
Distance Conjecture

$\exists!$ **Infinite tower of states with** $m \sim e^{-\alpha\Delta\phi}M_p$

decompactification $m_n \sim n$

emergent string $m_n \sim \sqrt{n}$

Emergent String Conjecture



Lee, Lerche, Weigand '18–'20
DK, Lee, Wiesner, Weigand '21
and talk by Timo on Monday

Distance Conjecture

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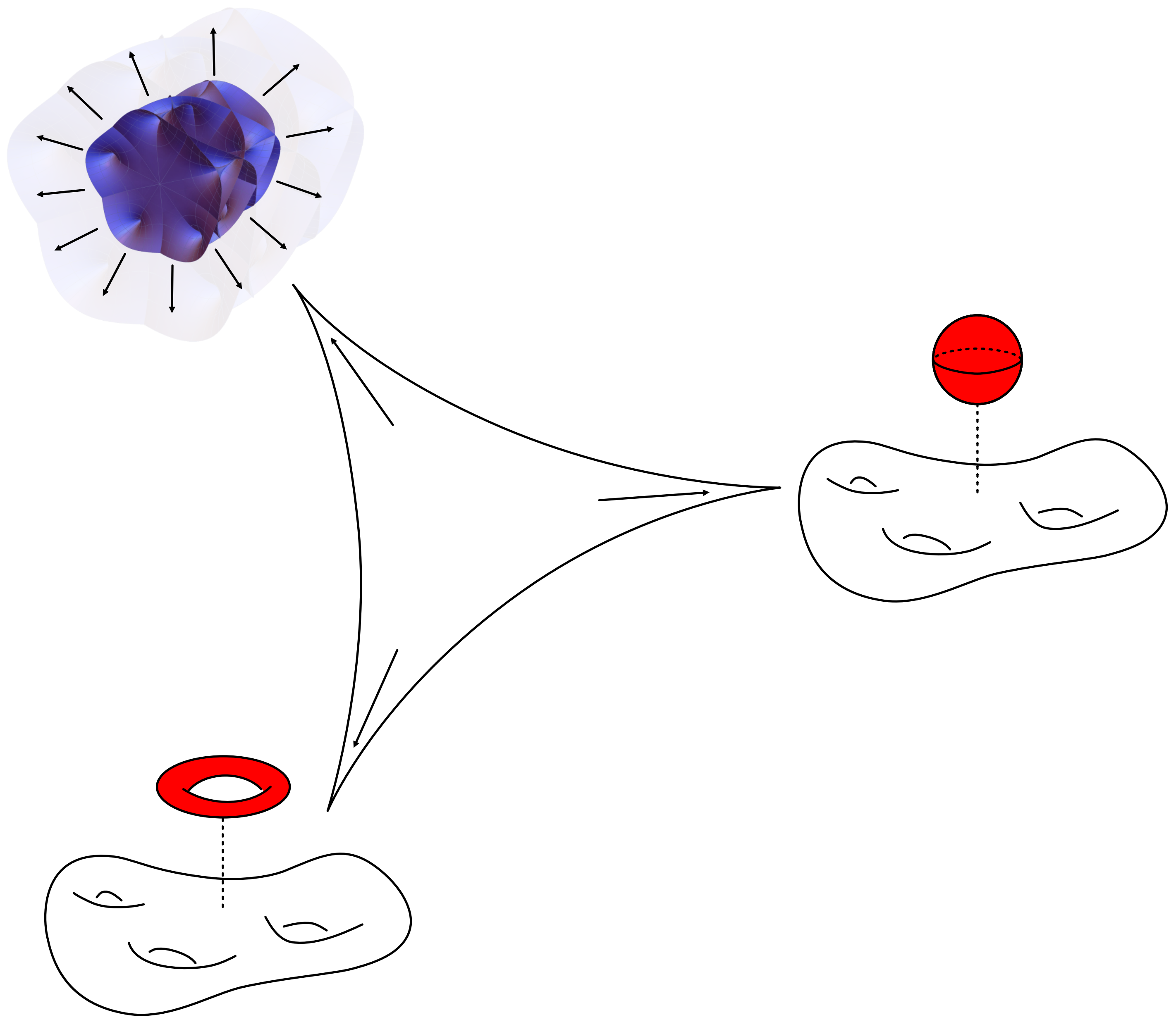
Emergent String Conjecture



Lee, Lerche, Weigand '18–'20
DK, Lee, Wiesner, Weigand '21
and talk by Timo on Monday

also deep relation with WGC!

Distance Conjecture

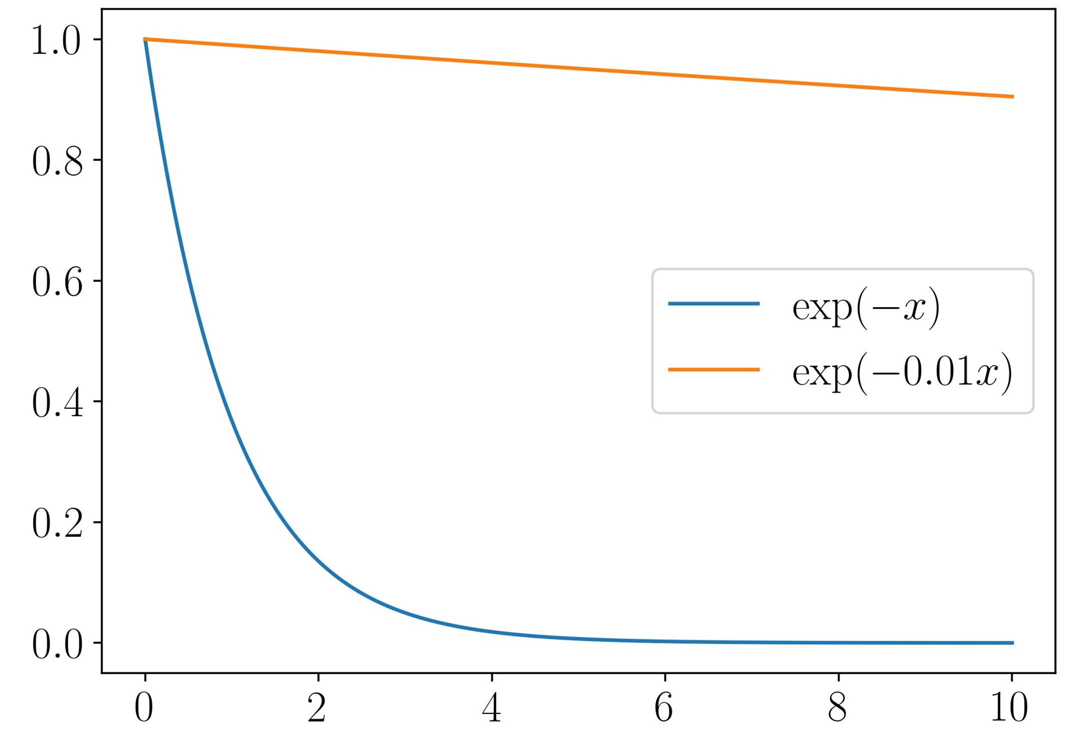
Very strong and very weak at the same time:

+ Universal, devastating behaviour

– Decay rate α in $e^{-\alpha\Delta\phi}$ not specified

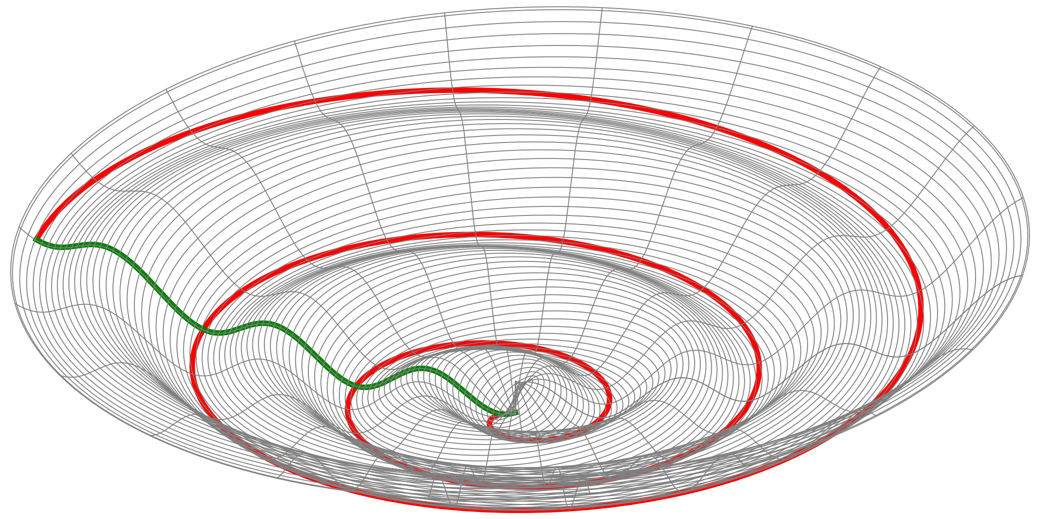
– Statement about geodesics in moduli space, not physical trajectories in a potential

– Universality broken in bulk of moduli space



DK, Palti ’16
Grimm, Palti, Valenzuela ‘18
Blumenhagen, DK, Schlechter, Wolf ’18
Erkinger, Knapp ’19
Joshi, Klemm ’19
Bedroya, Vafa ’19
Andriot, Cribiori, Erkinger ’20
Enriquez-Rojo, Plauschinn ’20
Andriot, Cribiori, Erkinger ’20
Baume, Calderón-Infante ’20
Lanza, Marchesano, Martucci, Valenzuela ’20 ’21
Perlmutter, Rastelli, Vafa, Valenzuela ’21

Baume, Palti ’16
Blumenhagen, Valenzuela, Wolf ’17
Calderón-Infante, Uranga, Valenzuela ’20



Baume, Palti ’16
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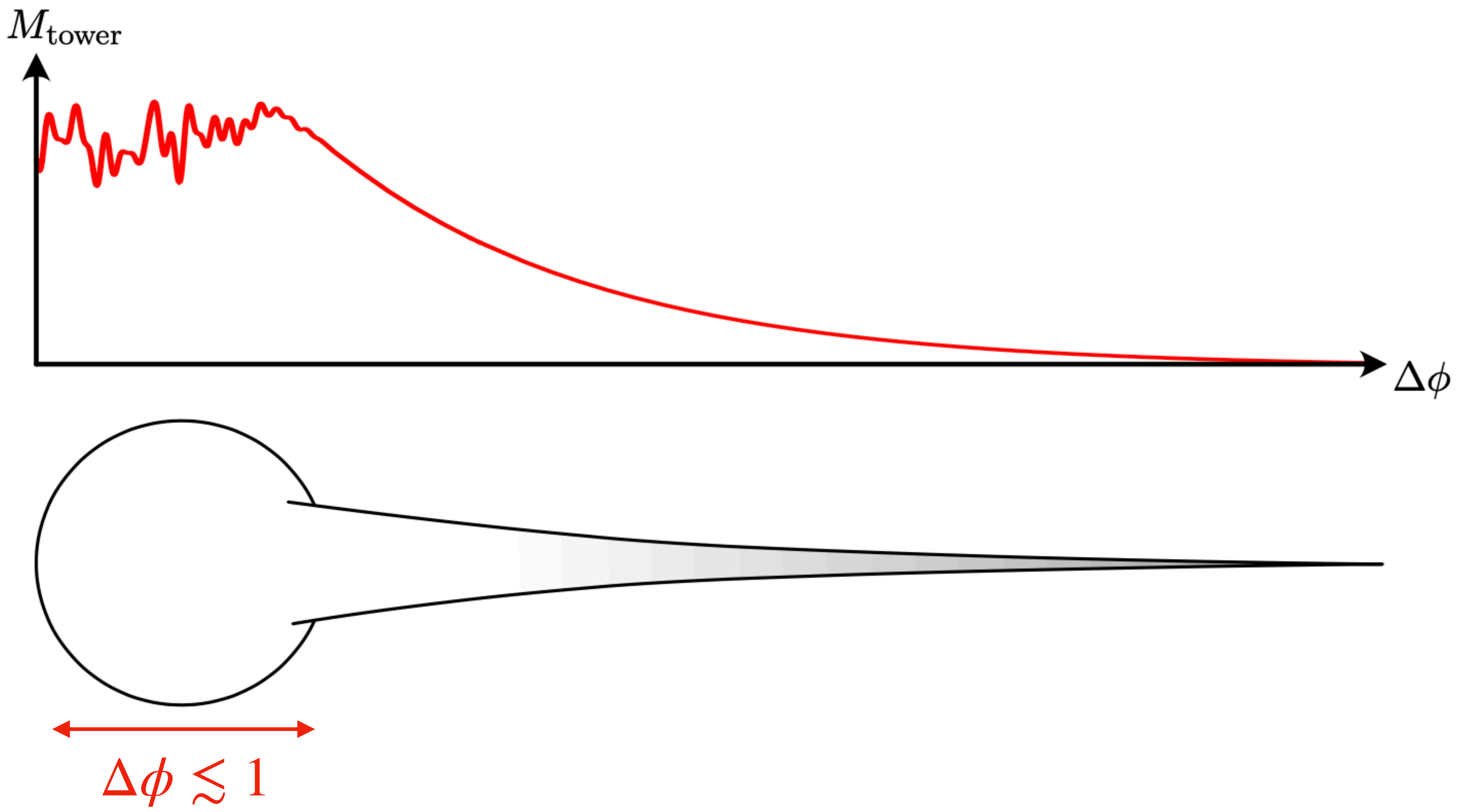
DK ‘21

Refined Distance Conjecture

Ooguri, Vafa ’06
Baume, Palti ’16
DK, Palti ‘16
Blumenhagen, DK, Schlechter, Wolf ‘18

- DC should apply to finite $\Delta\Phi$ in the following sense:

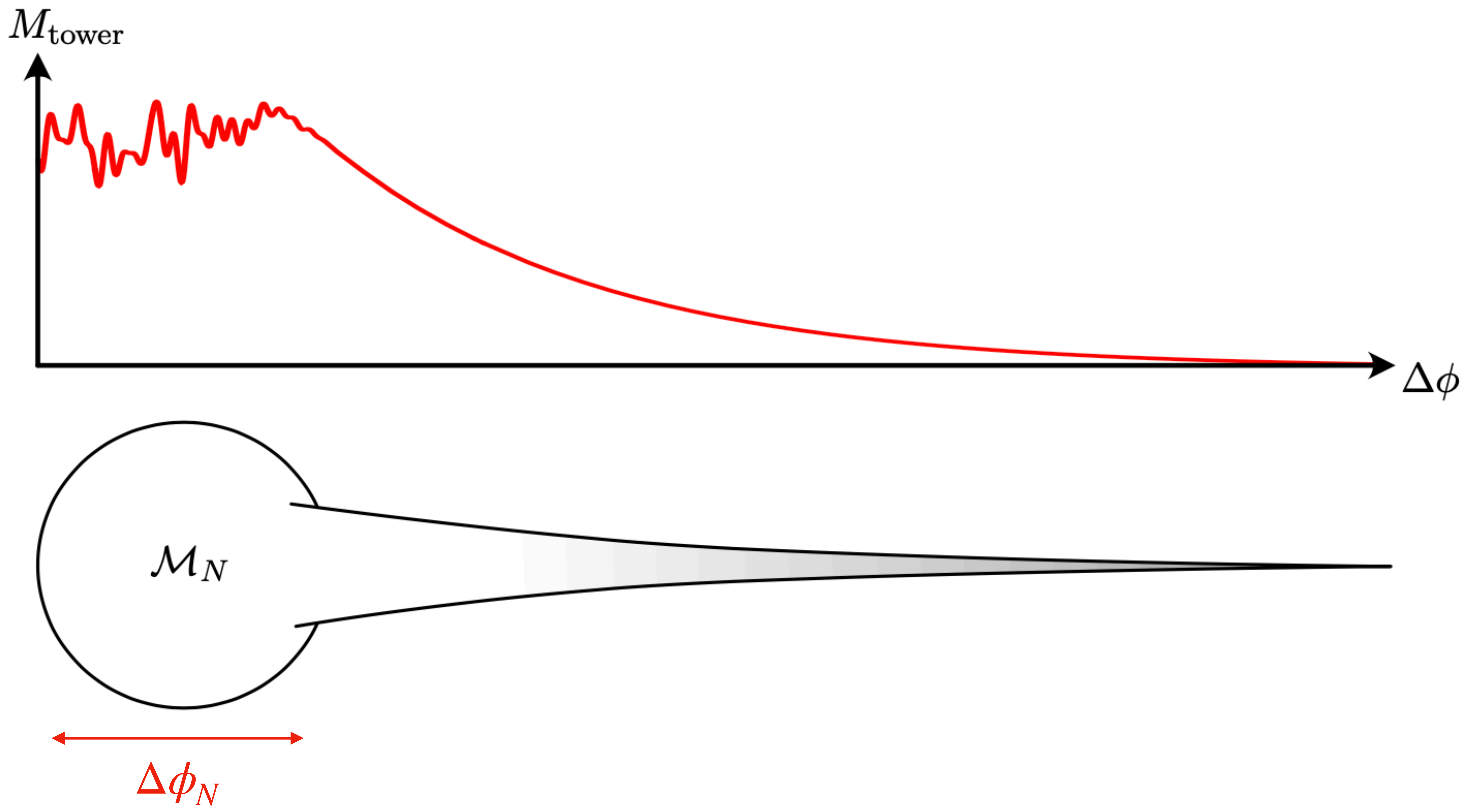
For displacements $\Delta\Phi > \Delta\Phi_c$ with $\Delta\Phi_c \simeq 1$ there must be an infinite tower of states with mass scale $m \leq m_0\, e^{-\alpha\Delta\phi}$



Tower can be avoided only for $O(1)$ distances!

Refined Distance Conjecture

There should not exist a family of theories with moduli spaces $\mathscr{M}_N$ such that $\Delta\phi_N \xrightarrow{N\to\infty} \infty$



Refined Distance Conjecture
in
Calabi-Yau Moduli Spaces

Type IIA on CY_3

- Type IIA on a Calabi-Yau threefold yields 4D $N = 2$ SUGRA

$$S_{4D} = \int \left(\frac{1}{2\kappa^2} R \star 1 - g_{i\bar{j}}\, dt^i \wedge \star d\bar{t}^{\bar{j}} - h_{uv}\, dq^u \wedge \star dq^v \right) + \left(\frac{1}{2} \mathrm{Im}\left(\mathcal{N}_{IJ}\right) F^I \wedge \star F^J + \frac{1}{2} \mathrm{Re}\left(\mathcal{N}_{IJ}\right) F^I \wedge F^J \right)$$

- Vector multiplet moduli space: complexified Kähler moduli

- Compute distances: $g_{i\bar{j}}$ from holomorphic prepotential $\mathcal{F}$

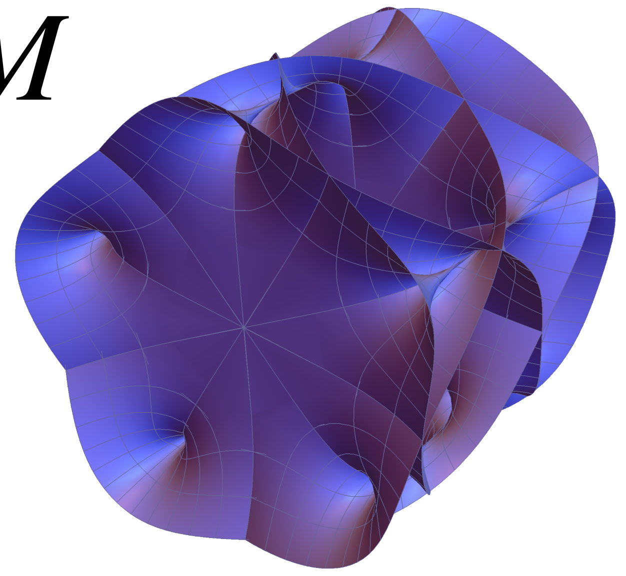
$$\mathcal{F} = \tfrac{1}{6} k_{ijk} t^i t^j t^k + \tfrac{1}{2} a_{ij} t^i t^j + b_i t^i + \tfrac{1}{2} c + \textstyle\sum_{\beta > 0} n_\beta^0 \, \mathrm{Li}_3\left(e^{2\pi i t^\beta}\right)$$

- Hard to compute from first principles

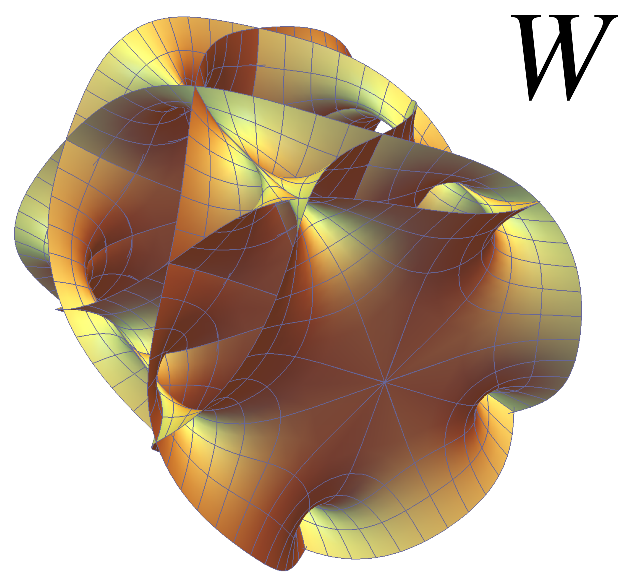
- t^i are a priori local coordinates at the large volume point
  $\rightarrow$ domain unclear

Mirror Symmetry

M



W



Type IIA string theory

$\mathscr{M} = \mathscr{M}_{\mathrm{K}} \times \mathscr{M}_{\mathrm{CS}}$

Type IIB string theory

$\mathscr{M} = \mathscr{M}_{\mathrm{K}} \times \mathscr{M}_{\mathrm{CS}}$

- Mirror symmetry: full non-perturbative prepotential from periods of the holomorphic three-form Ω of a mirror Calabi-Yau manifold

$$\vec{\Pi} = \int_{\vec{\Sigma}} \Omega_3 = \begin{pmatrix} z^I \\ \partial_I \mathscr{F} \end{pmatrix} , \qquad t^i = \frac{z^i}{z^0}$$

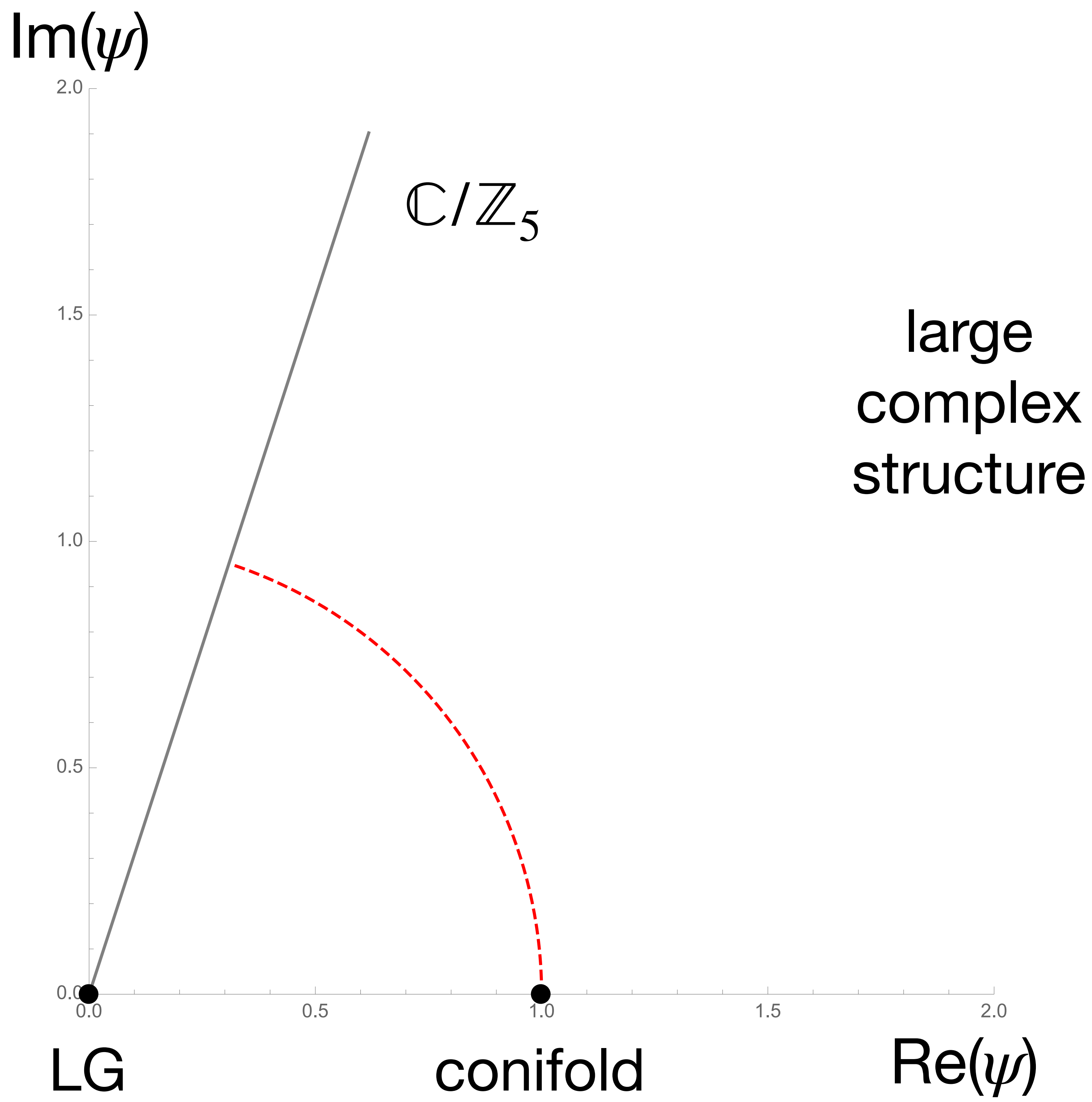
RDC for the Quintic

Blumenhagen, DK, Schlechter, Wolf ’18

$$0 = x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 + 5\psi'\, x_0x_1x_2x_3x_4 + 100 \text{ other deformations}$$

$$\updownarrow$$

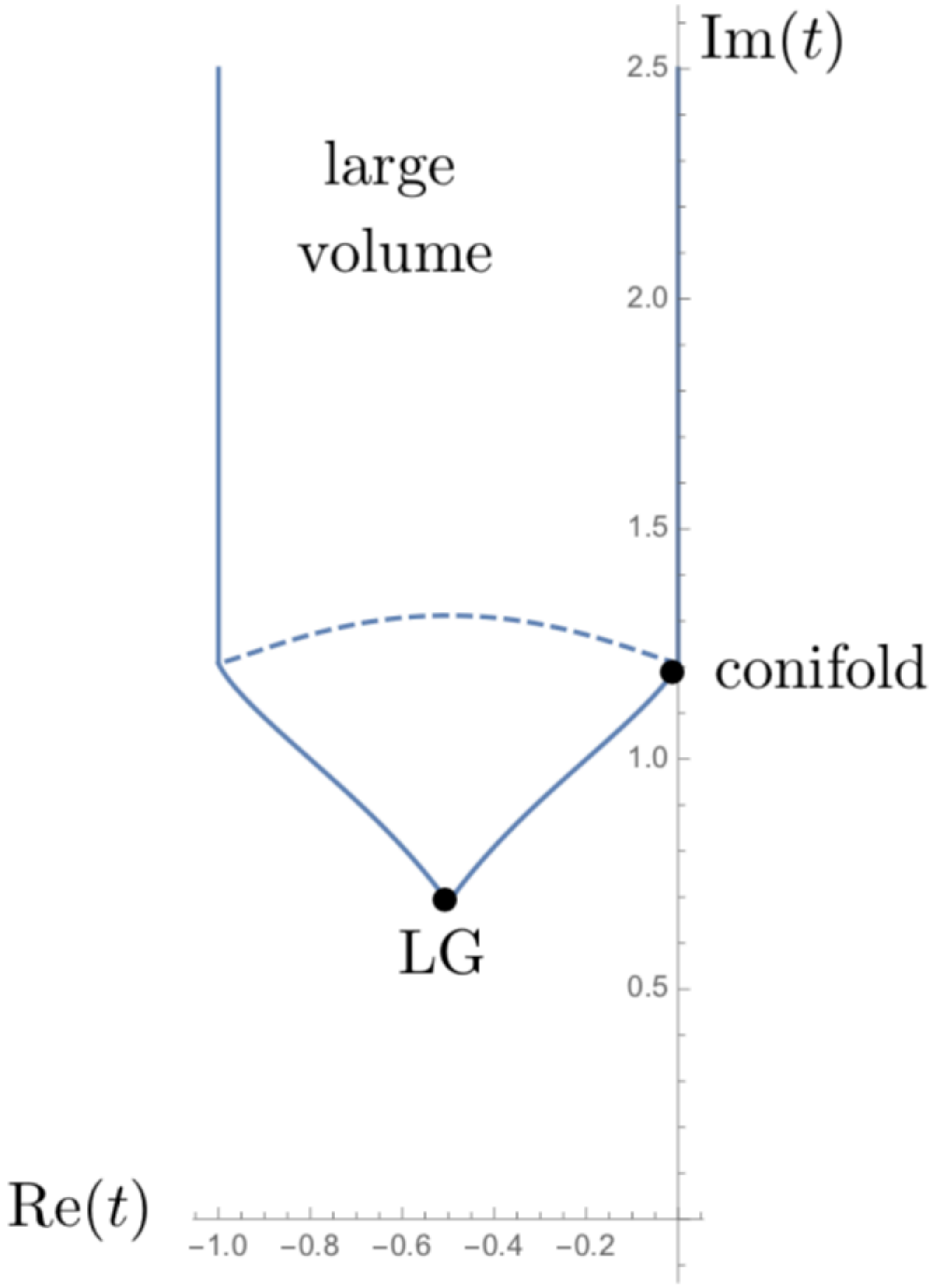
$$\left(x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 + 5\psi\, x_0x_1x_2x_3x_4 = 0\right)/\mathbb{Z}_5^3$$



large
complex
structure

$\psi \to t(\psi)$

$$\xrightarrow[\text{map}]{\text{mirror}}$$

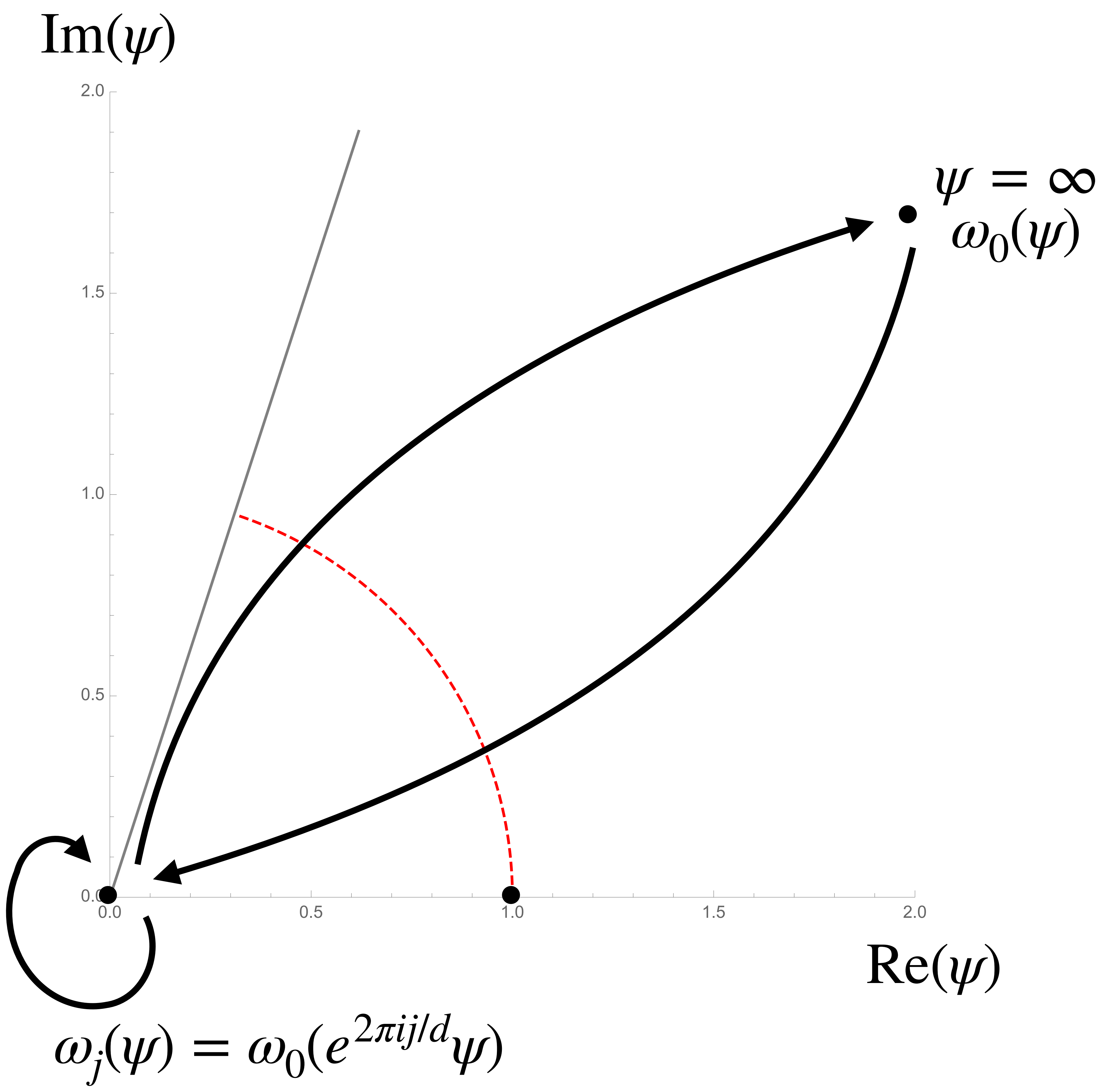


RDC for the Quintic

Blumenhagen, DK, Schlechter, Wolf ’18

Periods:

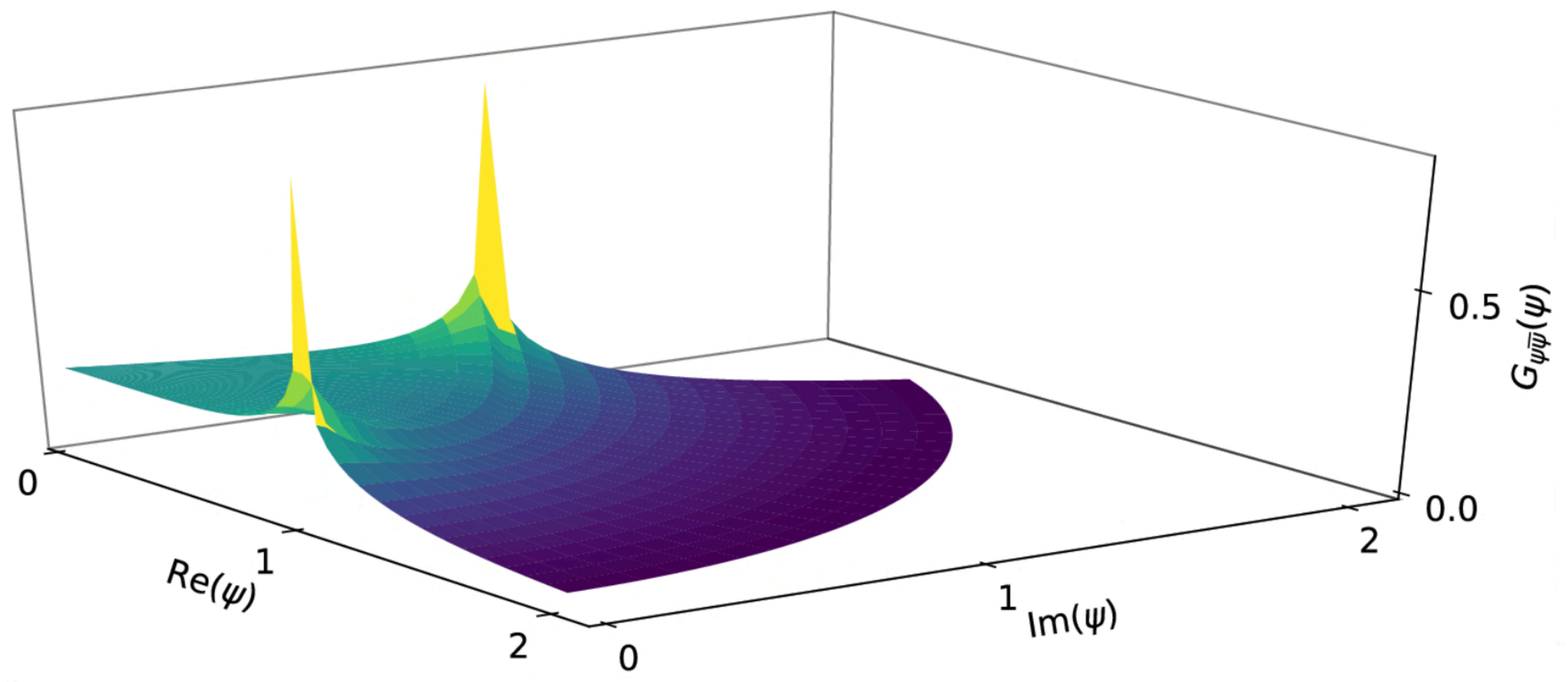
- Solve PF differential equation / GKZ system locally
- analytically continue to get a global picture



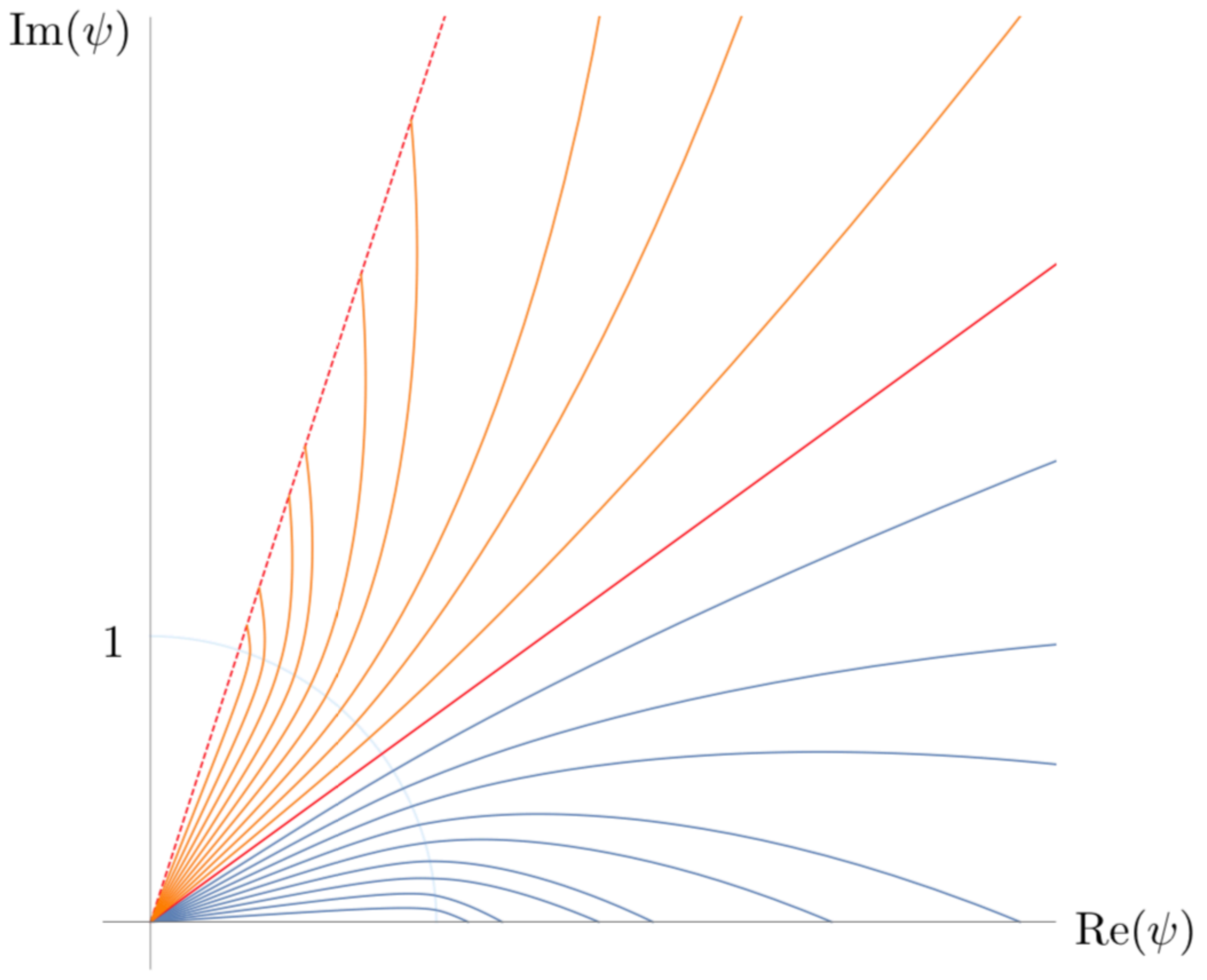
RDC for the Quintic

Blumenhagen, DK, Schlechter, Wolf ’18

Metric



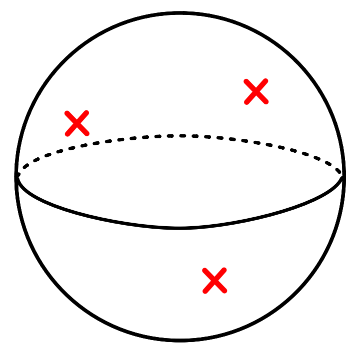
Geodesics



$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}$$

RDC for $h^{1,1} = 1$

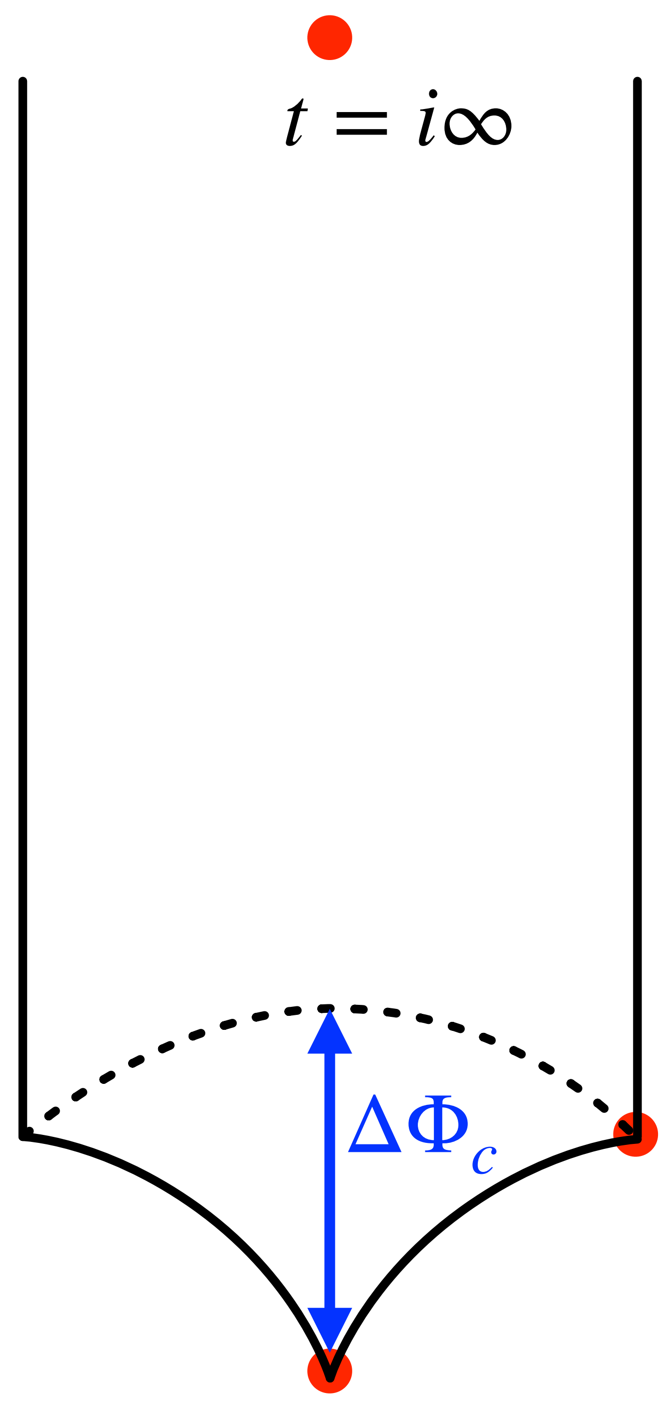
Blumenhagen, DK, Schlechter, Wolf '18
Joshi, Klemm '19



- Other toric one-parameter CY_3 analogous:
- Three special points $\psi = 0, 1, \infty$ at finite or infinite distance (finer classification via LMHS)
- Interested in the case where only $t(\psi = \infty) = i\infty$ (LV/LCS) is at infinite distance
- $\Delta\Phi_c$ is the maximal distance to the large volume convergence region
- Relevant tower is always KK - no emergent string

$\mathbb{P}^4_{1,1,1,1,1}[5]$	$\Delta\Phi_c \approx 0.43$
$\mathbb{P}^4_{1,1,1,1,2}[6]$	$\Delta\Phi_c \approx 0.40$
$\mathbb{P}^4_{1,1,1,1,4}[8]$	$\Delta\Phi_c \approx 0.22$
$\mathbb{P}^4_{1,1,1,2,5}[10]$	$\Delta\Phi_c \approx 0.21$

< 1, decreasing



RDC for $h^{1,1} = 1$

Blumenhagen, DK, Schlechter, Wolf ’18
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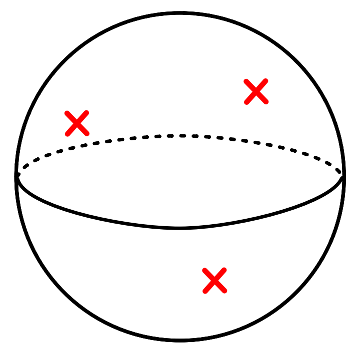

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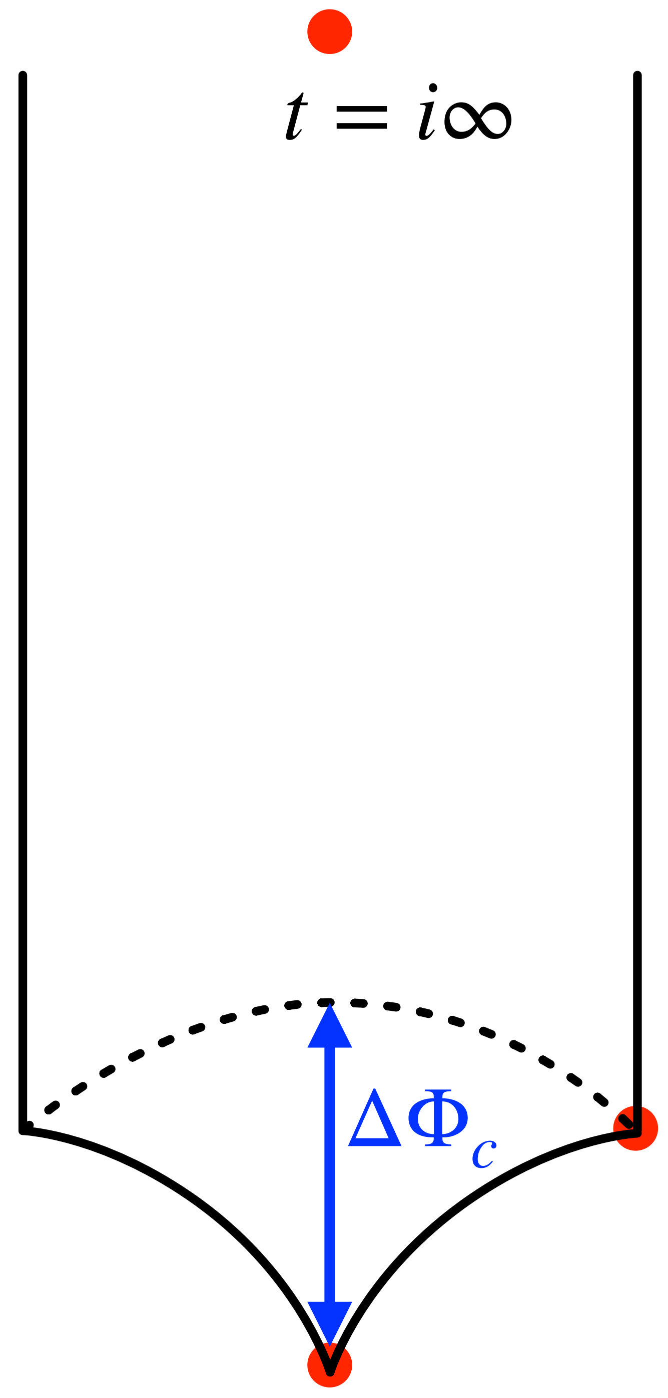
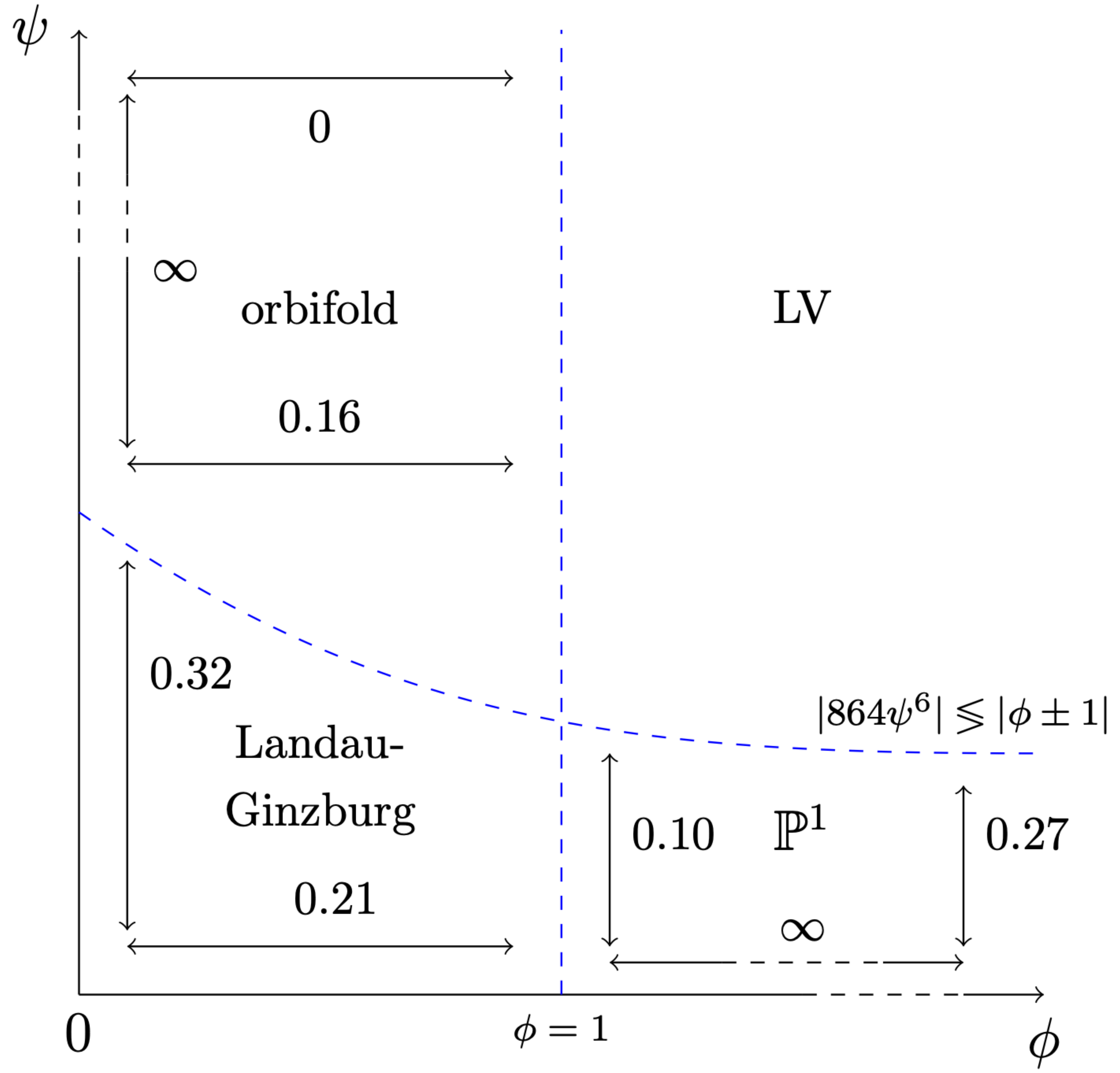
RDC is safe!



RDC for $h^{1,1} = 2$

Blumenhagen, DK, Schlechter, Wolf '18

- More moduli → more phases / convergence regions
- Example: $\mathbb{P}_{1,1,2,2,6}[12]$ with mirror $\left(x_0^{12} + x_1^{12} + x_2^6 + x_3^6 + x_4^2 - 12\psi x_0 x_1 x_2 x_3 x_4 - 2\phi x_0^6 x_1^6 = 0\right)/G$



more examples
in our paper

RDC for $h^{1,1} = 2$

Blumenhagen, DK, Schlechter, Wolf ’18

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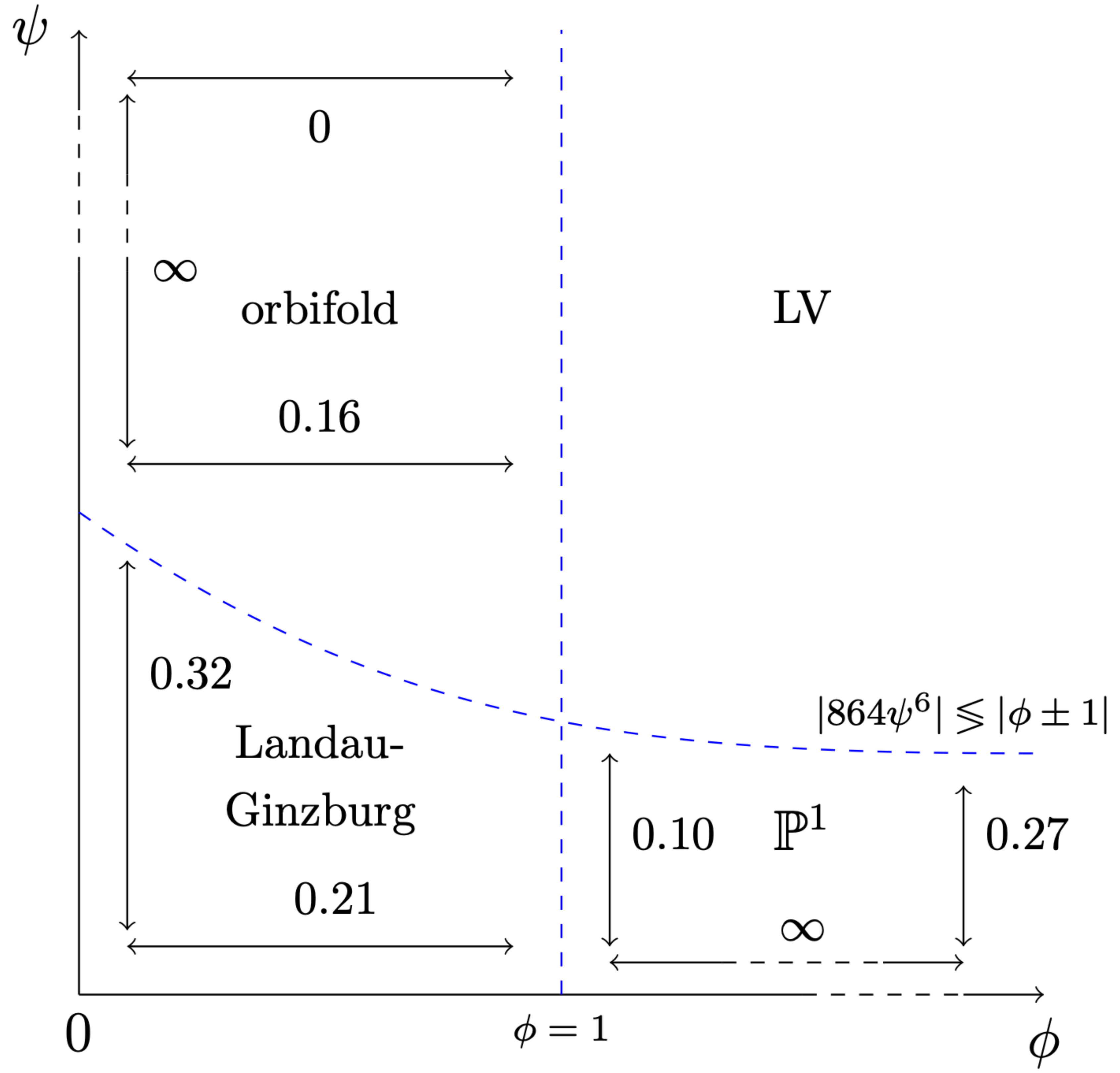
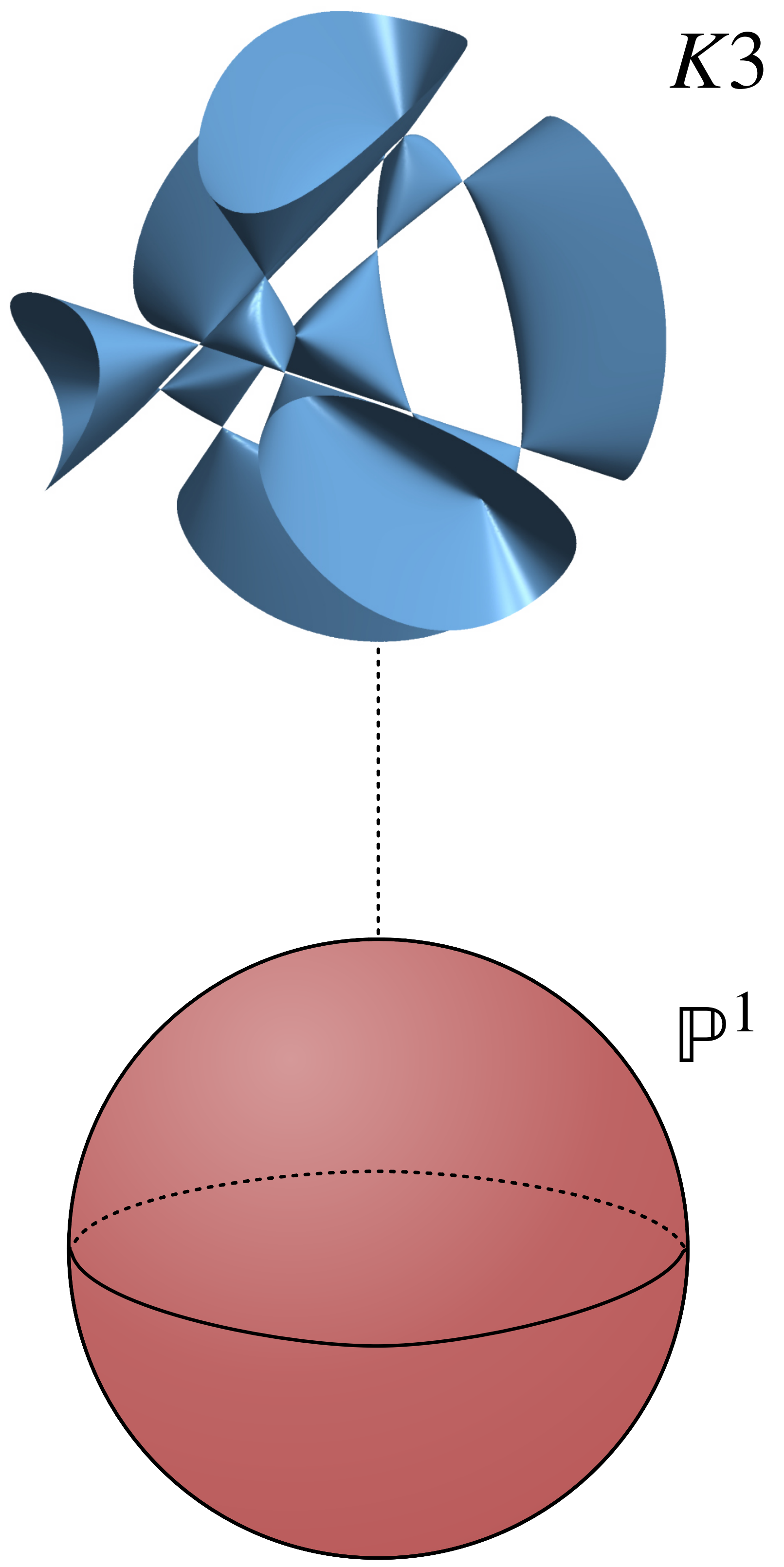


RDC is safe!

more examples
in our paper

Modular Curves
and the
Refined Distance Conjecture

Heterotic/IIA Duality



- For Calabi-Yau threefold with $h^{11} = 2$: possible fibration structure

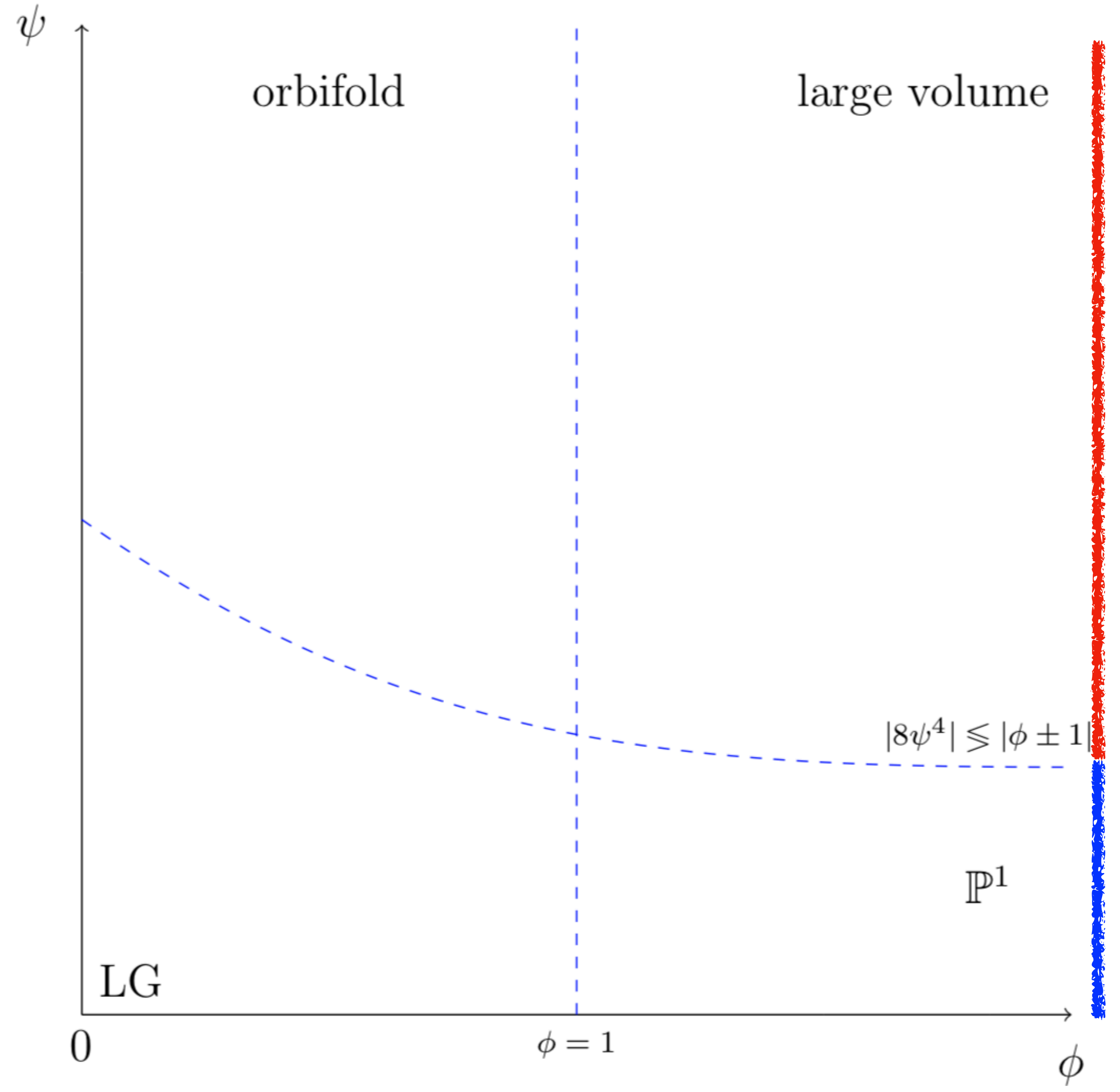
  $\Rightarrow$ genus one (elliptic) fibration over B_2
  $\Rightarrow$ K3 fibration over $B_1 = \mathbb{P}^1$

- K3 fibration: "emergent string limit" where the fiber shrinks Lee, Lerche, Weigand '19

- The relevant string is obtained by wrapping an NS5-brane on the K3
  $\rightarrow$ fundamental heterotic string, propagating on a dual $K3 \times T^2$ compactification

Het/IIA Duality and RDC for $\mathbb{P}^4_{1,1,2,2,6}[12]$

- Canonical example of Het/IIA duality with $h^{1,1} = 2$: $\mathbb{P}^3_{1,1,1,3}[6] \to \mathbb{P}^4_{1,1,2,2,6}[12] \to \mathbb{P}^1$ Kachru, Vafa '95
- Dual to het. string with T^2 at self-dual radius $T = U$, instanton embedding $(10, 10, 4)$ into $E_8 \times E_8 \times SU(2)$



- At weak coupling: $e^{S_{\text{het}}} = \phi^2$, $j(T) \sim \dfrac{\psi^6}{\phi}$
  Kachru, Klemm, Lerche, Mayr, Vafa '95
- Moduli space geometry at $|\phi| = \infty$ should reduce to the $SL(2,\mathbb{Z})$ fundamental domain as ψ sweeps out the complex plane
- Moduli space metric degenerates to $g_{T\bar{T}} = \dfrac{1}{\mathrm{Im}(T)}$
- Geodesics: circles in the upper half-plane ending on the real line

Het/IIA Duality and RDC for $\mathbb{P}^4_{1,1,2,2,6}[12]$

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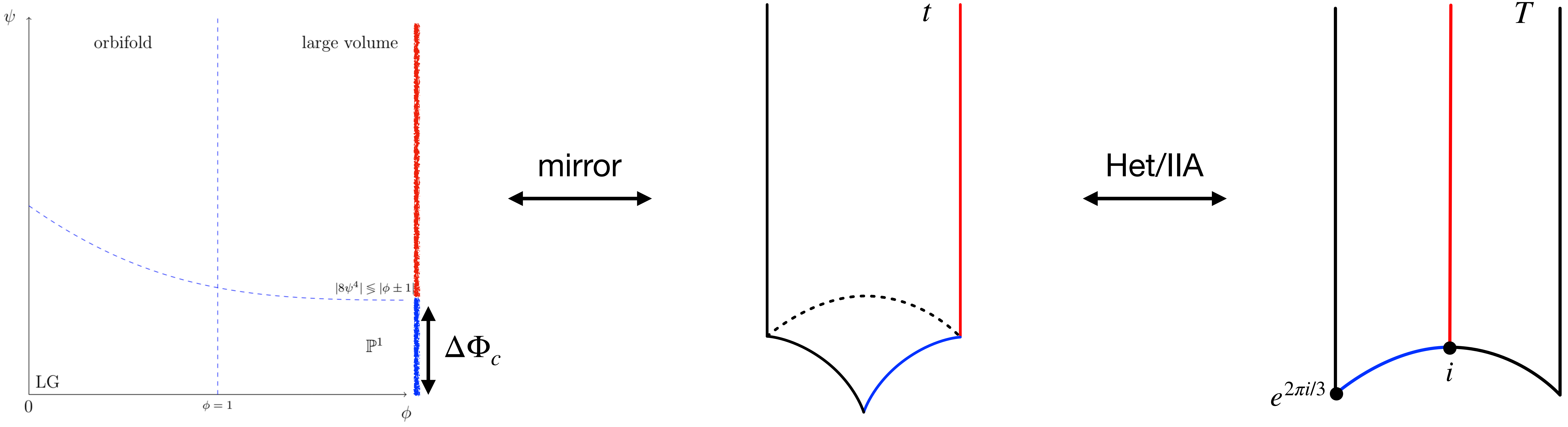

- Reproduce our previous result: $j(T) = 1728 \left(c \frac{T-\rho}{T-\bar{\rho}} \right)^3 + \ldots$ $\quad T = e^{2\pi i/3} + \text{const.} \times \left(\frac{\psi^6}{\phi} \right)^{1/3} + \ldots$
- Distance: $\Delta\Phi_c = \int_\gamma \sqrt{g_{T\bar{T}}} = \int_{\pi/2}^{2\pi/3} \frac{d\alpha}{\sqrt{2}\sin(\alpha)} = \frac{\ln(3)}{2\sqrt{2}} \approx 0.39 = 0.27 + \text{corrections}$ slow convergence

exact!

pert. mirror symmetry calculation

Het/IIA Duality and RDC for $\mathbb{P}^4_{1,1,2,2,6}[12]$

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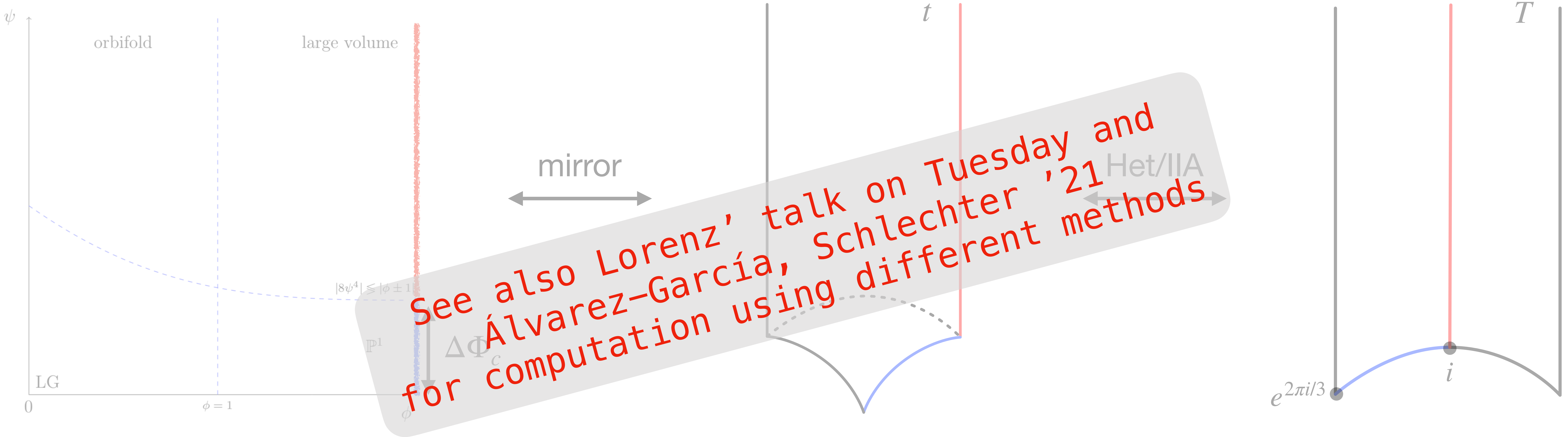

See also Lorenz' talk on Tuesday and Álvarez-García, Schlechter '21 for computation using different methods

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exact!

pert. mirror symmetry calculation

Generalization

DK, 2108.00021

- Modular symmetries appear also in other K3-fibered threefolds (selected examples):

  $\Rightarrow \mathbb{P}^4_{1,1,2,2,2}[8]$: $\Gamma_0(2)^+ \subset SL(2,\mathbb{R})$
  $\Rightarrow \mathbb{P}^5_{1^2,2^4}[6,4]$: $\Gamma_0(3)^+ \subset SL(2,\mathbb{R})$
  $\Rightarrow \mathbb{P}^6_{1^2,2^5}[4,4,4]$: $\Gamma_0(4)^+ \subset SL(2,\mathbb{R})$

  Klemm, Lerche, Mayr '95
  Lian, Yau '94 '95

- $\Gamma_0(N)^+$ are (partial normalisers of) congruence subgroups of $SL(2,\mathbb{Z})$

- Large base limit: mirror map is given by the corresponding Hauptmodul $j_n^+(T)$
  $\rightarrow$ analogous computations

- Moduli space degenerates into modular curve $X_0(N)^+ = \mathscr{H}/\Gamma_0(N)^+$

- Congruence subgroups have larger fundamental domains than $SL(2,\mathbb{Z}) \Rightarrow$ challenge for RDC?

$\Gamma_0(2)^+, \Gamma_0(3)^+, \Gamma_0(4)^+$

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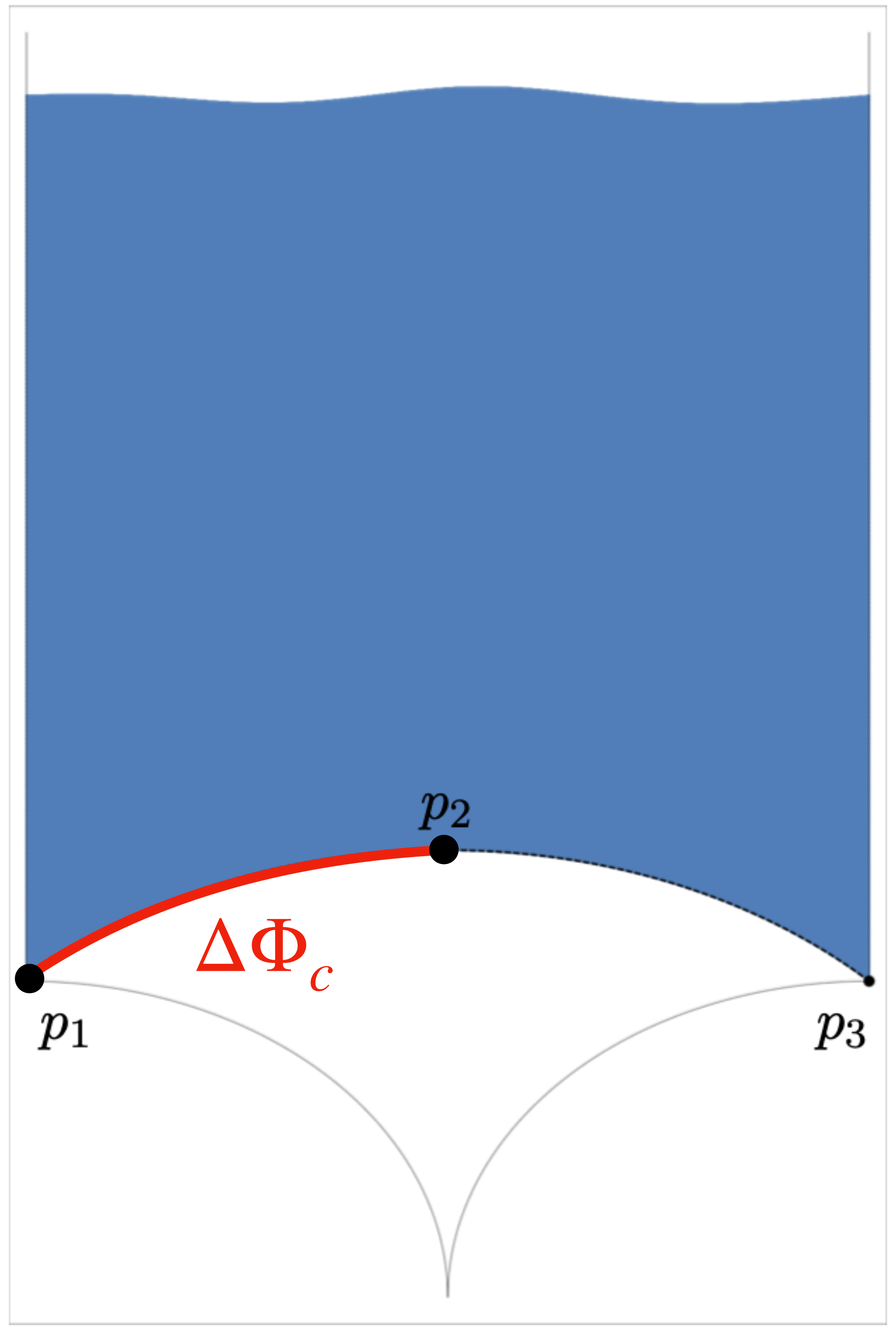
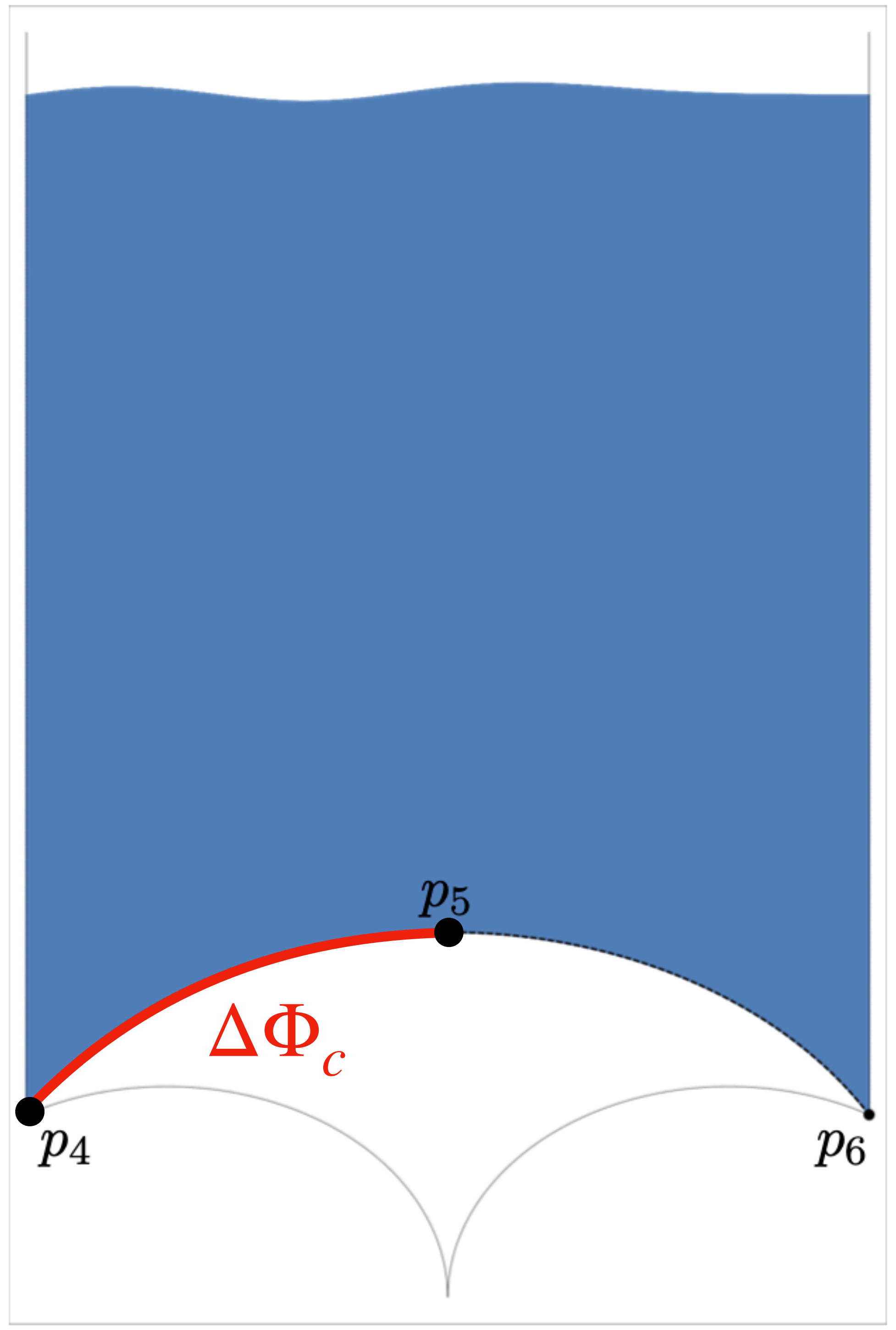
$\mathbb{P}^4_{1,1,2,2,2}[8]$



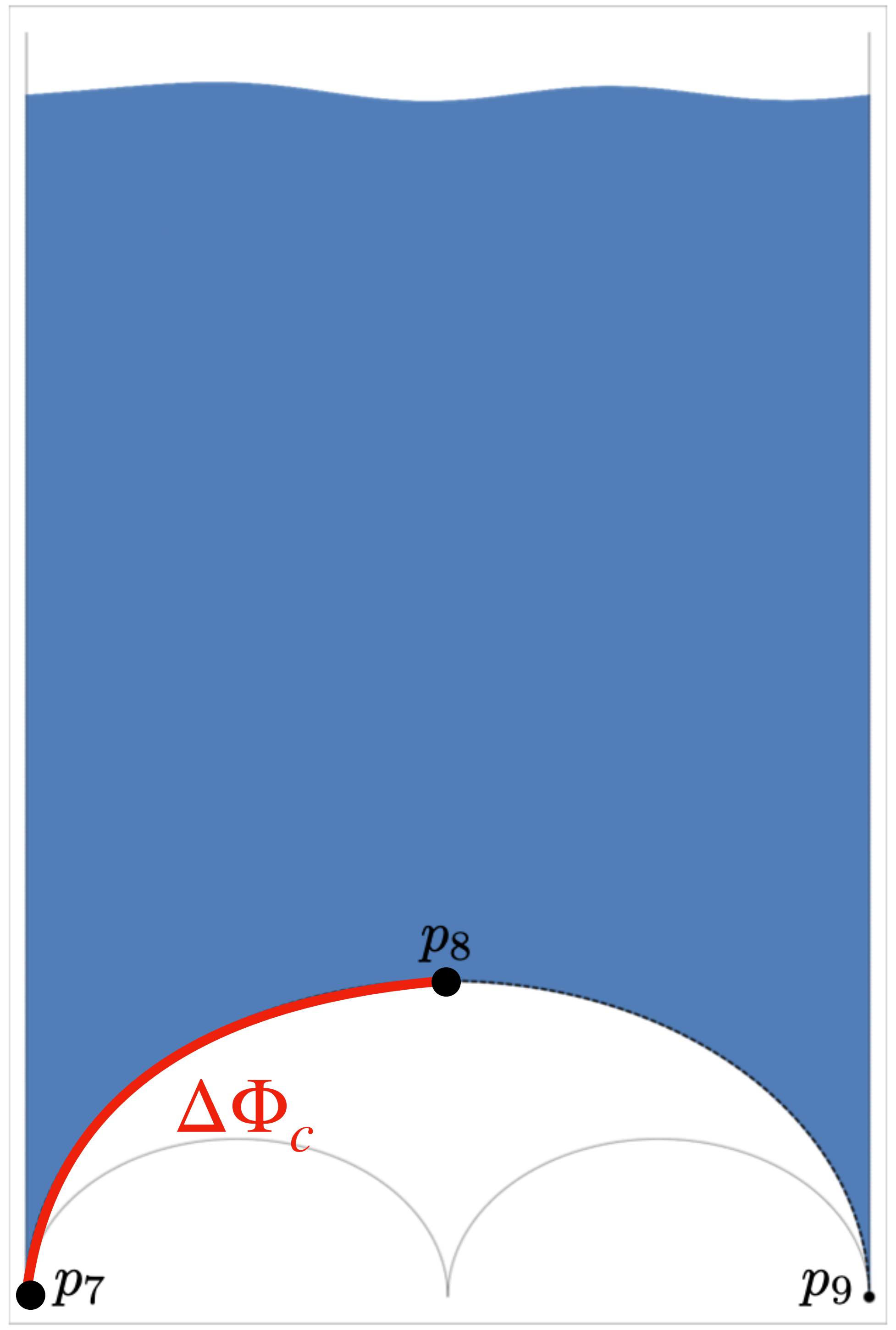
$$\mathrm{dist}(\overline{p_1p_2}) = \frac{1}{\sqrt{2}} \log\left(\cot\left(\frac{\pi}{8}\right)\right) \approx 0.62$$

$\mathbb{P}^5_{1^2,2^4}[4,6]$



$$\mathrm{dist}(\overline{p_4p_5}) = \sqrt{2}\coth^{-1}(\sqrt{3}) \approx 0.93$$

$\mathbb{P}^6_{1^2,2^5}[4,4,4]$

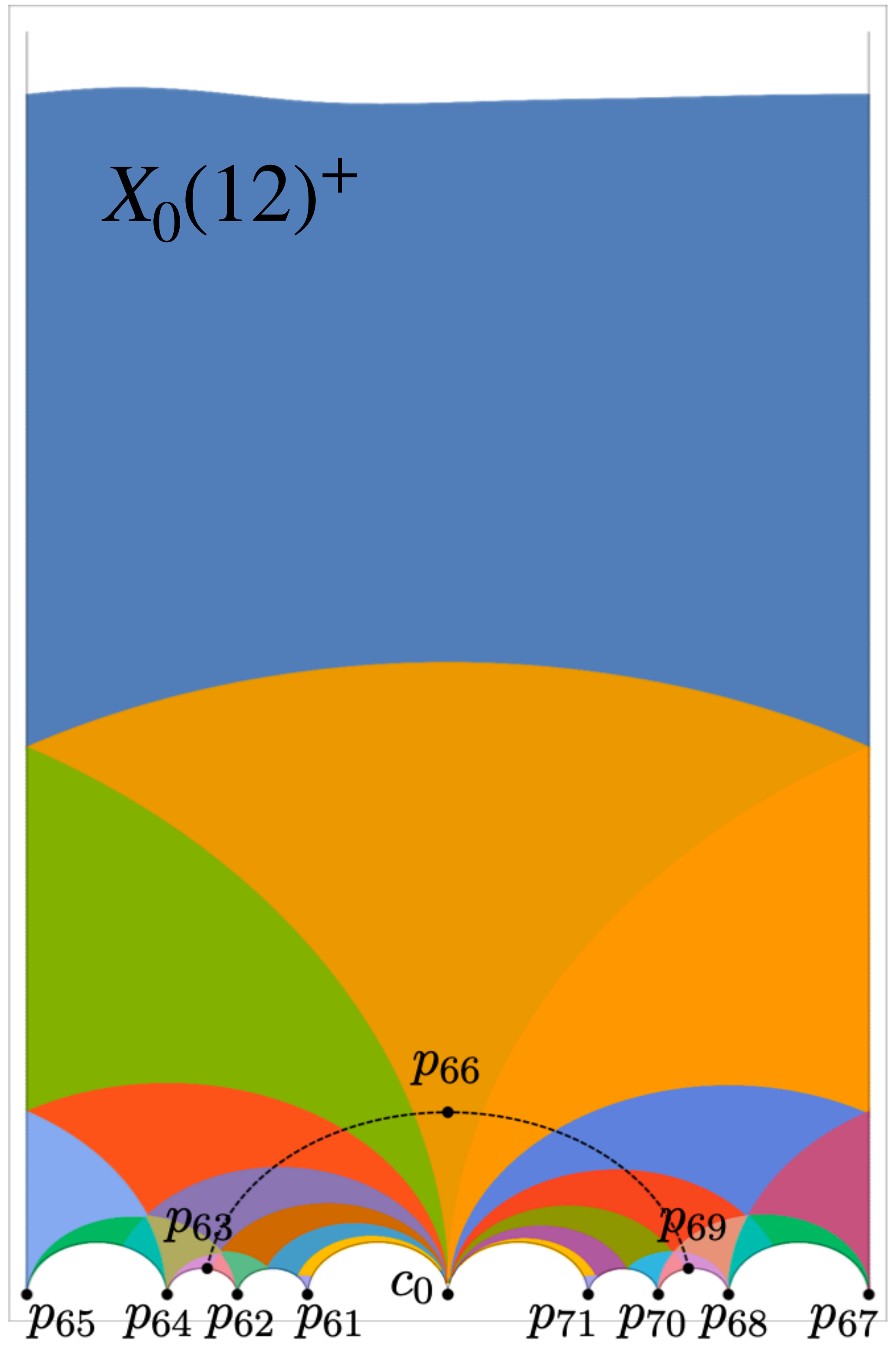


$$\mathrm{dist}(\overline{p_7p_8}) = \infty$$

$\Gamma_0(5)^+$ and beyond…

DK, 2108.00021

- For $\Gamma_0(N)^+$ with $N > 4$ the fundamental domains get more and more complicated. Often multiple finite distance geodesics:

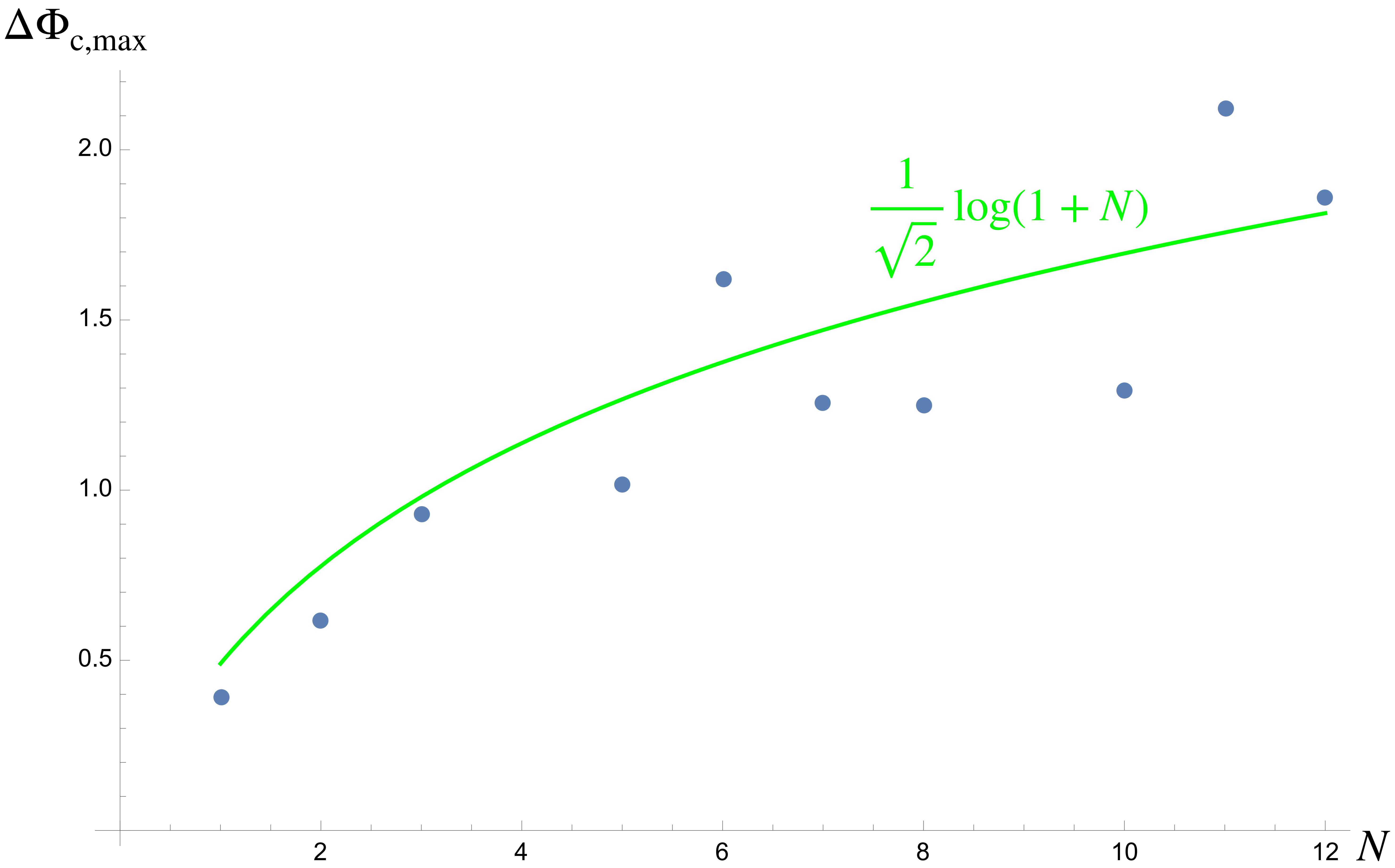


Congruence Subgroup	Geodesic	Distance
$\Gamma_0(5)^+$	$\overline{p_{13}\,p_{11}}$	$\frac{1}{\sqrt{2}}\sinh^{-1}(2) \approx 1.02$
	$\overline{p_{11}\,p_{12}}$	$\frac{1}{\sqrt{2}}\ln(\cot(\frac{1}{2}\sec^{-1}(\frac{3}{\sqrt{5}}))) \approx 0.68$
$\Gamma_0(6)^+$	$\overline{p_{20}\,p_{18}}$	$\frac{1}{2\sqrt{2}}\ln(49+20\sqrt{6}) \approx 1.62$
$\Gamma_0(7)^+$	$\overline{p_{27}\,p_{25}}$	$\frac{1}{\sqrt{2}}\ln(\cot(\frac{1}{2}\cos^{-1}(-\frac{5}{2\sqrt{7}}))) \approx 1.26$
	$\overline{p_{25}\,p_{26}}$	$\frac{1}{\sqrt{2}}\ln(\cot(\frac{1}{2}\cos^{-1}(\frac{2}{\sqrt{7}}))) \approx 0.70$
$\Gamma_0(8)^+$	$\overline{p_{34}\,p_{32}}$	$\frac{1}{2\sqrt{2}}\log(17+12\sqrt{2}) \approx 1.25$
$\Gamma_0(10)^+$	$\overline{p_{49}\,p_{47}}$	$\frac{1}{2\sqrt{2}}\log(19+6\sqrt{10}) \approx 1.29$
$\Gamma_0(11)^+$	$\overline{p_{57}\,p_{55}}$	$\frac{1}{2\sqrt{2}}\log(199+60\sqrt{11}) \approx 2.12$
	$\overline{p_{55}\,p_{56}}$	$\frac{1}{2\sqrt{2}}\log(97+56\sqrt{3}) \approx 1.86$
$\Gamma_0(12)^+$	$\overline{p_{66}\,p_{63}}$	$\frac{1}{2\sqrt{2}}\log(97+56\sqrt{3}) \approx 1.86$

$\sim \log(N)$ growth

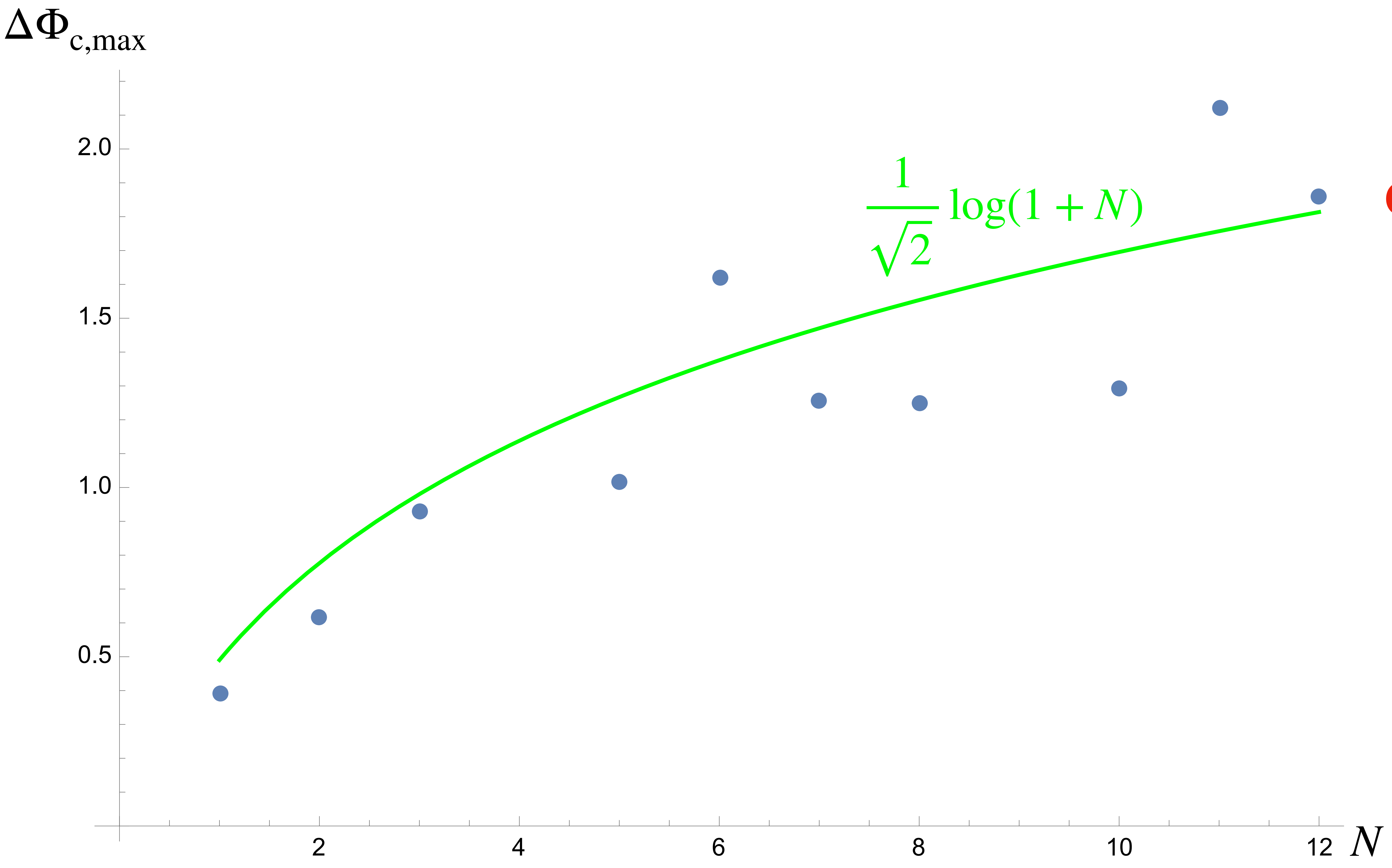
$\Gamma_0(5)^+$ and beyond...

DK, 2108.00021



$\Gamma_0(5)^+$ and beyond...

DK, 2108.00021



Can this be realized?

$\Gamma_0(5)^+$ and beyond…

DK, 2108.00021

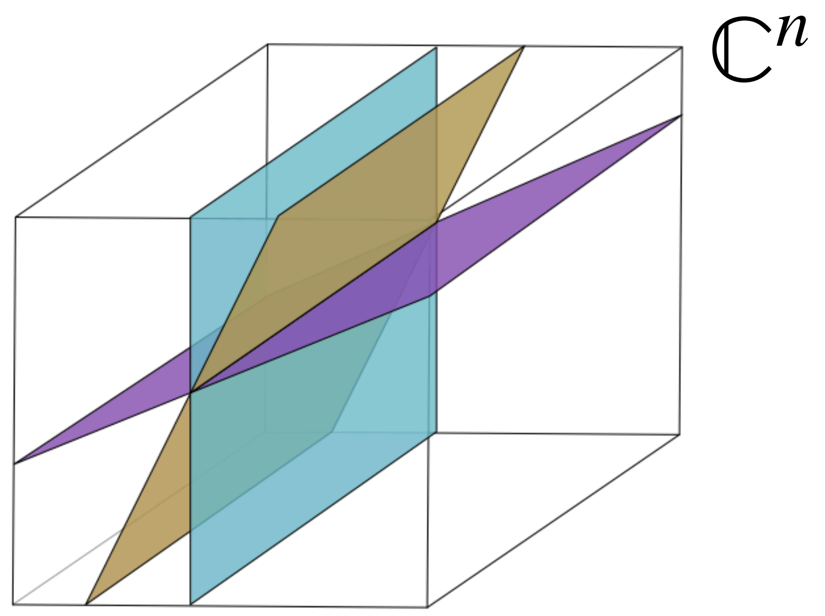
- Are $X_0(N)^+$ for $N > 4$ realised in Calabi-Yau moduli spaces?
- They **do** arise as moduli spaces of lattice polarised K3 surfaces mirror to general degree $2N$ polarised K3 surfaces (K3 with primitive ample divisor that satisfies $D^2 = 2N$) Dolgachev ’95 Doran, Harder, Novoseltsev, Thompson ‘17
- Toric constructions of such K3 surfaces are limited to $N \leq 4$
- Polarised K3 surfaces of higher degree can be constructed in Grassmannian ambient spaces Mukai ’88–‘16
- They arise as complete intersections defined by the zero locus of a section of a vector bundle
- **Example:** Degree 10 K3 surface

ambient:	$Y = Gr(2,5)$	(6-dimensional)
vector bundle:	$V = O(1)^{\oplus 3} \oplus O(2)$	(rank 4)
section:	$s_0 \in \Gamma(Y, V)$ generic	
K3:	$\{s_0 = 0\}$	(co-dimension 4)

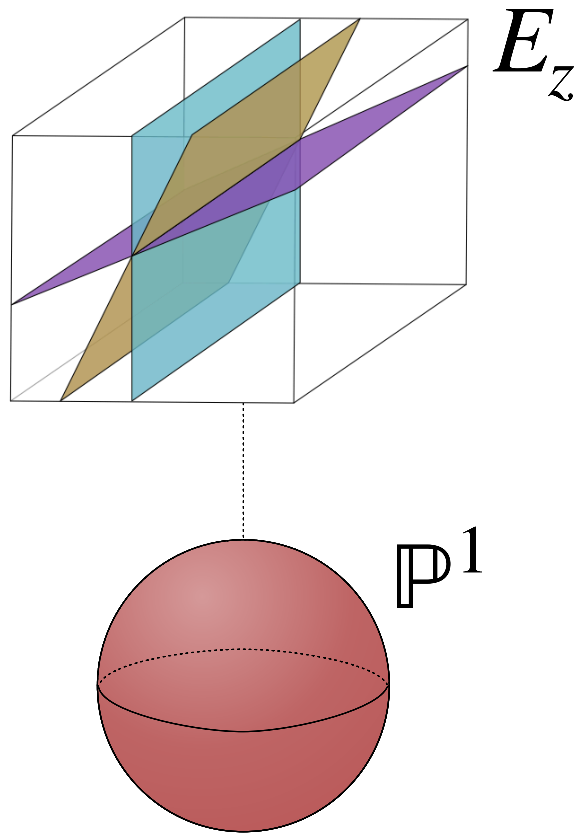
$\Gamma_0(5)^+$ and beyond...

DK, 2108.00021

- To obtain a CY fibration, we promote the ambient space to the Grassmann bundle of a rank n vector bundle $E \simeq \bigoplus_i \mathcal{O}(t_i)$ over $\mathbb{P}^1$



$Gr(k,n)$ $\longrightarrow$ $Gr(k,E)$



- The vector bundle V has to be twisted appropriately by $\mathcal{O}_{\mathbb{P}^1}(1)$ such that the canonical class of the resulting threefold is trivial

- Provided this results in a smooth threefold, we obtain the desired Calabi-Yau

$\Gamma_0(5)^+$ and beyond…

DK, 2108.00021

- We find configurations of vector bundles that tentatively realise values of N up to 19, e.g.

N=5: $Y = Gr\left(2, \mathcal{O}^{\oplus 4} \oplus \mathcal{O}(1)\right) \quad V_Y = \left(\mathcal{O}_Y(1)^{\oplus 3} \oplus \mathcal{O}_Y(2)\right) \otimes p^*\mathcal{O}(1)$

See also Knapp, Scheidegger, Schimannek ’21 for this example and similar construction of genus one fibered CY3 with 5-section

$$J_2^3 = 7 \quad J_2^2 \cdot J_1 = 10 \quad c_2 \cdot J_1 = 24 \quad c_2 \cdot J_2 = 58 \quad \chi = -102$$

N=6: $Y = Gr\left(2, \mathcal{O}^{\oplus 5}\right) \quad V_Y = \left(\mathcal{O}_Y(1) \otimes p^*\mathcal{O}(1)\right)^{\oplus 2} \oplus S^\vee \otimes \det(Q)$

$$J_2^3 = 24 \quad J_2^2 \cdot J_1 = 12 \quad c_2 \cdot J_1 = 24 \quad c_2 \cdot J_2 = 72 \quad \chi = -92$$

$\vdots$

$\Longrightarrow$ RDC violated by these models?

Heterotic Duals?

- Would be great to understand heterotic duals in detail
- Vacua can not descend from toroidal compactification of 6D theory ($h^{11} = 2$!)
- Rather, they can likely be thought of as arising from fibering special 8D heterotic vacua over K3
- It is natural to expect that the 8D theory sits at a gauge enhancement point
- Because $\Gamma_0(N)^+$ appear, we expect enhancement to occur at a reduced self-dual radius $R_{\rm sd} = 1/\sqrt{N}$, induced e.g. by Wilson lines
- Such 8D heterotic vacua have been classified recently `Font, Fraiman, Graña, Núñez, Freitas ’20 ‘21`
- Possible match for $N = 2$: vacuum with gauge group $\big(E_7 \times E_7 \times SU(4)\big)/\mathbb{Z}_2$
- Very naive calculation of spectrum gives match het <-> IIA

Summary & Future Directions

- We have tested the refined distance conjecture using mirror symmetry and het/IIA duality
  $\longrightarrow h^{11} = 1$: mirror symmetry is a very efficient tool RDC holds
  $\longrightarrow h^{11} = 2$: for K3 fibrations (het. dual), there is a large base/weak coupling limit in which the moduli space degenerates into a moduli space for a congruence subgroup $\Gamma_0(N)^+$ tension with RDC for large N

- Large N necessarily involves constructions beyond toric geometry, we have suggested a construction that utilises Grassmann bundles

Questions:

- For which values of N do smooth Calabi-Yau threefolds exist? What is the maximal value? Genus zero property of $\Gamma_0(N)^+$? see also Hajouji, Oehlmann '19 Dierigl, Heckman '20

- Global structure of the moduli space?

- Heterotic duals?

- Toric degenerations and extremal transitions? Kreuzer, Batyrev '08