Problem (4):

(a) $[\gamma^\mu P_\pm] \, Tr[\gamma_\mu \frac{1}{2}(1 \pm \gamma_5)\gamma^\nu] = C_1 \frac{1}{4} \gamma^\mu (1 \pm \gamma_5) \gamma^\nu \gamma_\mu (1 \pm \gamma_5)$

$\gamma^\lambda \gamma_\lambda = 4$

$Tr[$ungerade Anzahl von γ's$] = 0$; $Tr[\gamma^\alpha \gamma^\beta] = 4 g^{\alpha\beta}$; $P_\pm^2 = P_\pm$;

$\gamma^\lambda \gamma^\alpha \gamma_\lambda = -2\gamma^\alpha$; $\gamma^\lambda \gamma^\alpha \gamma^\beta \gamma_\lambda = 4 g^{\alpha\beta}$; $Tr[\gamma_5] = 0$; $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$

$$[\gamma^\mu P_\pm] \frac{1}{2} 4 g_\mu^{\ \nu} = C_1 \gamma^\mu \gamma^\nu \gamma_\mu P_\pm = C_1 (-2\gamma^\nu) P_\pm$$

$$2\gamma^\nu P_\pm = -2 C_1 \gamma^\nu P_\pm \implies \underline{\underline{C_1 = -1}}$$

(b) $2\gamma^\nu P_\pm = C_2 P_\mp \gamma^\nu P_\pm = C_2 \gamma^\nu P_\pm^2 = C_2 \gamma^\nu P_\pm$

$$\implies \underline{\underline{C_2 = 2}}$$

(b') $\sigma^{\mu\nu} P_\mp \gamma^\lambda \sigma_{\mu\nu} P_\pm = \sigma^{\mu\nu} \gamma^\lambda \sigma_{\mu\nu} P_\pm =$

$$= -\frac{1}{4} \underbrace{[\gamma^\mu, \gamma^\nu] \gamma^\lambda [\gamma_\mu, \gamma_\nu]} P_\pm = 0$$

$$\underbrace{\gamma^\mu \gamma^\nu \gamma^\lambda \gamma_\mu}_{4 g^{\nu\lambda}} \gamma_\nu - \gamma^\nu \underbrace{\gamma^\mu \gamma^\lambda \gamma_\mu}_{-2\gamma^\lambda} \gamma_\nu - \gamma^\mu \underbrace{\gamma^\nu \gamma^\lambda \gamma_\nu}_{-2\gamma^\lambda} \gamma_\mu + \underbrace{\gamma^\nu \gamma^\mu \gamma^\lambda \gamma_\nu}_{4 g^{\mu\lambda}} \gamma_\mu$$

$$= 4\gamma^\lambda - 4\gamma^\lambda - 4\gamma^\lambda + 4\gamma^\lambda = 0$$

(c)(i) $P_\pm \, Tr(P_\pm) = C_3 P_\pm P_\pm + C_4 \sigma^{\mu\nu} P_\pm \sigma_{\mu\nu} P_\pm$

$$P_\pm \frac{1}{2} 4 = C_3 P_\pm + C_4 \sigma^{\mu\nu} \sigma_{\mu\nu} P_\pm$$

$$2 = C_3 - \frac{1}{4} C_4 \underbrace{[\gamma^\mu, \gamma^\nu][\gamma_\mu, \gamma_\nu]}$$

$$\underbrace{\gamma^\mu \gamma^\nu \gamma_\mu}_{-2\gamma^\nu} \gamma_\nu - \gamma^\nu \underbrace{\gamma^\mu \gamma_\mu}_{4} \gamma_\nu - \gamma^\mu \underbrace{\gamma^\nu \gamma_\nu}_{4} \gamma_\mu + \underbrace{\gamma^\nu \gamma^\mu \gamma_\nu}_{-2\gamma^\mu} \gamma_\mu =$$

$$= -8 - 16 - 16 - 8 = -48$$

$$\Rightarrow 2 = C_3 + 12\, C_4$$

(ii) $P_\pm P_\pm = C_3\, P_\pm \, Tr[P_\pm] + C_4\, \sigma^{\mu\nu} P_\pm \, Tr[\sigma_{\mu\nu} P_\pm]$

$$P_\pm = 2C_3\, P_\pm + C_4\, \sigma^{\mu\nu} P_\pm \underbrace{Tr\, \sigma_{\mu\nu}}_{0}$$

$0 \leftarrow$ Spur von Kommutator

$$\Rightarrow \underline{\underline{C_3 = \frac{1}{2}}}$$

$$\Rightarrow 2 = \frac{1}{2} + 12\, C_4 \quad \Rightarrow \quad 3 = 24\, C_4$$

$$\underline{\underline{C_4 = \frac{1}{8}}}$$

(d) (i) $\sigma^{\mu\nu} P_\pm \underbrace{Tr[\sigma_{\mu\nu} P_\pm]}_{0} = C_3\, P_\pm P_\pm + C_4\, \sigma^{\mu\nu} P_\pm \sigma_{\mu\nu} P_\pm$

$$0 = C_3\, P_\pm + C_4 \underbrace{\sigma^{\mu\nu} \sigma_{\mu\nu}}_{12} P_\pm$$

$$\Rightarrow C_3 = -12\, C_4$$

(ii) $\sigma^{\mu\nu} P_\pm \sigma_{\mu\nu} P_\pm = C_3\, P_\pm \cdot Tr(P_\pm) + C_4 \cdot 0$

$$+12 = C_3\, 2$$

$$\underline{\underline{C_3 = +6}}$$

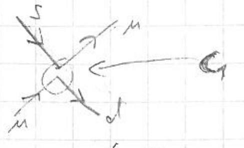
$$\Rightarrow \underline{\underline{C_4 = \frac{C_3}{-12} = -\frac{1}{2}}}$$

Exercises

(2)

$$\delta\mathcal{L}^{ne} \sim \sum_i C_i^0 O_i^0 = \sum_{ij} Z_q^2 \Big[\overbrace{Z^c_{ij}\, C_j}^{C_i^0}\Big] O_i$$

$$Z_{ij} = \begin{pmatrix} 1+\delta Z_{11} & \delta Z_{12} \\ \delta Z_{21} & 1+\delta Z_{22} \end{pmatrix} = \mathbb{1} + \delta Z_{ij}$$

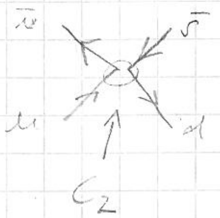
e.g.: 

$$\delta\mathcal{L}^{ne} = -\frac{4G_F}{\sqrt{2}} V_{us} V_{ud}^* \left[C_1^0 O_1^0 + C_2^0 O_2^0\right] + h.c.$$

$$O_1 = (\bar{u}^a \gamma^\mu P_L s^a)(\bar{d}^b \gamma_\mu P_L u^b)$$

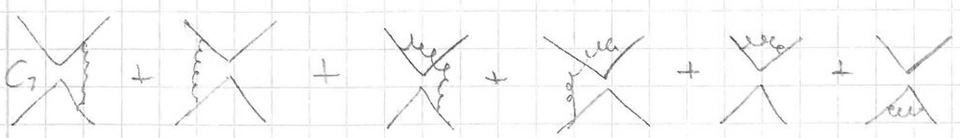
$$O_2 = (\bar{u}^a \gamma^\mu P_L s^b)(\bar{d}^b \gamma_\mu P_L u^a)$$

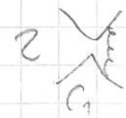
$$\overset{\text{Fierz}}{=} +(\bar{u}^a \gamma^\mu P_L u^a)(\bar{d}^b \gamma_\mu P_L s^b)$$

$\longrightarrow$ 

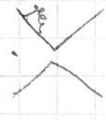
1-loop renormalization:

• always drop universal factor $\left[-\frac{4G_F}{\sqrt{2}} V_{us}V_{ud}^*\right]$ from now on!

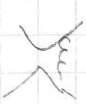
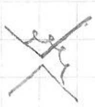
 + CT's $\overset{!}{=}$ finite

= 2  + 2  + 2  $+ \left[2\delta Z_q + \delta Z_{11}\right] C_1 \langle iO_1\rangle + \delta Z_{21} C_1 \langle iO_2\rangle$

(the last three diagram terms) $= 0$ (QED Ward identity $\times C_F$)

$2\delta Z_q = 4\cdot\frac{1}{2}\cdot$  ; $\delta Z_q = C_F\, \delta Z_q^{QED} = -\frac{C_F \alpha_s}{4\pi\epsilon}$

$$\boxed{T^A T^A = C_F \mathbb{1}}$$

• Need to compute  and  to fix δZ_{11} and δZ_{21}!

• loop integrals dimensionless → set ext. momenta to zero!

momentum flow ≠ fermion flow

$$= C_1\, i\, (ig\tilde{\mu}^{\epsilon})^2 \int d^dk \left[\bar{u}^a \gamma^\nu T^A_{ab} \frac{i\not{k}}{k^2}\gamma^\mu P_L s^b\right]\left[\bar{d}^c \gamma_\nu T^A_{cd} \frac{-i\not{k}}{k^2}\gamma_\mu P_L u^d\right]\frac{-i}{k^2-\lambda^2}$$

drop it!

integral: $\int d^dk \, \frac{k^\alpha k^\beta}{k^4(k^2-\lambda^2)} = \frac{1}{d} g^{\alpha\beta} \underbrace{\int d^dk \frac{1}{k^2(k^2-\lambda^2)}}_{\frac{i}{16\pi^2\epsilon}+O(\epsilon^0)}$

IR-regulator ("gluon mass")

$$= \underbrace{\left[-g^2\left(\frac{i}{d\,16\pi^2\epsilon}\right)\right]}_{\frac{-i\alpha}{16\pi\epsilon}}\left[\bar{u}^a\gamma^\nu T^A_{ab}\gamma^\alpha\gamma^\mu P_L s^b\right]\left[\bar{d}^c\gamma_\nu T^A_{cd}\gamma_\alpha\gamma_\mu P_L u^d\right]$$

Color Fierz: $T^A_{ab}T^A_{cd} = \frac{1}{2}\delta_{ad}\delta_{bc} - \frac{1}{2N_c}\delta_{ab}\delta_{cd}$

$$= \frac{-i\alpha}{16\pi\epsilon}\left[\frac{1}{2}\left(\bar{u}^a_L\gamma^\nu\gamma^\alpha\gamma^\mu s^b_L\right)\left(\bar{d}^b_L\gamma_\nu\gamma_\alpha\gamma_\mu u^a_L\right) - \frac{1}{2N_c}\left(\bar{u}^a_L\gamma^\nu\gamma^\alpha\gamma^\mu s^a_L\right)\left(\bar{d}^b_L\gamma_\nu\gamma_\alpha\gamma_\mu u^b_L\right)\right]$$

$$\left[\gamma^\mu\gamma^\nu\gamma^\lambda = \overset{4}{g^{\mu\nu}\gamma^\lambda} + \overset{4}{g^{\nu\lambda}\gamma^\mu} - \overset{-2}{g^{\mu\lambda}\gamma^\nu} - i\epsilon^{\sigma\mu\nu\lambda}\gamma_\sigma\gamma_5\right]$$

$$\Rightarrow \gamma^\mu\gamma^\nu\gamma^\lambda\otimes\gamma_\mu\gamma_\nu\gamma_\lambda = 10\gamma^\mu\otimes\gamma_\mu - i\epsilon^{\sigma\mu\nu\lambda}\gamma_\sigma\gamma_5\otimes\gamma_\mu\gamma_\nu\gamma_\lambda$$

$$= 10\gamma^\mu\otimes\gamma_\mu - \epsilon^{\sigma\mu\nu\lambda}\gamma_\sigma\gamma_5\otimes\epsilon_{\kappa\mu\nu\lambda}\gamma^\kappa\gamma_5 =$$

$$\epsilon^{\sigma\mu\nu\lambda}\epsilon_{\kappa\mu\nu\lambda} = -6\delta^\sigma_\kappa$$

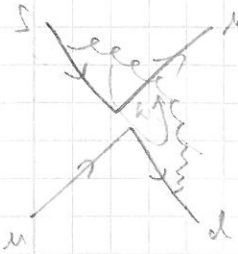
$P_L = \frac{1}{2}(1-\gamma_5)$

$$= 10\gamma^\mu\otimes\gamma_\mu + 6\gamma^\mu\gamma_5\otimes\gamma_\mu\gamma_5$$

$[\gamma_5 P_L = -P_L]$

$$\Rightarrow \gamma^\mu\gamma^\nu\gamma^\lambda P_L\otimes\gamma_\mu\gamma_\nu\gamma_\lambda P_L = 16\gamma^\mu P_L\otimes\gamma_\mu P_L$$

$$= -\frac{i\alpha}{16\pi\epsilon}\, 8\Big[\underbrace{(\bar u^a\gamma^\mu P_L s^b)(\bar d^b\gamma_\mu P_L u^a)}_{\langle O_2\rangle} - \frac{1}{N_c}\underbrace{(\bar u^a\gamma^\mu P_L s^a)(\bar d^b\gamma_\mu P_L u^b)}_{\langle O_1\rangle}\Big]$$



$=$

$=$

$$= i(ig\mu^{\varepsilon})^2\, \frac{-i}{k^2-\lambda^2} \int d^dk \left[\bar u^a\gamma^\mu P_L \frac{i\not k}{k^2} T^A_{ab}\gamma^\nu s^b_L\right]\left[\bar d^c\gamma_\nu T^A_{cd} \frac{i\not k}{k^2}\gamma_\mu P_L u^d\right]$$

$\vdots$

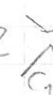
$$= \left[+\frac{i\alpha}{16\pi\epsilon}\right]\left[\frac{1}{2}(\bar u^a_L\gamma^\mu\gamma^\sigma\gamma^\nu s^b_L)(\bar d^b_L\gamma_\nu\gamma_\sigma\gamma_\mu u^a_L) - \frac{1}{2N_c}(\bar u^a_L\gamma^\mu\gamma^\sigma\gamma^\nu s^a_L)(\bar d^b_L\gamma_\nu\gamma_\sigma\gamma_\mu u^b_L)\right]$$

$$\gamma^\mu\gamma^\sigma\gamma^\nu P_L\otimes\gamma_\nu\gamma_\sigma\gamma_\mu P_L = -\overbrace{\gamma^\mu\gamma^\sigma\gamma^\nu P_L\otimes\gamma_\nu\gamma_\sigma\gamma_\mu P_L}^{16\gamma^\mu P_L\otimes\gamma_\mu P_L}$$

$$+\, 8\gamma^\mu P_L\otimes\gamma_\mu P_L + 4\gamma^\mu P_L\otimes\gamma_\mu P_L + 8\gamma^\mu P_L\otimes\gamma_\mu P_L =$$

$$= 4\gamma^\mu P_L\otimes\gamma_\mu P_L$$

$$= \left[+\frac{i\alpha}{16\pi\epsilon}\right] 2\left[\langle O_2\rangle - \frac{1}{N_c}\langle O_1\rangle\right]$$

2  $+\,2$  $+\ \delta Z_{11} C_1 i\langle O_1\rangle + \delta Z_{21} C_1 i\langle O_2\rangle = O(\epsilon^0)$

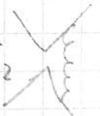
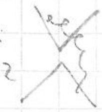
$$\Rightarrow 2\cdot(-6)\left[\frac{i\alpha}{16\pi\epsilon}\right]\left[\langle O_2\rangle - \frac{1}{N_c}\langle O_1\rangle\right] + \delta Z_{11}\, i\langle O_1\rangle + \delta Z_{21}\, i\langle O_2\rangle = 0$$

$$\Rightarrow \delta Z_{21} = +\frac{12\alpha}{16\pi\epsilon} = \frac{3\alpha}{4\pi\epsilon}$$

$$\delta Z_{11} = -\frac{3\alpha}{4\pi\epsilon}\frac{1}{N_c}$$

Analogous calculation for O_2: replace $u \leftrightarrow s$!

C_2  etc. [QCD is flavor blind!]

$\Rightarrow \; 2\,C_2$  $+ \; 2\,C_2$  $= 2(-6)\left[\frac{i\alpha}{16\pi\epsilon}\right]\left[\langle O_1\rangle - \frac{1}{N_c}\langle O_2\rangle\right]$

$$\Rightarrow \quad \delta Z_{22} = -\frac{3\alpha}{4\pi\epsilon}\frac{1}{N_c}$$

$$\delta Z_{12} = \frac{3\alpha}{4\pi\epsilon} \qquad\qquad N_c = 3$$

$$\Rightarrow \quad (Z_{ij}) = 1\!\!1 + (\delta Z_{ij}) = 1\!\!1 + \underbrace{\frac{\alpha}{4\pi\epsilon}\begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix}}_{(\delta Z_{ij})}$$