EFT EXERCISES FOR LECTURE 7 : Heavy Quark Effective Theory (May 3, 2011)

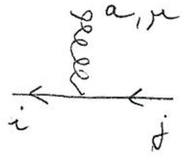
(i) Heavy antiquarks in HQET

$p_{\bar{Q}}$: physical momentum

Heavy antiquark propagator  $-p_{\bar{Q}} = -m_Q v - k$

$$i\,\frac{-\not{p}_{\bar{Q}}+m_Q}{p_{\bar{Q}}^2 - m_Q^2 + i\epsilon} = i\,\frac{m_Q(1-\not{v}) - \not{k}}{2m_Q\, v\cdot k + k^2 + i\epsilon} \simeq i\left(\frac{1-\not{v}}{2}\right)\frac{1}{v\cdot k + i\epsilon}$$

$\frac{k}{m_Q} \ll 1$

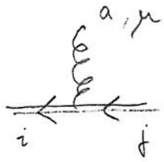
Antiquark - gluon vertex  $= -ig_s\gamma^\mu T^a_{ij}$ (QCD Feynman rule)

$\gamma^\mu\not{v} = -\not{v}\gamma^\mu + 2v^\mu$

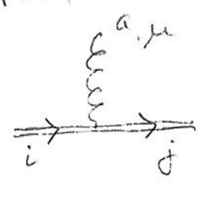
$$\left(\frac{1-\not{v}}{2}\right)\gamma^\mu\left(\frac{1-\not{v}}{2}\right) = \frac{1-\not{v}}{2}\,\frac{1+\not{v}}{2}\,\gamma^\mu - \frac{1-\not{v}}{2}\,v^\mu = \frac{1-\not{v}}{2}(-v^\mu)\frac{1-\not{v}}{2}$$

$\frac{1-\not{v}}{2} = \left(\frac{1-\not{v}}{2}\right)^2$

HQET propagator or antiquark spinor in each side

HQET  $= ig_s v^\mu T^a_{ij}$

If we treat quark and antiquark in the same footing, the fermion flow will be reversed

$\bar{Q} \rightarrow$  $= ig_s v^\mu (T^{aT})_{ji}$

same ordering as for quarks

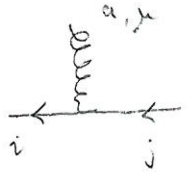
At the level of Feynman amplitudes, the change in the fermion flow for antiquarks is achieved by using the charge conjugate spinors for the antiquarks

$$v(k) = C\,\bar{u}^{c\,T}(k), \quad \text{with } C \equiv i\gamma^0\gamma^2, \quad C^2 = -1$$

Using $(\gamma^\mu)^\dagger = \gamma_0 \gamma^\mu \gamma_0$ and $(\gamma^\mu)^* = \gamma^2 \gamma^\mu \gamma^2$, the following properties of the charge conjugate matrix C can be proven:

$$C^T = -C = C^\dagger \,, \quad C\gamma_\mu C^{-1} = -\gamma_\mu^T \,, \quad C^\dagger \gamma^\mu C = C^T \gamma^\mu C = \gamma^{\mu T}$$

$$v = C\bar{u}^{cT}, \; \bar{v} = u^T \gamma^0 C \gamma^0$$



$$= \bar{v}_i \left(\frac{1-\not{v}}{2}\right)(-ig_s v^\mu T^a_{ij}) \frac{1-\not{v}}{2} v_j \;\stackrel{\downarrow}{=}$$

$$= u^{cT}_i \gamma^0 C \gamma^0 \frac{1-\not{v}}{2}(-ig_s v^\mu T^a_{ij}) \frac{1-\not{v}}{2} C \gamma^0 u^{c*}_j$$

$C\frac{1-\not{v}}{2}C^{-1}$
$= \frac{1}{2}(1+\not{v}^T)$

$$= u^{cT}_i \frac{1+\not{v}^T}{2}(-ig_s v^\mu T^a_{ij}) \frac{1+\not{v}^T}{2} \gamma^0 u^{c*}_j$$

take transpose in spin space

$$\text{"="} (-)\, u^{c\dagger}_j \gamma^0 \frac{1+\not{v}}{2}(-ig_s v^\mu T^a_{ij}) \frac{1+\not{v}}{2} u^c_i$$

$$\equiv \bar{u}^c \frac{1+\not{v}}{2} \underbrace{\left[i g v^\mu (T^{aT}) \right]}_{= -igv^\mu(-T^{aT})} \frac{1+\not{v}}{2} u^c$$

from Fermi statistics. True if charge conjugation is taken at the level of fields $\langle e^+ | \chi^{cT}_i \chi^{c*}_j | e^+ \rangle$
$= -\langle e^+ | \chi^{c\dagger}_j \chi^c_i | e^+ \rangle$

$-T^{aT} \equiv \bar{T}^a$ generator of the $\bar{3}$ SU(3) representation

QCD quark field: $Q^{(x)} \sim \sum_p e^{-ipx} a_p u(p) + e^{ipx} b^\dagger_p v(p)$

$$\equiv \psi(x) + \chi^{c\dagger}(x)$$

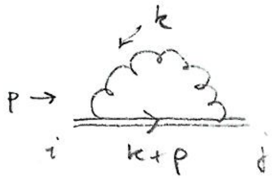
then $\psi(x)$ and $\chi^c(x)$ are both fields that annihilate particles (quarks and antiquarks respectively). Since ψ, χ^c transform under the SU(3) fundamental representation, i.e $U_g \psi$ with $U_g = e^{ig_s\theta^a T^a}$, then χ^c transforms under

$$U_g^* \chi^c = e^{-ig_s\theta^a T^{a*}} \chi^c = e^{ig_s\theta^a(-T^{a*})} \chi^c = e^{ig_s\theta^a \bar{T}^a} \chi^c$$

$$\longrightarrow D_\mu \chi^c = \partial_\mu \chi^c + ig \bar{T}^a A^a_\mu \chi^c$$

(2) Operator renormalization in HQET

$\overline{\text{MS}}$ wave-function renormalization constant of a heavy quark



$$= \int \underbrace{đ^d k}_{= \frac{d^d k}{(2\pi)^d}} \left[-ig T^A \tilde{\mu}^{\epsilon}\right]_{jk} v_\mu \frac{i}{v(k+p)+i\epsilon} \left[-ig T^A \tilde{\mu}^{\epsilon}\right]_{ki} \frac{-i}{k^2+i\epsilon}$$

$$= -\underbrace{(T^A T^A)_{ji}}_{= C_F \delta_{ji}} g^2 \tilde{\mu}^{2\epsilon} \int đ^d k \frac{1}{[v\cdot(k+p)]_+ [k^2]_+} = -C_F 4\pi \alpha_s \delta_{ij} \tilde{\mu}^{2\epsilon} I(1,1)$$

$$I(\alpha,\beta) = \int đ^d k \frac{1}{[v(k+p)]_+^{\alpha} [k^2]_+^{\beta}} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty d\lambda\, \lambda^{\alpha-1} \int đ^d k \frac{1}{\underbrace{\left(v(k+p)\lambda + k^2\right)_+^{\alpha+\beta}}}$$

note that Feynman parameter has dimensions

$$= \left(k + v\frac{\lambda}{2}\right)^2 + v\cdot p\lambda - \frac{v^2\lambda^2}{4}$$

$$= k'^2 + \lambda\left(v\cdot p - \frac{\lambda}{4} + i\epsilon\right)$$

$$= i(-)^{\alpha+\beta} \frac{\Gamma(\alpha+\beta-d/2)}{\Gamma(\alpha+\beta)(4\pi)^{d/2}} \underbrace{\int_0^\infty d\lambda\, \lambda^{\alpha-1} \left(-\lambda(v p) + \frac{\lambda^2}{4} - i\epsilon\right)^{d/2-\alpha-\beta}}$$

$$= \int_0^\infty d\lambda\, \lambda^{\frac{d}{2}-1-\beta} \left(\frac{\lambda}{4} - v\cdot p - i\epsilon\right)^{\frac{d}{2}-\alpha-\beta}$$

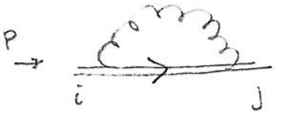
$$= (-v p)_-^{\frac{d}{2}-\alpha-\beta} \int_0^\infty d\lambda\, \lambda^{\frac{d}{2}-1-\beta} \left(\frac{\lambda}{4(-v\cdot p)_-} + 1\right)^{\frac{d}{2}-\alpha-\beta}$$

$$\tilde{\lambda} = \frac{\lambda}{4(-v p)_-} \qquad = (-v\cdot p)_-^{d-\alpha-2\beta} (4)^{\frac{d}{2}-\beta} \int_0^\infty d\tilde{\lambda}\, \tilde{\lambda}^{\frac{d}{2}-1-\beta} (1+\tilde{\lambda})^{\frac{d}{2}-\alpha-\beta}$$

$$= \frac{\Gamma\left(\frac{d}{2}-\beta\right)\Gamma(2\beta+\alpha-d)}{\Gamma(\alpha+\beta-d/2)}$$

$$= i(-)^{\alpha+\beta} \frac{2^{d-2\beta}}{(4\pi)^{d/2}} \frac{\Gamma\left(\frac{d}{2}-\beta\right)\Gamma(2\beta+\alpha-d)}{\Gamma(\alpha+\beta)} (-v\cdot p)_-^{d-\alpha-2\beta}$$

$$I(1,1) = i \frac{2^{d-2}}{(4\pi)^{d/2}} \frac{\Gamma\left(\frac{d}{2}-1\right)\Gamma(3-d)}{\Gamma(2)} (-v\cdot p)^{d-3} \underset{d=4-2\epsilon}{=} \frac{-i}{8\pi^2\epsilon}(-v\cdot p) + \mathcal{O}(\epsilon^0)$$



$$= -i C_F \frac{\alpha_s}{2\pi\epsilon} v\cdot p\, \delta_{ij} + \ldots \qquad i\, v\cdot p\, \delta Z_h$$

counterterm from $\mathcal{L}_{ct} = i\delta Z_h \bar{Q}_v v\cdot D Q_v$

$$\Rightarrow \boxed{\delta Z_h = C_F \alpha_s / 2\pi\epsilon}$$

heavy-to-light currents $\bar{q}\Gamma Q_v$ renormalization

bare quantities

$$j_\Gamma = C_\Gamma^0 \, \bar{q}^0 \, \Gamma \, Q_v^0 = Z_{C_\Gamma} Z_q^{1/2} Z_h^{1/2} C_\Gamma \, \bar{q} \, \Gamma \, Q_v$$

dirac structure

current Wilson coefficient

$$= \left(1 + \delta Z_{C_\Gamma} + \underbrace{\tfrac{1}{2}\delta Z_q}_{} + \underbrace{\tfrac{1}{2}\delta Z_h}_{} + \dots\right) C_\Gamma \, \bar{q}\,\Gamma\, Q_v$$

$Z = 1 + \delta Z + \dots$

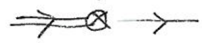
light quark wave-function renormalization from QCD ($\overline{MS}$)

$$\delta Z_q = -\frac{C_F \alpha_s}{4\pi\epsilon}$$

heavy quark wave-function renormalization, just computed

$$\delta Z_h = \frac{C_F \alpha_s}{2\pi\epsilon}$$

provides → counterterm



$i C_\Gamma (\delta Z + \dots)\Gamma$

One-loop renormalization

$$= \int d^dk \, [-ig T^A \tilde{\mu}^\epsilon \gamma^\mu] \frac{i(\not{k} + \not{p}_q)}{[(k+p_q)^2]_+} C_\Gamma \Gamma \frac{i}{[v(k+p_Q)]_+} [-ig T^A \tilde{\mu}^\epsilon v_\mu] \frac{-i}{[k^2]_+}$$

drop p_q, p_Q for calculating the UV divergence

$$= -i C_\Gamma C_F g^2 \tilde{\mu}^{2\epsilon} \int d^dk \frac{\not{v}\not{k}\Gamma}{[v\cdot k]_+ [k^2]_+^2}$$

$$\int d^dk \frac{k^\mu}{[v\cdot k]_+ [k^2]_+^2} = v^\mu A \rightarrow \overset{=1}{v^2} A = \int d^dk \frac{v\cdot k}{[v\cdot k]_+ [k^2]_+^2} = 0$$

scaleless integral both IR and UV div.

→ insert IR regulator by hand to extract only the UV div.

easiest: (same UV behaviour as original integral)

$$\int d^dk \frac{v\cdot k}{[v\cdot k]_+ [k^2 - m^2]_+^2} = \int d^dk \frac{1}{[k^2 - m^2]_+^2} = i \frac{\Gamma(2 - d/2)}{(4\pi)^{d/2}} (m^2 - i\epsilon)^{\frac{d}{2} - 2}$$

$$= i \frac{\Gamma(\epsilon)}{(4\pi)^{2-\epsilon}} (m^2)^{-\epsilon}$$

$$= -i C_\Gamma C_F g^2 \not{v}\not{v}\Gamma \, i \frac{\Gamma(\epsilon)}{(4\pi)^{2-\epsilon}} (m^2)^{-\epsilon} + \text{finite and IR-div. terms}$$

$$= C_\Gamma C_F \frac{\alpha_s}{4\pi\epsilon} \Gamma + \dots$$

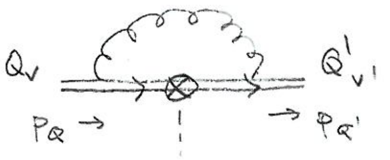
$$\delta Z_{C_\Gamma} - \frac{C_F \alpha_s}{8\pi\epsilon} + \frac{C_F \alpha_s}{4\pi\epsilon} + \frac{C_F \alpha_s}{4\pi\epsilon} = 0 \rightarrow \boxed{\delta Z_{C_\Gamma} = -\frac{3}{8}\frac{C_F \alpha_s}{\pi\epsilon}}$$

heavy-to-heavy currents $\bar{Q}'_{v'}\Gamma Q_v$ renormalization

$$j_\Gamma = C_\Gamma^0 \, \bar{Q}'^{0}_{v'} \, \Gamma \, Q_v^0 = Z_{C_\Gamma} Z_h \, \bar{Q}'_{v'} \, \Gamma \, Q_v$$

heavy quarks can have different v

$$= (1 + \delta Z_{C_\Gamma} + \delta Z_h + \dots) \, C_\Gamma \, \bar{Q}'_{v'} \, \Gamma \, Q_v$$



$$= \int d^dk \, [-ig T^A \tilde{\mu}^\epsilon v'_\mu] \frac{i}{[v'\cdot(k+p_{Q'})]_+} C_\Gamma \Gamma \frac{i}{[v\cdot(k+p_Q)]_+} [-ig T^A \tilde{\mu}^\epsilon v^\mu] \frac{(-i)}{[k^2]_+}$$

$$= -ig^2 C_F \underbrace{v\cdot v'}_{\equiv\, w} C_\Gamma \Gamma \tilde{\mu}^{2\epsilon} \underbrace{\int d^dk \frac{1}{[v'\cdot(k+p_{Q'})]_+ [v\cdot(k+p_Q)]_+ [k^2]_+}}_{= I}$$

$$I = \int_0^1 dx \int d^dk \frac{1}{[v\cdot(k+p_Q)x + v'(k+p_{Q'})\underbrace{(1-x)}_{\equiv\, \bar{x}}]_+^2} \frac{1}{[k^2]_+}$$

$$= \Gamma(3) \int_0^1 dx \int_0^\infty d\lambda \, \lambda \int d^dk \frac{1}{[k^2 + (v\cdot k\, x + v'\cdot k\, \bar{x} + v\cdot p_Q x + v' p_{Q'} \bar{x})\lambda]_+^3}$$

$$= \left(k + \lambda \frac{x}{2} v + \lambda \frac{\bar{x}}{2} v'\right)^2 - \lambda^2 \left(\frac{vx}{2} + \frac{v'x}{2}\right)^2 + v\, p_Q x + v' p_{Q'} \bar{x}$$

$$= k'^2 - \lambda^2 \left(\frac{vx}{2} + \frac{v'x}{2}\right)^2 + v\cdot p_Q x + v' p_{Q'} \bar{x}$$

$$= i(-) \frac{\cancel{\Gamma(3)}\, \Gamma(3-d/2)}{\cancel{\Gamma(3)}\, (4\pi)^{d/2}} \int_0^1 dx \int_0^\infty d\lambda \, \lambda \left[\lambda^2 \left(\frac{vx}{2} + \frac{v'x}{2}\right)^2 + v\cdot p_Q x + v'\cdot p_{Q'} \bar{x}\right]^{\frac{d}{2}-3}$$

$$(*) \int_0^\infty d\lambda \, \lambda^2 [\lambda a + b] = a^{-\frac{d}{2}+1} b^{d-4} \frac{\Gamma(4-d)\Gamma(\frac{d}{2}-1)}{\Gamma(3-d/2)}$$

use $(*)$

$$= \left[\left(\frac{vx}{2} + \frac{v'x}{2}\right)^2\right]^{-\frac{d}{2}+1} (v\cdot p_Q x + v' p_{Q'} \bar{x})^{d-4} \times \frac{\Gamma(4-d)\Gamma(\frac{d}{2}-1)}{\Gamma(3-d/2)}$$

note that if we had taken $p_Q = p_{Q'} = 0$ we would have obtained zero; $p_Q \neq 0$ is needed to regulate the IR divergence

UV div. here

$$I = -i\,\frac{\Gamma(4-d)\,\Gamma(d/2-1)}{(4\pi)^{d/2}} \underbrace{\int_0^1 dx \left[\frac{x^2}{4}+\frac{\bar{x}^2}{4}+\frac{x\bar{x}}{2}\,v\cdot v'\right]^{-1+\epsilon}}_{\text{finite for } \epsilon\to 0\text{, call it } 4\,r(\omega)} (v\cdot p_\alpha\, x + v'\cdot p_\alpha\, \bar{x})^{-2\epsilon}$$

$$r(\omega) = \int_0^1 dx\, \frac{1}{\underbrace{x^2+(1-x)^2+2x(1-x)\underbrace{v\cdot v'}_{\omega}}_{2x^2\underbrace{(1-\omega)}_{\equiv\,\bar{\omega}}-2x(1-\omega)+1}} = \frac{1}{2\bar{\omega}}\int_0^1 dx\, \frac{1}{\underbrace{x^2-x+\frac{1}{2\bar{\omega}}}_{=(x-x_+)(x-x_-)}}$$

$$x_\pm = \frac{1}{2}\left(1\pm\sqrt{1-\frac{2}{\bar{\omega}}}\right) = \frac{1}{2}\left(1\pm\sqrt{\frac{\omega+1}{\omega-1}}\right)$$

$$x_+-x_- = \sqrt{\frac{\omega+1}{\omega-1}}$$

$$= \frac{1}{2\bar{\omega}}\,\frac{1}{x_+-x_-}\int_0^1 dx\left(\frac{1}{x-x_+}-\frac{1}{x-x_-}\right)$$

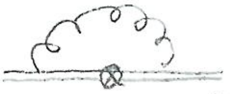
$$= \frac{1}{2\bar{\omega}}\sqrt{\frac{\omega-1}{\omega+1}}\left(\ln\underbrace{\frac{1-x_+}{-x_+}}_{\frac{1/2(1-\sqrt{\;})}{-1/2(1+\sqrt{\;})}} - \ln\underbrace{\frac{1-x_-}{-x_-}}_{\frac{1/2(1+\sqrt{\;})}{-1/2(1-\sqrt{\;})}}\right)$$

$$= -\sqrt{\frac{1}{\omega^2-1}}\,\ln\frac{\sqrt{\;}-1}{\sqrt{\;}+1}$$

$$\to = \frac{\sqrt{\omega+1}-\sqrt{\omega-1}}{\sqrt{\omega+1}+\sqrt{\omega-1}} \times\frac{\sqrt{\omega+1}+\sqrt{\omega-1}}{\sqrt{\omega+1}+\sqrt{\omega-1}} = \frac{\omega+1-(\omega-1)}{\omega+1+\omega-1+2\sqrt{\omega^2-1}} = \frac{1}{\omega+\sqrt{\omega^2-1}}$$

$$= \sqrt{\frac{1}{\omega^2-1}}\,\ln\left(\omega+\sqrt{\omega^2-1}\right)$$

$$I = -i\,\frac{\Gamma(2\epsilon)\,\Gamma(1-\epsilon)}{(4\pi)^{d/2}}\,4\,r(\omega) + \text{finite and IR-div terms} = -\frac{i}{8\pi^2\epsilon}\,r(\omega)+\ldots$$



$$= -ig^2 C_F\, C_\Gamma\, \Gamma\, \frac{(-i)}{8\pi^2\epsilon}\,\omega\, r(\omega)+\ldots = -\frac{C_F\alpha_s}{2\pi\epsilon}\, C_\Gamma\,\Gamma\,\omega\, r(\omega)+\ldots$$

$$\Rightarrow\quad \delta Z_{C_\Gamma} + \underbrace{\delta Z_h}_{\frac{C_F\alpha_s}{2\pi\epsilon}} = \frac{C_F\alpha_s}{2\pi\epsilon}\,\omega\, r(\omega) \;\longrightarrow\; \boxed{\delta Z_{C_\Gamma} = \frac{C_F\alpha_s}{2\pi\epsilon}\left[\omega r(\omega)-1\right]}$$

result is flavour and Γ - independent

- Consider $v=v'$ ($\omega=1$) ('Non-recoil' limit). Then $\omega r(\omega)\to 1$ and $\delta Z_{C_\Gamma}=0 \Rightarrow \bar{Q}'_{v}\,\Gamma\, Q_v$ is RG-invariant. In particular $\bar{Q}_v\,\gamma^0\, Q_v$ = # of heavy quarks is RG-invariant