EFT EXERCISES FOR LECTURE 6: Composite Operators and Effective Weak Hamiltonians (April 10, 2011)

(1) Non-Leptonic b Quark Decays and Penguin transitions

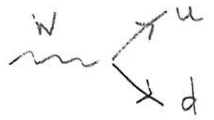
Decays of a b quark into u, d, s and c quarks

Preliminaries

* no FCNC's in the SM at tree-level. Tree-level amplitudes for b decay involve W-exchange

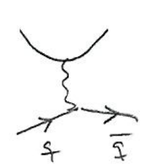
* Charged currents generate up $\leftrightarrow$ down quark transitions among the three families, with strength given by the CKM matrix

$$\mathcal{L}_{cc} = \frac{g_2}{\sqrt{2}}\, \bar{u}_i^L\, V_{ij}^{CKM}\, \gamma^\mu\, d_j^L\, W_\mu + \text{h.c.}$$

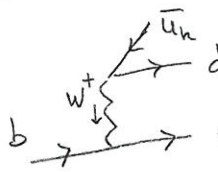
Feynman rule 

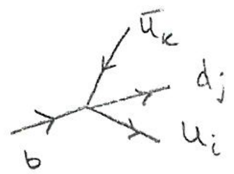
$i\frac{g_2}{\sqrt{2}} V_{ij}^{CKM}$ (i → up quark, j → down quark) $\quad V_{CKM} \sim \begin{pmatrix} 1 & \lambda & \lambda^3 \\ \lambda & 1 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix}$

* penguin-like amplitudes involve a $q_i \bar{q}_i$ pair (i.e. a quark and antiquark of the same flavour



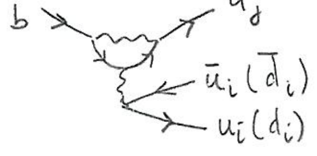
* Charged currents mediate b-decays of general form $b \to u_i \bar{u}_k d_j$, with $i,j,k = 1,2$ (top is heavier than bottom quark)



EFT →


current - current (CC) operator

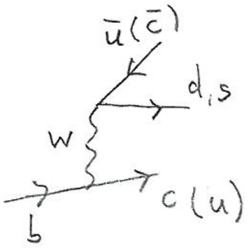
* penguin amplitudes generate decays $b \to u_i \bar{u}_i d_j$, $i,j = 1,2$, (suppressed with respect CC, since penguins are loop contributions), but also $b \to \bar{d}_i d_i d_j$



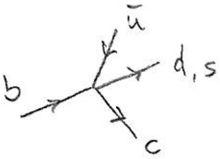
Therefore, the various decay modes can be classified into three classes

- Class I : $b \to u_i \bar{u}_k d_j$ with $i \neq k$. Only CC ops.
- Class II : $b \to u_i \bar{u}_i d_j$, CC and penguin ops.
- Class III : $b \to \bar{d}_i d_i d_j$, only penguin ops.

Class I . $b \to u_i \bar{u}_k d_j$, $i \neq k$



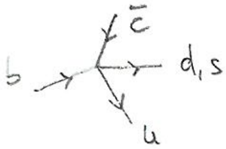
① $b \to c\bar{u}d,\ c\bar{u}s$



$$\sim \frac{g_2^2}{2M_W^2} V_{cb} \begin{Bmatrix} V_{ud}^* \\ V_{us}^* \end{Bmatrix} \sim G_F \begin{Bmatrix} \lambda^2 \\ \lambda^3 \end{Bmatrix}$$

V_{cb}: $\mathcal{O}(\lambda^2)$, V_{ud}^*: $\mathcal{O}(1)$, V_{us}^*: $\mathcal{O}(\lambda)$

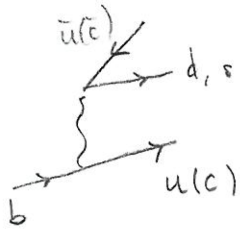
② $b \to u\bar{c}d,\ u\bar{c}s$



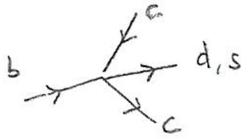
$$\sim \frac{g_2^2}{2M_W^2} V_{ub} \begin{Bmatrix} V_{cd}^* \\ V_{cs}^* \end{Bmatrix} \sim G_F \begin{Bmatrix} \lambda^4 \\ \lambda^3 \end{Bmatrix}$$

Class II $b \to u_i \bar{u}_i d_j$

* C.C.

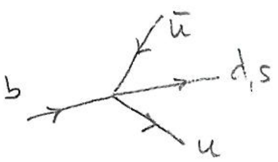


① $b \to c\bar{c}d,\ c\bar{c}s$



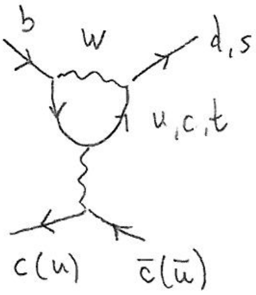
$$\sim \frac{g_2^2}{2M_W^2} V_{cb} \begin{Bmatrix} V_{cd}^* \\ V_{cs}^* \end{Bmatrix} \sim G_F \begin{Bmatrix} \lambda^3 \\ \lambda \end{Bmatrix}$$

② $b \to u\bar{u}d,\ u\bar{u}s$

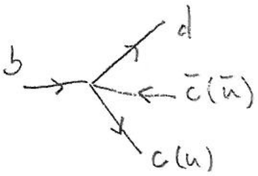


$$\sim \frac{g_2^2}{2M_W^2} V_{ub} \begin{Bmatrix} V_{ud}^* \\ V_{us}^* \end{Bmatrix} \sim G_F \begin{Bmatrix} \lambda^3 \\ \lambda^4 \end{Bmatrix}$$

* penguin ops.



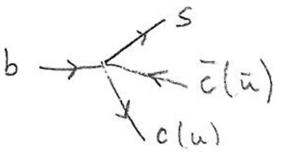
① $b \to c\bar{c}d,\ u\bar{u}d$ — loop suppression (α for EW penguins, α_s for QCD ones)



$$\sim G_F\,\alpha\,\Big(\underbrace{V_{ub}V_{ud}^*}_{\mathcal{O}(\lambda^3)}F(x_u) + \underbrace{V_{cb}V_{cd}^*}_{\mathcal{O}(\lambda^3)}F(x_c) + \underbrace{V_{tb}V_{td}^*}_{\mathcal{O}(\lambda^3)}F(x_t)\Big)$$

$$\sim G_F\,\alpha\,\lambda^3 \qquad x_i \equiv \frac{m_i^2}{M_W^2}$$

② $b \to c\bar{c}s,\ u\bar{u}s$

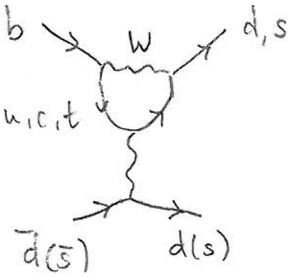


$$\sim G_F\,\alpha\,\Big(\underbrace{V_{ub}V_{us}^*}_{\lambda^4}F(x_u) + \underbrace{V_{cb}V_{cs}^*}_{\lambda^2}F(x_c) + \underbrace{V_{tb}V_{ts}^*}_{\lambda^2}F(x_t)\Big)$$

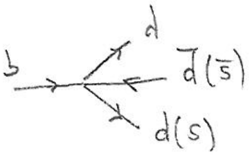
$$\sim G_F\,\alpha\,\lambda^2$$

Note that setting $F(x_u) \simeq F(x_c) \simeq F(x_t)$ (i.e. all internal masses equal), then unitarity of the CKM matrix dictates $V_{ub}V_{us}^* + V_{cb}V_{cs}^* + V_{tb}V_{ts}^* = 0$, and the penguin amplitudes above vanish (GIM mechanism)

Class III $\quad b \to d_i\bar{d}_i d_j$



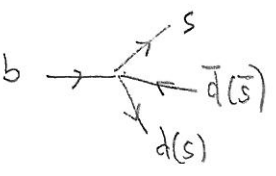
① $b \to d\bar{d}d,\ s\bar{s}d$



$$\sim G_F\,\alpha\,\big(V_{ub}V_{ud}^*F(x_u) + V_{cb}V_{cd}^*F(x_c) + V_{tb}V_{td}^*F(x_t)\big)$$

$$\sim G_F\,\alpha\,\lambda^3$$

② $b \to d\bar{d}s,\ s\bar{s}s$



$$\sim G_F\,\alpha\,\big(V_{ub}V_{us}^*F(x_u) + V_{cb}V_{cs}^*F(x_c) + V_{tb}V_{ts}^*F(x_t)\big)$$

$$\sim G_F\,\alpha\,\lambda^2$$

(2) The decay $b \to s\gamma$ and QCD equations of motion (e.o.m)

a) QCD - QED Lagrangian

$$\mathcal{L}^{QED+QCD} = \sum_q \bar{q}(i\not{D} - m_q)q - \frac{1}{4} G^{A,\mu\nu} G^A_{,\mu\nu} - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

$D_\mu = \partial_\mu + igT^A A^A_\mu + ieQA_\mu$, covariant derivative in the SU(3) fundamental representation

Euler - Lagrange e.o.m : $\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0$

- $\bar{q}_i$: $(i\not{D} - m_q) q_i = 0 \rightarrow i\not{D} q_i = m_q q_i$

  for the RH, LH components

  $$P_{L,R}\, i\not{D} q_i = P_{L,R}\, m_q q_i$$
  $$= i\not{D} P_{R,L}$$

  $$\rightarrow i\not{D} q_{i,R,L} = m_q q_{i,L,R}$$

- photon field A_μ : $F^{\mu\nu} = (\partial^\mu A^\nu) - (\partial^\nu A^\mu)$

$$\frac{\partial}{\partial(\partial^\alpha A^\beta)} F_{\mu\nu}F^{\mu\nu} = 2F_{\mu\nu} \frac{\partial F^{\mu\nu}}{\partial(\partial^\alpha A^\beta)} = 2F_{\mu\nu}\left(g^\mu{}_\alpha g^\nu{}_\beta - g^\mu{}_\beta g^\nu{}_\alpha\right)$$
$$= 4F_{\alpha\beta}$$

$$0 = \partial^\alpha \frac{\partial \mathcal{L}}{\partial(\partial^\alpha A^\beta)} - \frac{\partial \mathcal{L}}{\partial A^\beta} = -\partial^\alpha F_{\alpha\beta} \underbrace{- \sum_q ieQ\bar{q} i\gamma_\beta q}_{\text{from } D_\mu} \rightarrow \boxed{\partial^\alpha F_{\alpha\beta} = \underbrace{\sum_q eQ\bar{q}\gamma_\beta q}_{\equiv j_\beta}}$$

The dual field strenght tensor, $\tilde{F}_{\alpha\beta} \equiv \frac{1}{2}\epsilon_{\alpha\beta\mu\nu}F^{\mu\nu}$, satisfies the operator identity

$$\boxed{\partial^\alpha \tilde{F}_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\mu\nu}\, \partial^\alpha(\partial^\mu A^\nu - \partial^\nu A^\mu) = 0}$$

(sym. in α,ν; sym. in $\alpha \leftrightarrow \mu$)

- gluon field A^A_μ : $ig G_{\mu\nu} = [D_\mu, D_\nu] = /\ldots$ plug $D_\mu = \partial_\mu + igT^A A^A_\mu \ldots /$

$$= ig\, T^A \underbrace{\left\{(\partial_\mu A^A_\nu) - (\partial_\nu A^A_\mu) - g_s f^{ABC} A^A_\mu A^B_\nu\right\}}_{\equiv G^A_{\mu\nu}}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A^A_\nu)} = -\frac{1}{4}\, 2\, G^B_{\rho\sigma}\, \frac{\partial G^{B,\rho\sigma}}{\partial(\partial_\mu A^A_\nu)} = -G^{A,\mu\nu}$$

$$\frac{\partial \mathcal{L}}{\partial A^A_\nu} = -\frac{1}{2} G^B_{\rho\gamma}\, \underbrace{\frac{\partial G^{B,\rho\gamma}}{\partial A^A_\nu}}_{-g\left(f^{ACB} g^{\rho\nu} A^{C,\gamma} + f^{CAB} \delta^{\gamma\nu} A^{C,\rho}\right)} \overbrace{- \sum_q g\, \bar{q} \gamma^\nu T^A q}^{\equiv j^{\nu,A}} = \frac{g}{2} f^{ACB} \overbrace{\left(G^{B,\nu}{}_\gamma A^{C,\gamma} - G^{B}_\rho{}^{\nu} A^{C,\rho}\right)}^{-2 f^{ABC} G^{B\nu}{}_\gamma A^{C,\gamma}} - j^{\nu,A}$$

$$0 = -\partial_\mu G^{A,\mu\nu} + g f^{ABC} G^{\nu}{}_{\gamma}{}^{B} A^{C,\gamma} + j^{\nu,A}$$

$$\rightarrow \quad \underbrace{\left(\partial_\mu \delta^{AB} - g f^{ACB} A^C{}_\mu\right)}\, G^{B,\mu\nu} = j^{\nu,A}$$

$$\equiv D^{AB}_\mu G^{B,\mu\nu} = \left(D^{(ad)}_\mu G^{\mu\nu}\right)^A$$

covariant derivative
acting on the adjoint group space

$$\boxed{\left(D^{(ad)}_\mu G^{\mu\nu}\right)^A = j^{\nu,A}}$$

$$D^{(ad)}_\mu = [D_\mu, \;]$$

The dual cromomagnetic field strenght tensor, $\tilde{G}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} G_{\alpha\beta}$, satisfies the operator identity $\boxed{D^{(ad)}_\mu \tilde{G}^{\mu\nu} = 0}$, also known as the Bianchi identity:

$$0 = D^{(ad)}_\mu \tilde{G}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} D^{(ad)}_\mu G_{\alpha\beta} = \frac{1}{6}\epsilon^{\mu\nu\alpha\beta} \underbrace{\left(D^{(ad)}_\mu G_{\alpha\beta} + D^{(ad)}_\beta G_{\mu\alpha} + D^{(ad)}_\alpha G_{\beta\mu}\right)}_{=0}$$

(The Bianchi identity can be proven using the Jacobi identity for the standard cov. derivative

$$[D_\mu, [D_\alpha, D_\beta]] + [D_\beta, [D_\mu, D_\alpha]] + [D_\alpha, [D_\beta, D_\mu]] = 0$$

and then $[D_\mu, D_\nu] = ig\, G_{\mu\nu}$ and $[D_\mu, G_{\alpha\beta}] = T^C D^{CB}_\mu G^B{}_{\alpha\beta}$)

b) Reduce the dim 6 operators $O_1 - O_{10}$ to linear combinations of O_8, O_9 and 4-quark operators using the e.o.m. Take $m_s = 0$.

- $O_1 = \underbrace{\bar{s}_L \not{D}}_{=0,\ \not{D} s_{L,R}=0} D_\mu D^\mu b_L = 0$

- $O_4 = \underbrace{\bar{s}_L \not{D}}_{=0} \not{D} \not{D} b_L = 0$, $O_{10} = m_b \underbrace{\bar{s}_L \not{D}}_{=0} \not{D} b_L = 0$

- $O_5 = g \bar{s}_L T^A \gamma^\mu b_L (D^{(ad)\nu} G_{\mu\nu})^A$

  $= g \bar{s}_L T^A \gamma^\mu b_L (-\sum_q g_s \bar{q} \gamma_\mu T^A q_i) \to$ 4 quark op.

- $O_2 = \bar{s}_L D_\mu \not{D} D^\mu b_L - \frac{1}{2} \underbrace{\bar{s}_L D_\mu D^\mu \not{D} b_L}_{\equiv Y} = O_3 - \frac{1}{2} Y$

  $Y = \bar{s}_L D_\mu D^\mu \not{D} b_L = (-im_b) \bar{s}_L D_\mu D^\mu b_R = -\frac{im_b}{2} \bar{s}_L \{\gamma_\mu, \not{D}\} D^\mu b_R$

  (e.o.m. for b_L; $D_\mu = \frac{1}{2}\{\gamma_\mu, \not{D}\}$; $\bar{s}_L \not{D} = 0$)

  $= -\frac{i}{2} m_b \bar{s}_L \gamma_\mu \not{D} D^\mu b_R = -\frac{i}{2} m_b \bar{s}_L \underbrace{\gamma_\mu \gamma_\nu D^\mu D^\nu}_{\gamma_\mu\gamma_\nu (D^\nu D^\mu - [D^\nu D^\mu]),\ \not{D}\not{D}} b_R = \frac{i}{2} m_b \bar{s}_L \gamma_\mu \gamma_\nu \underbrace{[D^\mu D^\nu]}_{= igG^{\mu\nu} + ieQF^{\mu\nu}} b_R$

  $= -\frac{m_b}{2} g \bar{s}_L \underbrace{\gamma_\mu \gamma_\nu G^{\mu\nu}}_{= \frac{1}{2}[\gamma_\mu, \gamma_\nu] G^{\mu\nu} = -i\sigma_{\mu\nu} G^{\mu\nu}} b_R - \frac{m_b}{2} e Q_b \bar{s}_L \underbrace{\gamma_\mu \gamma_\nu F^{\mu\nu}}_{= -i\sigma_{\mu\nu} F^{\mu\nu}} b_R$

  $= \frac{i}{2}(O_9 + O_8)$

  $\longrightarrow$ $\boxed{O_2 = O_3 - \frac{i}{4} O_8 - \frac{i}{4} O_9}$

- $O_3 = \bar{s}_L D_\mu \not{D} D^\mu b_L = \bar{s}_L (D_\mu \not{D} - \not{D} D_\mu) D^\mu b_L = \bar{s}_L [D_\mu, D_\nu] \gamma^\nu D^\mu b_L$

  $= ig \bar{s}_L G^A_{\mu\nu} T^A \gamma^\nu D^\mu b_L^{\ 0} + ieQ_b \underbrace{\bar{s}_L F_{\mu\nu} \gamma^\nu D^\mu b_L}_{\equiv X} = -iO_6 + ieQ_b X$

$$X = \bar s_L F_{\mu\nu}\gamma^\nu D^\mu b_L = \frac{1}{2}\bar s_L F_{\mu\nu}\gamma^\nu\{\gamma^\mu, \not D\} b_L$$

$$= \frac{1}{2}\bar s_L \underbrace{F_{\mu\nu}\gamma^\nu\gamma^\mu}_{F_{\mu\nu}(-i\sigma^{\nu\mu})}(-i m_b) b_L + \frac{1}{2}\bar s_L F_{\mu\nu}\underbrace{\gamma^\nu \not D}_{-\not D\gamma^\nu + 2D^\nu}\gamma^\mu b_L \qquad (\not D \text{ term} \to 0)$$

$$= \underbrace{\frac{1}{2} m_b \bar s_L F_{\mu\nu}\sigma^{\mu\nu} b_L}_{O_8/e} + \bar s_L \underbrace{F_{\mu\nu} D^\nu\gamma^\mu}_{= -F_{\nu\mu}D^\nu\gamma^\mu = -F_{\mu\nu}D^\mu\gamma^\nu} b_L$$

$$\longrightarrow \quad 2X = \frac{O_8}{2e}$$

$$\longrightarrow \quad \boxed{O_3 = -i O_6 + i e Q_b \frac{O_8}{4e}}$$

- $$O_6 = g G^A_{\mu\nu}\bar s_L T^A\gamma^\mu D^\nu b_L = \frac{g}{2} G^A_{\mu\nu} T^A\gamma^\mu\{\gamma^\nu, \not D\} b_L$$

$$= \frac{g}{2} G^A_{\mu\nu}\bar s_L T^A \underbrace{\gamma^\mu\gamma^\nu}_{-i\sigma^{\mu\nu}}\underbrace{\not D b_L}_{-i m_b b_R} + \frac{g}{2} G^A_{\mu\nu}\bar s_L T^A \underbrace{\gamma^\mu\not D}_{-\not D\gamma^\mu + 2D^\mu}\gamma^\nu b_L$$

$$= \underbrace{-\frac{g m_b}{2} G^A_{\mu\nu}\bar s_L T^A\sigma^{\mu\nu} b_L}_{-\frac{1}{2}O_9} + \underbrace{g G^A_{\mu\nu}\bar s_L T^A\gamma^\nu D^\mu b_L}_{= -O_6}$$

$$\longrightarrow \quad \boxed{O_6 = -1/4\, O_9}$$

$\gamma_5 \frac{1-\gamma_5}{2} b = -b_L$, $\quad \gamma_5^2 = 1$

- $$O_7 = g \underbrace{\tilde G^A_{\mu\nu}}_{\frac{1}{2}\epsilon_{\mu\nu\lambda\rho}G^{A,\lambda\rho}}\bar s_L T^A\gamma^\mu D^\nu b_L = \frac{i}{2} g G^{A,\lambda\rho}\bar s_L T^A\underbrace{(-i\epsilon_{\mu\nu\lambda\rho}\gamma^\mu\gamma_5)}_{}D^\nu\gamma_5 b_L$$

$$= i\epsilon_{\nu\lambda\rho\mu}\gamma^\mu\gamma_5$$

$$= -\gamma_\nu\gamma_\lambda\gamma_\rho + g_{\nu\lambda}\gamma_\rho + g_{\lambda\rho}\gamma_\nu - g_{\nu\rho}\gamma_\lambda$$

sym. in λ,ρ

$$= \frac{ig}{2} G^{A,\lambda\rho}\bar s_L T^A(\gamma_\nu\gamma_\lambda\gamma_\rho - g_{\nu\lambda}\gamma_\rho - g_{\lambda\rho}\gamma_\nu + g_{\nu\rho}\gamma_\lambda) D^\nu b_L$$

$$= \frac{i}{2} g G^{A,\lambda\rho}\bar s_L T^A(\not D\gamma_\lambda\gamma_\rho - D_\lambda\gamma_\rho + D_\rho\gamma_\lambda) b_L$$

move $\not D$ to the right $\longrightarrow$

$$= -\gamma_\lambda\not D\gamma_\rho + 2D_\lambda\gamma_\rho - D_\lambda\gamma_\rho + D_\rho\gamma_\lambda$$

$$= \gamma_\lambda\gamma_\rho\not D - 2\gamma_\lambda D_\rho + D_\lambda\gamma_\rho + D_\rho\gamma_\lambda$$

$$= \gamma_\lambda\gamma_\rho\not D + D_\lambda\gamma_\rho - D_\rho\gamma_\lambda$$

$$= \frac{i}{2} g G^{A,\lambda\rho} \bar{s}_L T^A \Big(\underbrace{\tfrac{1}{2} [\gamma_\lambda, \gamma_\rho]}_{-i\sigma_{\lambda\rho}} \not{D} - 2 D_\rho \gamma_\lambda \Big) b_L$$

$$= \underbrace{\frac{1}{2} g G^{A,\lambda\rho} \bar{s}_L T^A \sigma_{\lambda\rho} (-i m_b) b_R}_{= -i O_9} - i \underbrace{g G^{A,\lambda\rho} \bar{s}_L T^A \gamma_\lambda D_\rho b_L}_{= O_6}$$

$$\longrightarrow \quad \boxed{O_7 = -\frac{i}{2} O_9 - i O_6}$$