EFT EXERCISES FOR LECTURE 6 : Composite Operators and Effective Weak Hamiltonian (Mar 8, 2011)

(1) strong coupling evolution (numerical) $\rightarrow$ see notebook

(2) Strong coupling evolution (analytical)

$$\frac{d\alpha_s(\mu)}{d\ln\mu} = \beta(\alpha_s(\mu)) = -2\alpha_s(\mu)\sum_{n=0}^{\infty}\beta_n\left(\frac{\alpha_s(\mu)}{4\pi}\right)^{n+1}$$

$$\downarrow$$

$$d\ln\mu = \int\frac{d\alpha_s}{\beta(\alpha_s)} \rightarrow \ln\frac{\mu}{\mu_0} = \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)}\frac{d\alpha_s}{\left(-\frac{\alpha_s^2\beta_0}{2\pi}\right)}\left\{1+\frac{\beta_1}{\beta_0}\frac{\alpha_s}{4\pi}+\frac{\beta_2}{\beta_0}\left(\frac{\alpha_s}{4\pi}\right)^2+\dots\right\}^{-1}$$

Define $t = -\frac{2\pi}{\beta_0\alpha_s}$, $dt = \frac{2\pi}{\beta_0\alpha_s^2}d\alpha_s$, $\frac{\alpha_s}{4\pi} = -\frac{1}{2\beta_0 t}$

$$\ln\frac{\mu}{\mu_0} = \int_{t_0}^{t}(-dt)\left\{1-\frac{\beta_1}{2\beta_0^2}\frac{1}{t}+\frac{\beta_2}{4\beta_0^3}\frac{1}{t^2}-\frac{\beta_3}{8\beta_0^4}\frac{1}{t^3}+\dots\right\}^{-1}$$

perturbative expansion in $\alpha_s \Leftrightarrow \frac{1}{t}$

$\frac{1}{1-\Delta} = 1+\Delta+\Delta^2+\dots$

$$= \int_{t}^{t_0}dt\left\{1+\frac{\beta_1}{2\beta_0^2}\frac{1}{t}-\frac{\beta_2}{4\beta_0^3}\frac{1}{t^2}+\frac{\beta_3}{8\beta_0^4}\frac{1}{t^3}+\dots + \left(-\frac{\beta_1}{2\beta_0^2}\frac{1}{t}+\frac{\beta_2}{4\beta_0^3}\frac{1}{t^2}+\dots\right)^2+\left(-\frac{\beta_1}{2\beta_0^2}\frac{1}{t}+\dots\right)^3+\dots\right\}$$

keep terms up to $1/t^3$

$$= \int_{t}^{t_0}dt\left\{1+\frac{1}{t}\underbrace{\left(\frac{\beta_1}{2\beta_0^2}\right)}_{\equiv\hat{b}_1}+\frac{1}{t^2}\underbrace{\left(-\frac{\beta_2}{4\beta_0^3}+\frac{\beta_1^2}{4\beta_0^4}\right)}_{\equiv\hat{b}_2} + \frac{1}{t^3}\underbrace{\left(\frac{\beta_3}{8\beta_0^4}-\frac{\beta_1\beta_2}{4\beta_0^5}-\frac{\beta_1^3}{8\beta_0^6}\right)}_{\equiv\hat{b}_3}\right\}$$

$$= G(t_0)-G(t)$$

with $G(t) = t+\hat{b}_1\ln(-t)-\frac{\hat{b}_2}{t}-\frac{1}{2t^2}\hat{b}_3$ $\rightarrow$ results for $\hat{b}_i$ checked in notebook

$$\Rightarrow \ln\frac{\mu}{\mu_0} = t_0-t+\hat{b}_1\underbrace{\left[\ln(-t_0)-\ln(-t)\right]}_{\ln(t_0/t)}-\frac{\hat{b}_2}{t_0}\left(1-\frac{t_0}{t}\right)-\frac{\hat{b}_3}{2t_0^2}\left(1-\frac{t_0^2}{t^2}\right)$$

Solve the equation $\ln \frac{\mu}{\mu_0} = G(t_0) - G(t)$ iteratively

Rewrite the equation as

$$t = t_0 - \ln \frac{\mu}{\mu_0} + \hat{b}_1 \ln \frac{t_0}{t} - \frac{\hat{b}_2}{t_0}\left(1 - \frac{t_0}{t}\right) - \frac{\hat{b}_3}{2t_0^2}\left(1 - \frac{t_0^2}{t^2}\right)$$

$$\hookrightarrow \frac{t}{t_0} = \underbrace{1 - \frac{1}{t_0} \ln \frac{\mu}{\mu_0}}_{\equiv X_f} + \frac{\hat{b}_1}{t_0} \ln \frac{t_0}{t} - \frac{\hat{b}_2}{t_0^2}\left(1 - \frac{t_0}{t}\right) - \frac{\hat{b}_3}{2t_0^3}\left(1 - \frac{t_0^2}{t^2}\right)$$

$$x \equiv t/t_0 : \quad x = X_f - \frac{\hat{b}_1}{t_0} \ln x - \frac{\hat{b}_2}{t_0^2}\left(1 - \frac{1}{x}\right) - \frac{\hat{b}_3}{2t_0^2}\left(1 - \frac{1}{x^2}\right)$$

and solve for x perturbatively, taking $X_f \sim \mathcal{O}(1)$, $\frac{1}{t_0} \sim \mathcal{O}(\alpha_s)$ and so on (in the notebook I introduced an auxiliar expansion parameter 'eps' to keep track of the perturb. order; eps $\leftrightarrow \alpha_s$; make eps $\to 1$ at the end)

$\mathcal{O}(\alpha_s^0)$: $\quad x^0 = X_f$

$\mathcal{O}(\alpha_s)$: $\quad x^1 = X_f - \frac{\hat{b}_1}{t_0} \ln X_f \nearrow$ it is sufficient to insert $x = x^0$ here, since this term is $\mathcal{O}(\alpha_s)$ already

$\mathcal{O}(\alpha_s^2)$: $\quad x^2 = X_f - \frac{\hat{b}_1}{t_0} \underbrace{\ln\left(X_f - \frac{\hat{b}_1}{t_0} \ln X_f\right)}_{= \ln X_f - \frac{\hat{b}_1}{t_0 X_f} \ln X_f + \dots} - \frac{\hat{b}_2}{t_0^2}\left(1 - \frac{1}{X_f}\right)$

$$= x^1 + \frac{1}{t_0^2 X_f}\left(\hat{b}_1^2 \ln X_f + \hat{b}_2 (1 - X_f)\right)$$

$\mathcal{O}(\alpha_s^3)$: $\quad x^3 = x^2 + \frac{1}{16\beta_0^6 t_0^3 X_f^2} \Big[\beta_0^2 \beta_3 (1 - X_f^2) - 2\beta_0\beta_1\beta_2 \left(X_f(1 - X_f) - \log X_f\right) + \beta_1^3 \left(\log^2 X_f - (1 - X_f)^2\right)$

$\uparrow$ see Mathematica notebook

(3) The hadronic scale Λ_{QCD}

Show that $\Lambda_{QCD} \equiv \mu\, e^{G(t)}$ is a RG-invariant quantity

From $\ln \frac{\mu}{\mu_0} = G(t_0) - G(t)$ we have

$$\ln \mu + G(t) = \ln \mu_0 + G(t_0)$$

exponentiate $\longrightarrow$ $\mu\, e^{G(t)} = \mu_0\, e^{G(t_0)} = cte$, μ-independent

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LL expression ; $G^{LL}(t) = t$

$$\Lambda^{LL}_{QCD} = \mu\, e^{t} \xrightarrow[t = -\frac{2\pi}{\beta_0 \alpha_s}]{} \ln \frac{\Lambda^{LL}_{QCD}}{\mu} = -\frac{2\pi}{\beta_0 \alpha_s(\mu)}$$

$$\alpha_s(\mu) = \frac{2\pi}{\beta_0 \ln\left(\frac{\mu}{\Lambda^{LL}_{QCD}}\right)}$$

Λ^{LL}_{QCD} is the scale μ at which $\alpha_s(\mu) \to \infty$