5th lecture 04/12/05

Composite Operators & Operator Mixing – Renormalization & Loops Part II

→ Suppose the following situation:

You want to describe the effects of weak interactions in the decay of hadrons.

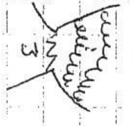
- weak interactions mediated by e.g. W, Z bosons

← semileptonic decays

② **but:** W, Z and also t, H cannot resonate because hadron masses $m_H \ll M_W, M_Z, m_t, m_H$ (e.g. kaons: D-mesons: B-mesons: pions: } $m_H \ll M_W, m_t, m_H, M_Z, \ldots$)

① problems in a naive SM computation

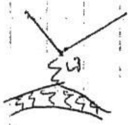
also:



QCD corrections contain large corrections $\left(\alpha_s \ln\left(\frac{M_W^2}{m_q^2}\right)\right)^n \sim (\alpha_s \times \quad)^n$ that spoil the pert. expansion in the SM.

light

also: the decays of the hadrons also involve non-perturbative effects due to confinement that cannot be computed in the SM by means of Feynman diagrams



→ Let's go step by step: Now we first switch to an effective theory where the W, Z, t, H are integrated out.

insert next page

~~⇒ We have: QCD (quarks: u, d, s, c, b and gluons)~~ → $\mathcal{L}_{QCD}$ renormalizable + flavor conserving

~~**AND:** higher dimensional operators describing the genuine effects of weak interactions~~

Note: the renormalizable QCD action is not sufficient here because dim < 4 operators in QCD cannot describe weak decays
⇒ the EFT we need must also include operators of dim > 4.

Note: we only need to consider those higher dim operators that are relevant for the decays we are interested in. (here: we only treat $O(\frac{1}{M_W^2})$ effects ↔ dim 6 op.'s)

Note: we treat the mass differences between t, H, W, Z as small and integrate them out all at once at $\mu = M_W$
→ matching conditions contain $\ln(M_W/m_t)$, etc. terms
→ those logs could also be summed by EFT methods, but we do not do that here

In practice these logs are not large.

06/01/04 ①

Addendum: (to the semileptonic current renormalization)

Start here on 06/02/04

general remarks

→ effective action of QCD + electroweak interactions for scales below $M_W, M_Z, m_t, \dots$ treating electroweak effects at first order ($O(G_F)$, but neglect $O(G_F^2)$)

$$\mathcal{L}_{EFT} = \mathcal{L}_{QCD} \quad \leftarrow \text{dim 4 terms (unrenormalized)}$$

(we neglect $\mathcal{L}_{QED}$ $\frac{\alpha_{em}}{\alpha_s}$ ")

$$+ \, \delta\mathcal{L}_{weak}^{\text{semileptonic}} \quad \leftarrow \text{dim 6 operators for semileptonic decays}$$

$$+ \, \delta\mathcal{L}_{weak}^{\text{non-leptonic}} \quad \leftarrow \text{ — " — for non-leptonic decays}$$

$$\delta\mathcal{L}_{weak}^{s.l.} = -\frac{4G_F}{\sqrt{2}} \, c^0 \sum_{q \neq q', l} [\bar{l}\gamma_\mu P_L \nu_e][\bar{q}^0 \gamma^\mu P_L q'^0] V_{qq'} + h.c.$$

$\uparrow$ lepton fields not renormalized, like "classic external fields" in QCD

$\uparrow$ quark fields are being renormalized

fermion counterterm diagrams

$$q^0 = Z_q^{1/2} q(\mu), \quad c^0 = Z_c \, c(\mu)$$

"Composite field renormalization"

* The coupling c^0 is required to renormalize potential UV-divergences that arise from putting 2 quark fields operators at the same point x $[\bar{\psi}(x) \Gamma \psi(x)]$ in addition to the ones already renormalized by the fields themselves.

→ "effective Hamiltonian" weak

$$\delta H_{eff, weak} = -\delta\mathcal{L}_{weak}$$

→ the counterterm diagrams compensate UV-divergences in matrix elements of the EFT.
In many cases these matrix elements cannot be computed perturbatively.
In this case one can in fact ignore the renormalization constants and write the effective Hamiltonian as

e.g. $$\delta H_{eff, weak}^{s.l.} = \frac{4G_F}{\sqrt{2}} \, c(\mu) \, [\bar{l}\gamma_\mu P_L \nu_e][\bar{q}\gamma^\mu P_L q] V_{qq'} + h.c.$$

$\uparrow$ short-distance Wilson coefficient $\mu \sim O(m_q)$; sums potentially large logs $\ln\left(\frac{M_W}{m_q}\right)$ ⇒ computed by matching and RGE's

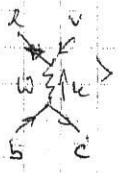
$\uparrow$ matrix elements (non-perturbative) evaluated at $\mu \sim O(m_q)$; contains no large logs ⟹ can be obtained from non-pert. methods

"effective Hamiltonian"

~~note: ~~frequently~~ This "loose" way of showing results is used ~~frequently~~,~~
~~but you should always ~~at~~ keep in mind the precise meaning of it in terms of renormalized quantities and renormalization constants!~~

e.g.: $\boxed{\text{semileptonic decays}}$ $b \to c\,\ell\nu$, $u\,\ell\nu$

<u>full theory</u>



matrix elements

$$\langle\ \cdots\ \rangle = \left(i\frac{g_2}{\sqrt{2}}\right)^2 \frac{-i\left(g^{\mu\nu}-\frac{k^\mu k^\nu}{M_W^2}\right)}{k^2-M_W^2}\left[\bar u(p_c)\gamma_\mu P_L u(p_b)\right]\left[\bar u(p_\ell)\gamma_\nu P_L v(p_\nu)\right]V_{cb}$$

unitary gauge, gauge dependent (drops out in S-matrix) & is $O(\frac{1}{M_W^4})$

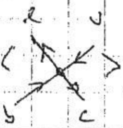
$P_L=\frac{1}{2}(1-\gamma_5)$

$k=p_b-p_c=p_\ell+p_\nu \Rightarrow \frac{k^\mu k^\nu}{M_W^2}$ can be neglected at $O(\frac{1}{M_W^2})$

$e = g_2\sin\theta = g_1\cos\theta$
$= \frac{g_1 g_2}{\sqrt{g_1^2+g_2^2}}$

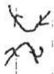
<u>Fermi constant</u>: $G_F=\frac{\sqrt{2}\,g_2^2}{8M_W^2}=1.167\cdot 10^{-5}\,\text{GeV}^{-2}$

<u>EFT:</u>



$$\Rightarrow \underbrace{\langle\ \cdots\ \rangle = -i\frac{4G_F}{\sqrt{2}}\left[\bar u_c\gamma_\mu P_L u_b\right]\left[\bar u_\ell\gamma^\mu P_L u_\nu\right]V_{cb}}$$

<u>also:</u>



indicates the W exchange

- this is the dim 6 operator we need to describe semileptonic decays $b\to c\,\ell\nu$
- since G_F is very small we can almost always work at LO in G_F i.e. no $(\frac{1}{M_W^2})^2$ effects or operators with dim >6 need to be accounted for.
- this is a term in the so called "<u>Effective weak Hamiltonian</u>"

$$\delta\mathcal{L}^{s.l.}_{weak}=\delta\mathcal{L}^{(6),s.l.}_{QCD}=-\delta H^{s.l.}_{weak}=-\frac{4G_F}{\sqrt{2}}\left[\bar c\gamma_\mu P_L b\right]\left[\bar\ell\gamma^\mu P_L \nu\right]V_{cb}\cdot C(\mu)+h.c.$$

this is the coefficient of the dim 6 operator in QCD, $\boxed{\text{"Wilson coefficient"}}$

$\boxed{\text{Matching at LL order}}$ Born matching: $\Rightarrow \underline{C(\mu=M_W)=1}$

$\boxed{\text{Renormalization at 1 loop}}$ $\rightarrow$ $\boxed{\text{aim: we need to evolve } C \text{ down to } \mu=m_b}$

- we only have to consider the renormalization of the currents $[\bar c\gamma_\mu b]$ and $[\bar c\gamma_\mu\gamma_5 b]$ within QCD
- since both currents did not exist in $\mathcal{L}^{d\le4}_{QCD}$ they are called "<u>composite operators</u>" being built from fields contained in $\mathcal{L}^{d\le4}_{QCD}$
- ~~the renormalization procedure works exactly as we have done it before~~
  ~~⇒ compute Greensfunctions with one insertion of the current and introduce counterterms to absorb the divergences.~~
- only one insertion of the current necessary at order $\frac{1}{M_W^2}$
- QCD is flavor-blind ⇒ the flavors in the current are not relevant

⇒ unrenormalized current: $j^\mu_{(5)} = C^0(\mu)\,\bar\psi^0\gamma^\mu(\gamma^5)\psi^0$
$= Z_c Z_\psi\, C\bar\psi\gamma^\mu(\gamma^5)\psi$

⇒ So, we already know the result: $Z_c=1$ (to all orders this is in fact true) for $\bar\psi\gamma^\mu\psi$

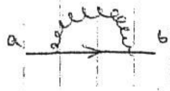
→ the following is <u>the</u> standard procedure

5th lecture

04/12/03

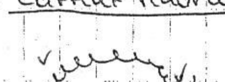
a) Wave function renormalization → of all fields in the composite operator

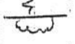
- we have already done the calculation in the 3rd lecture in QED; we just have to include the SU(3) colorfactors



$$= T^A_{ac} T^A_{cb} = C_F \delta_{ab} \,, \quad C_F = \frac{N_c^2-1}{2N_c} = \frac{4}{3} \Rightarrow \boxed{\delta Z_\psi = -C_F \frac{\alpha_s}{4\pi\epsilon}}$$

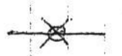
b) Current renormalization



- up to colorfactors already done in QED for  we had $\gamma^\nu \not k \gamma^\mu \not k \gamma_\nu$ in the numerator
- we have $\gamma^\nu \not k \gamma^\mu \gamma^5 \not k \gamma_\nu = \gamma^\nu \not k \gamma^\mu \not k \gamma_\nu \gamma^5$ (anticommuting γ^5)
  ⇒ the renormalization procedure is the same with or without the γ^5

⇒ 

$$= C_F \delta_{ab} \frac{\alpha_s}{4\pi\epsilon} \gamma^\mu (\gamma^5)$$



$$= (\delta Z_c + \delta Z_\psi) \gamma^\mu (\gamma^5) \delta_{ab} \quad \Bigg\} \quad \boxed{\delta Z_c = 0} \Rightarrow \frac{d}{d\ln\mu^2} C = 0$$

One says:

→ the (coupling) Wilson coefficient for vector and axial vector currents (composite operator currents) have zero anomalous dimension at one loop; they are not renormalized and do not run! (remember that but the fields themselves still do run!)

→ $$\boxed{\mathcal{L}^{\text{s.l.}}_{\text{weak}} = -\frac{4G_F}{\sqrt{2}} [\bar q \gamma_\mu P_L q'] [\bar \ell \gamma^\mu P_L \nu] V_{qq'} + \text{h.c.}}$$ is RG-invariant

(i.e. there are no large logs in the SM QCD ~~box~~ diagrams )

$\sim \ln\left(\frac{M_W^2}{m_q^2}\right)$

Note: for the vector current the result could have been anticipated because it is a conserved current

$$\int d^3x\, j^0(x) = Q_V \text{ is conserved charge} \Rightarrow [Q_V, \psi] = -\psi$$ valid to all orders

charge cannot get a renormalization factor as it is not scale dependent

Note: the axial-vector current is only conserved in the chiral limit $m_q = 0$, (then chiral symmetry relates vector ↔ axial vector) but for the UV-divergences $m_q \neq 0$ does not matter; therefore same arguments hold for j^μ_5

Note: μ-decay: $\mathcal{L}^{\ell}_{\text{weak}} = -\frac{4G_F}{\sqrt{2}} [\bar\nu_\mu \gamma_\mu P_L \mu] [\bar e \gamma^\mu P_L \nu_e] + \text{h.c.}$

is also not renormalized (i.e. no large logs $\left(\alpha_{em} \ln \frac{M_W^2}{m_\mu^2}\right)^n$) in QED, because the current $(\bar e \gamma_\mu P_L \mu)$ is conserved too
→ obtained after using Fierz identities

$$C(\mu < M_W) = C(M_W)$$

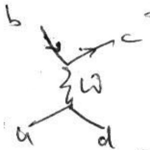
5th

Operator Mixing

~~Start here on~~
~~05/01/07~~

04/13/05 (39)

→ one step further:



you have to check for each case!

hadronic decays

e.g. $b \to c u d$ ($\Rightarrow B \to D\pi$)

a, b : color indices

influence the contribution

naive guess: $\delta\mathcal{L}^{h.l.}_{\rm eff} = -Z^0\frac{4G_F}{\sqrt{2}}[\bar c^0_a\gamma_\mu P_L b^0_a][\bar d^0_b\gamma^\mu P_L u^0_b]V_{cb}V^*_{ud} + h.c.$

$- \frac{4G_F}{\sqrt{2}}[\bar u^0_a\gamma_\mu P_L b^0_a][\bar d^0_b\gamma^\mu P_L c^0_b]V_{ub}V^*_{cd} + h.c.$

$\sim V_{ub}V^*_{cd}$! $\ll V_{cb}V^*_{ud}$] phenom. input!

→ we need more care this time and we have to write down all possible dim 6-operators

a) different γ-matrix structure $\Gamma = 1, \gamma^\mu\gamma^\nu, \ldots$
→ odd number of γ's are needed (zero otherwise) → e.g. $\bar c_L \Gamma b_L = 0$
→ can be reduced to one γ by using $\gamma_\alpha\gamma_\beta\gamma_\sigma = g_{\alpha\beta}\gamma_\sigma + g_{\beta\sigma}\gamma_\alpha - g_{\alpha\sigma}\gamma_\beta - i\epsilon_{\alpha\beta\sigma\rho}\gamma^\rho\gamma_5$

$V_{ub} \sim \lambda^3$
$V_{cd} \sim \lambda$
$V_{cb} \sim \lambda^2$
$V_{ud} \sim 1$
$\lambda \sim 0.22$
$\lambda^3 \sim 0.01$

b) different flavor ordering?
→ Fierz identities: $(\bar\psi^1_L\gamma_\mu\psi^2_L)(\bar\psi^3_L\gamma^\mu\psi^4_L) = -(\bar\psi^1_L\gamma_\mu\psi^4_L)(\bar\psi^3_L\gamma^\mu\psi^2_L)$
→ e.g. $[\bar c\gamma_\mu P_L u][\bar d\gamma^\mu P_L b]$ can be reduced to upper form

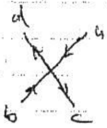
c) different color structure? → $[\bar c^a\gamma_\mu P_L b^a][\bar d^b\gamma^\mu P_L u^b]$
$[\bar c^a\gamma_\mu P_L b^b][\bar d^b\gamma^\mu P_L u^a]$ → impossible for s.l. decays but possible here!

unrenormalized dim 6 Lagrangian

$$\delta\mathcal{L}^{h.l.}_{\rm eff} = -\frac{4G_F}{\sqrt{2}}V_{cb}V^*_{ud}\left[C^0_1 O^0_1 + C^0_2 O^0_2\right] + h.c.$$

$$O_1 = [\bar c^a\gamma_\mu P_L b^a][\bar d^b\gamma^\mu P_L u^b]$$

$$O_2 = [\bar c^a\gamma_\mu P_L b^b][\bar d^b\gamma^\mu P_L u^a]$$



→ let's do the same program as for the semileptonic case:

Matching at LL order → Born

$C_1(\mu = M_W) = 1$

$C_2(\mu = M_W) = 0$

Renormalization at 1 loop

- complicated computation → see e.g. Peskin/Schröder Chap. 18
- two useful relations: color Fierz: $T^A_{ab}T^A_{cd} = \frac{1}{2}\delta_{ad}\delta_{bc} - \frac{1}{6}\delta_{ab}\delta_{cd}$

} Exercise!

γ-contractions: $(\bar\psi^1_L\gamma^\nu\gamma^\sigma\gamma^\mu\psi^2_L)(\bar\psi^3_L\gamma_\nu\gamma_\sigma\gamma_\mu\psi^4_L) = 16(\bar\psi^1_L\gamma^\mu\psi^2_L)(\bar\psi^3_L\gamma_\mu\psi^4_L)$

$\gamma_\alpha\gamma_\beta\gamma_\nu = g_{\alpha\beta}\gamma_\nu + g_{\beta\nu}\gamma_\alpha - g_{\alpha\nu}\gamma_\beta - i\epsilon_{\alpha\beta\nu\eta}\gamma^\eta\gamma_5$, $\epsilon_{0123} = 1$ (totally antisym

Counterterm contributions:

→ Operators can mix under renormalization

$$\delta\mathcal{L}_4^{n.l} \sim \sum_i C_i^0 O_i^0 = \sum_{ij} Z_4^2 (\overbrace{Z^c_{ij}\, C_j}^{C_i^0}) O_i \quad , \quad Z^c_{ij} = \begin{pmatrix} 1+\delta Z_{11} & \delta Z_{12} \\ \delta Z_{21} & 1+\delta Z_{22} \end{pmatrix}$$

$$= \left[C_1 (1 + \delta Z_{11} + 2\delta Z_4) + C_2\, \delta Z_{12} \right] O_1 + \left[C_1\, \delta Z_{21} + C_2 (1 + \delta Z_{22} + 2\delta Z_4) \right] O_2$$

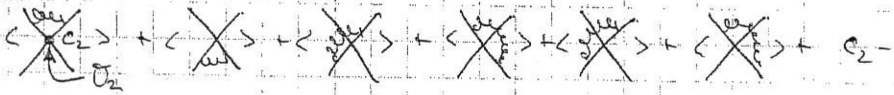
→ the results have the following form (I drop the factor $-\frac{4G_F}{\sqrt{2}} V_{cb} V_{ud}^*$)

C_1 insertions

$+ \; C_1$ - counterterms

$$= C_1\left[\left(\cancel{2C_F} + \frac{3}{N_c}\right)\frac{\alpha_s}{4\pi\epsilon} + \cancel{2\delta Z_4} + \delta Z_{11} \right] O_1 + C_1\left[-\frac{3\alpha_s}{4\pi\epsilon} + \delta Z_{21} \right] O_2$$

C_2-insertions



$+ \; C_2$ - counterterms

$$= C_2\left[-\frac{3\alpha_s}{4\pi\epsilon} + \delta Z_{12} \right] O_1 + C_2\left[\left(\cancel{2C_F} + \frac{3}{N_c}\right)\frac{\alpha_s}{4\pi\epsilon} + \cancel{2\delta Z_4} + \delta Z_{22} \right] O_2$$

$$\Rightarrow \boxed{Z^c = 1 + \frac{\alpha_s}{4\pi\epsilon}\begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix}} \quad \rightarrow \text{"Operator mixing"}$$

Note: there is no symmetry that prevents the mixing from happening. In general all operators with the same quantum numbers can mix into each other under renormalization.

Solving the RGE for the Wilson coefficients

$$0 = \frac{d}{d\ln\mu^2} C_i^0 = \left(\frac{d}{d\ln\mu^2} C_j\right) Z^c_{ij} + C_j \left(\frac{d}{d\ln\mu^2} Z^c_{ij}\right)$$

$$\rightarrow \boxed{\begin{aligned} \frac{d}{d\ln\mu^2}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} &= -\left(Z^c\right)^{-1}\left(\frac{d}{d\ln\mu^2} Z^c\right)\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \\ &= \frac{\alpha_s}{4\pi}\underbrace{\begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix}}_{\Gamma}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \end{aligned}}$$

$$\frac{d}{d\ln\mu^2} = \overbrace{\frac{d\alpha_s}{d\ln\mu^2}}^{-\epsilon\alpha_s + \ldots} \cdot \frac{d}{d\alpha_s}$$

"anomalous dimension matrix"

→ Standard: e.g. $C_\pm = \frac{1}{\sqrt{2}}(C_1 \pm C_2)$, $\boxed{C_{1/2} = \frac{1}{\sqrt{2}}(C_+ \pm C_-)}$

$$\Rightarrow \frac{d}{d\ln\mu^2}\begin{pmatrix} C_+ \\ C_- \end{pmatrix} = \frac{\alpha_s}{4\pi}\begin{pmatrix} 2 \\ -4 \end{pmatrix}\begin{pmatrix} C_+ \\ C_- \end{pmatrix}$$ → use formulas we have derived before

$$\boxed{C_+(\mu) = \frac{1}{\sqrt{2}}\left(\frac{\alpha_s(\mu)}{\alpha_s(M_W)}\right)^{-2/\beta_0} , \quad C_-(\mu) = \frac{1}{\sqrt{2}}\left(\frac{\alpha_s(\mu)}{\alpha_s(M_W)}\right)^{4/\beta_0}}$$

$$\Rightarrow \boxed{C_{1/2}(\mu) = \frac{1}{2}\left(\frac{\alpha_s(\mu)}{\alpha_s(M_W)}\right)^{-2/\beta_0} \pm \frac{1}{2}\left(\frac{\alpha_s(\mu)}{\alpha_s(M_W)}\right)^{4/\beta_0}}$$

- This solution sums logarithms $\left(\alpha_s \ln \frac{M_W^2}{\mu^2}\right)^n$. (LL order !)
- The result is the same for any of the non-leptonic operators in the dim 6 weak Lagrangian / Hamiltonian, e.g. $s \to uud$
- For the computation of hadron decays one has to compute matrix elements of the weak Hamiltonian between hadron states

$$\text{e.g. } \langle D\pi | \delta\mathcal{L}^{n\text{-}l}_{\text{weak}} | B\rangle \sim C_1(\mu)\langle D\pi | \tilde{O}_1(\mu) | B\rangle + C_2(\mu)\underbrace{\langle D\pi | \tilde{O}_2(\mu) | B\rangle}_{\text{non perturbative physics, matrix element}}$$

→ it is convenient to have all large perturbative $\ln\left(\frac{M_W}{m_b}\right)$ terms summed up in $C_{1/2}(\mu = m_b)$

→ <u>However</u>: in principle a choice $\mu > m_b$ is possible too, as long as the logs in the matrix element are not too large

→ μ is $\overset{\text{sometimes}}{\text{called}}$ <u>factorization scale</u>: $C_{1/2}(\mu)$ contains physics for scales $> \mu$

"<u>Factorization</u>" $\langle O_{1/2}(\mu)\rangle$ contain physics for scales $< \mu$

<u>$\mu = m_b$</u>: $C_1(m_b) = 1.103$, $C_2(m_b) = -0.241$ ($m_b = 4.2$ GeV)

$$C_1(m_b) = 1 + 0.056 + 0.028 + 0.011 + \ldots$$
$$C_2(m_b) = \underset{\text{Born}}{\underset{\uparrow}{}} - \underset{\mathcal{O}(\alpha_s)}{\underset{\uparrow}{0.169}} - \underset{\mathcal{O}(\alpha_s^2)}{\underset{\uparrow}{0.046}} - \underset{\mathcal{O}(\alpha_s^3)}{\underset{\uparrow}{0.016}} - \ldots$$

<u>$\mu = 1$ GeV</u>: $C_1(1\,\text{GeV}) = 1.238$, $C_2(1\,\text{GeV}) = -0.474$

$$C_1(1\,\text{GeV}) = 1 + 0.084 + 0.062 + 0.037 + \ldots$$
$$C_2(1\,\text{GeV}) = -0.252 - 0.102 - 0.052 - \ldots$$

<u>Note</u>: RGE's are only valid as long as perturbation theory works. You cannot scale μ down too low !

<u>Note</u>: The weak Hamiltonian <u>is</u> RG-invariant

$$\Rightarrow \frac{d}{d\ln\mu^2}\,\delta\mathcal{L}_W \sim \frac{d}{d\ln\mu^2}\left(C_1(\mu)\tilde{O}_1(\mu) + C_2(\mu)\tilde{O}_2(\mu)\right) = 0$$

→ the operators and their matrix elements $\overset{\text{also}}{\text{satisfy}}$ an RGE in order to compensate for the running of the Wilson coefficients.

$$\frac{d}{d\ln\mu^2}\begin{pmatrix}\tilde{O}_1\\ O_2\end{pmatrix} = \underbrace{-\frac{\alpha_s}{4\pi}\begin{pmatrix}-1 & 3\\ 3 & -1\end{pmatrix}}_{= -\Pi^T}\begin{pmatrix}O_1\\ \tilde{O}_2\end{pmatrix}$$

→ often in literature the anom. dims are presented in this way !

5th

Comparison with the SM computation

+ counterterms in the SM

→ neglect quark masses and expand in $\frac{1}{M_W}$; keep only log terms.

$$i\mathcal{A} \sim \left[1 + 2C_F \frac{\alpha_s}{4\pi} \ln\left(\frac{\mu^2}{-q^2}\right) + \frac{3}{N_C}\frac{\alpha_s}{4\pi} \ln\left(\frac{M_W^2}{-q^2}\right)\right] \mathcal{O}_1 + \left[-3\frac{\alpha_s}{4\pi} \ln\left(\frac{M_W^2}{-q^2}\right)\right] \mathcal{O}_2$$

→ Effective theory result:

low energy ~~contributions~~ agree
non-analytic terms ($\ln(-q^2)$)

$$i\mathcal{A}_{EFT} \sim C_1(\mu)\left\{\left[1 + \frac{\alpha_s}{4\pi}\left(2C_F + \frac{3}{N_C}\right)\ln\left(\frac{\mu^2}{-q^2}\right)\right]\mathcal{O}_1 + \left[-3\frac{\alpha_s}{4\pi}\ln\left(\frac{\mu^2}{-q^2}\right)\right]\mathcal{O}_2\right\}$$
$$+ C_2(\mu)\left\{\left[-3\frac{\alpha_s}{4\pi}\ln\left(\frac{\mu^2}{-q^2}\right)\right]\mathcal{O}_1 + \left[1 + \frac{\alpha_s}{4\pi}\left(2C_F + \frac{3}{N_C}\right)\ln\left(\frac{\mu^2}{-q^2}\right)\right]\mathcal{O}_2\right\}$$

Comparing one finds: (short distance contributions)

$\alpha_s = \alpha_s(\mu)$

$$C_1(\mu) = 1 - \frac{\alpha_s}{4\pi}\frac{3}{N_C}\ln\left(\frac{\mu^2}{M_W^2}\right) + \mathcal{O}(\alpha_s^2)$$
$$C_2(\mu) = 3\frac{\alpha_s}{4\pi}\ln\left(\frac{\mu^2}{M_W^2}\right) + \mathcal{O}(\alpha_s^2)$$

"Factorization of short- and long-distance effects"

this agrees with the $O(\alpha_s)$ terms obtained from the RGE's for $C_{1/2}$.

Summary Box

Note:
- The full theory and the EFT agree in the IR, which means that the $\ln(-q^2)$ terms (and in fact all non-analytic terms) agree. ⇒ only "hard" "off-shell" effects contained in matching conditions (→ $\mu \sim$ hard scale)
- The Wilson coefficients in the EFT do not contain IR-sensitive $\ln(-q^2)$ terms (→ $\mu \sim$ hard scale)
  (→ no large logs in matching condition)

→ If the computations and results show these features, this indicates that the EFT has the right low energy dof's and the proper set of operators.

→ Both statements are in fact equivalent in the correct EFT.

→ Having the correct EFT means that the two statements remain true at any order of approximation

Note: Renormalization of higher dimensional composite operators needs to be carried out when renormalizing higher order non-renormalizable terms in the effective Lagrangian.

Renormalization of higher dim. operators works in the same way as for renormalizable interactions

Factorization

$$\text{"}\int_{-p^2}^{M_W^2} dk^2 = \int_{\mu^2}^{M_W^2} dk^2 + \int_{-p^2}^{\mu^2} dk^2\text{"} \qquad \mu : \text{"factorization scale"}$$

$$\text{"}\underbrace{\left(1 + \alpha_s \ln\frac{M_W^2}{-p^2}\right)}_{\text{full}} = \underbrace{\left(1 + \alpha_s \ln\left(\frac{M_W^2}{\mu^2}\right)\right)}_{\text{short}} * \underbrace{\left(1 + \alpha_s \ln\left(\frac{\mu^2}{-p^2}\right)\right)}_{\text{long}}\text{"}$$

End
06/02/04

06/01/04 ②

Comments:

- When running from $\mu = M_W$ to $\mu \sim m_a$ one might include flavor thresholds
  e.g. kaon decay: $\mu \sim 1\,\mathrm{GeV}$ → charm $(n_f=3)$ and bottom $(n_f=4)$ integrated out] effects small

- Different renormalization schemes and regularization methods
  ($MS \leftrightarrow \overline{MS}$) (different versions of dim. reg)
  lead to scheme-dependencies in renormalized couplings and operators (usually starts at NLL)
  ⇒ predictions of observables are not affected
  scheme-dependent terms cancel

- When integrating out top, Higgs, W, Z in one step the higher order matching conditions are fct's of $m_W, m_t, m_Z, \ldots$
  → large logs might arise and might be treated by successively integrating out the heavy particles] effects small

- <u>renormalization scale dependence</u>

  Choice of $\mu \sim O(m_q)$ is <u>not</u> fixed
  Usually one varies μ between $\frac{1}{2} m_Q$ and $2 m_Q$.

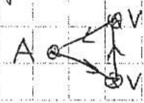
  → when higher order (NLL, NNLL..) corrections are included the residual μ-dependence becomes smaller
  because the μ-dependence of the $C(\mu)$'s and the matrix elements cancels more and more.

5th lecture

04/15/03 (43)

<u>Note:</u> γ_5 is a 4-dimensional object: $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$
$$= -\frac{i}{4!}\epsilon_{\mu\nu\sigma\rho}\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho$$
} Subtlety in dim. reg.! ← inconsistent!

- <u>Schemes:</u> <u>Naive dim. reg.</u>: $\{\gamma_5, \gamma_\mu\} = 0$
→ problematic for closed parity odd fermion loops



e.g. axial anomaly

<u>t'Hooft Veltman</u>: $d = 4-2\epsilon$, $g_{\mu\nu} = \tilde{g}_{\mu\nu} + \hat{g}_{\mu\nu}$
$$\gamma_\mu = \tilde{\gamma}_\mu + \hat{\gamma}_\mu$$
$$\tilde{g}_{\mu\nu}\,\tilde{g}^\nu_\rho = \tilde{g}_{\mu\rho}$$
$$\{\hat{\gamma}_\mu, \tilde{\gamma}_\rho\} = 0 \quad , \{\gamma_5, \tilde{\gamma}_\mu\} = 0 \quad , [\gamma_5, \hat{\gamma}_\nu] = 0$$
$$V-A = \frac{1}{2}\tilde{\gamma}_\mu(1-\gamma_5)$$

<u>Note:</u> <u>Evanescent operators</u>: beyond LL approximation the operator basis based on the γ-matrix structures
$\{1, \gamma_5, \gamma_\mu, \gamma_\mu\gamma_5, \sigma_{\mu\nu}\}$ is not sufficient in dim reg.
one needs additional operators that vanish for $\epsilon \to 0$
e.g. $[\bar{\psi}\gamma_\mu\gamma_\nu\gamma_\sigma(1-\gamma_5)\psi][\bar{\psi}\gamma^\mu\gamma^\nu\gamma^\sigma(1-\gamma_5)\psi]$
$$= (4-a\epsilon)[\bar{\psi}\gamma_\mu(1-\gamma_5)\psi][\bar{\psi}\gamma^\mu(1-\gamma_5)\psi] + \underset{\substack{\uparrow \\ \text{completely new structure!}}}{E} + O(\epsilon^2)$$

<u>Note:</u> For the computation of the RGEs and of matching conditions one can always use Green fct's or S-matrix elements for quark and gluon external states!

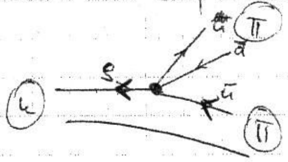
This works even if one eventually only evaluates hadronic amplitudes and S-matrix elements, because the matching conditions and the running are only applied for scales where perturbation theory works.

→ <u>Possible caveat</u>: duality violation → very difficult to quantify

06/01/04 ③

Now: ... it will become even more complicated !

e.g. $K \to \pi\pi$



← there are many more possibilities than this!

→ in general one needs 4-quark operators where 2 quarks have the same flavor

$\boxed{\text{Operator zoology}}$

see Buchalla, Buras, Lautenbacher
Rev. Mod. Phys. 68, 1125 (1996)
hep-ph/9512380

→ complete compilation of operators describing weak decays at energies $\ll M_W$

→ operators can be classified by the way they change flavor, isospin, etc.

e.g. $\Delta S = 1$, $\Delta C = 1$, ... ΔI ↖ isospin
ΔB
↑
strangeness, "charm #", bottom #, ..

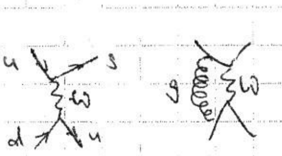
→ 6 different classes (only examples are shown : → s quark)

Current - current

$$Q_1 = (\bar{s}_i u_j)_{V-A} (\bar{u}_j d_i)_{V-A}$$

$$Q_2 = (\bar{s} u)_{V-A} (\bar{u} d)_{V-A}$$

Born (dominant!)



typical diagrams in the SM that generate the operators of the Q_1 type

QCD Penguins

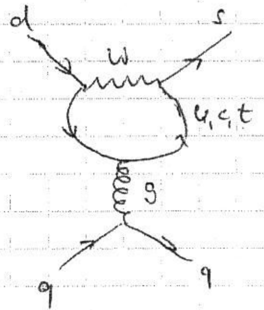
only loop-induced!

$$Q_3 = (\bar{s} d)_{V-A} \sum_q (\bar{q} q)_{V-A}$$

$$Q_4 = (\bar{s}_i d_j)_{V-A} \sum_q (\bar{q}_j q_i)_{V-A}$$

$$Q_5 = (\bar{s} d)_{V-A} \sum_q (\bar{q} q)_{V+A}$$

$$Q_6 = (\bar{s}_i d_j)_{V-A} \sum_q (\bar{q}_j q_i)_{V+A}$$



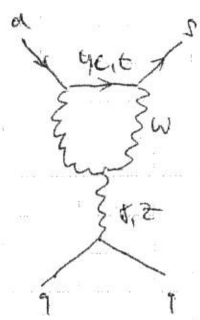
Electroweak Penguins

$$Q_7 = \frac{3}{2} (\bar{s} d)_{V-A} \sum_q e_q (\bar{q} q)_{V+A}$$

$$Q_8 = \frac{3}{2} (\bar{s}_i d_j)_{V-A} \sum_q e_q (\bar{q}_j q_i)_{V+A}$$

$$Q_9 = \frac{3}{2} (\bar{s} d)_{V-A} \sum_q e_q (\bar{q} q)_{V-A}$$

$$Q_{10} = \frac{3}{2} (\bar{s}_i d_j)_{V-A} \sum_q e_q (\bar{q}_j q_i)_{V-A}$$

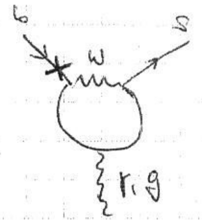


06/01/04 ④

Magnetic Penguins

$$Q_{7\gamma} = \frac{e}{8\pi^2} m_b \bar{s}_i \sigma^{\mu\nu}(1+\gamma_5) b_i F_{\mu\nu}$$

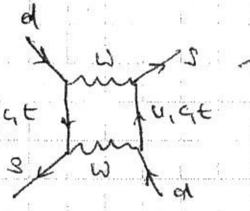
$$Q_{8g} = \frac{g}{8\pi^2} m_b \bar{s}_i \sigma^{\mu\nu}(1+\gamma_5) T^A_{ij} b_j G^A_{\mu\nu}$$



ΔS = 2, ΔB = 2 Ops.

$$Q_{\Delta S=2} = (\bar{s}d)_{V-A}(\bar{s}d)_{V-A}$$

$$Q_{\Delta B=2} = (\bar{b}d)_{V-A}(\bar{b}d)_{V-A}$$



→ $K^0-\bar{K}^0$ mixing (also $B^0-\bar{B}^0$)

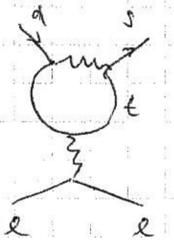
Semileptonic Ops.

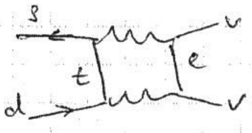
$$Q_{7V} = (\bar{s}d)_{V-A}(\bar{e}e)_V$$

$$Q_{7A} = (\bar{s}d)_{V-A}(\bar{e}e)_A$$

$$Q_{\bar{\nu}\nu} = (\bar{s}d)_{V-A}(\bar{\nu}\nu)_{V-A}$$

$$Q_{\bar{\mu}\mu} = (\bar{s}d)_{V-A}(\bar{\mu}\mu)_{V-A}$$





→ Important property of CKM matrix :

GIM mechanism

(Glashow - Iliopoulos - Maiani)

→ we encountered this issue for the absence of FCNC's in the SM at tree level

→ matching gives tree level contributions only to current-current operators

→ all other operators have coefficients arising from loop diagrams

$$\sim \sum_{i=u,c,t} \lambda_i F(x_i) \qquad \sim \sum_{i,j=u,c,t} \lambda_i \lambda_j F(x_i, x_j)$$

↑ Penguins, ↑ ΔS = 2, ΔB = 2 Op's

$$x_i = \frac{m_i^2}{M_W^2} \qquad \lambda_i = \begin{cases} V_{is}^* V_{id} & \text{K-decays} \\ V_{ib}^* V_{id} & \text{B-decays} \\ V_{ib}^* V_{is} & \text{B-decays} \end{cases}$$

→ unitarity of CKM matrix: $\lambda_u + \lambda_c + \lambda_t = 0$ →

- for equal internal $\xrightarrow{}$in loop quark masses no contribution, "GIM mechanism"

$$\sum_i \lambda_i F(x_i) = 0 \quad \text{for equal } x_i$$

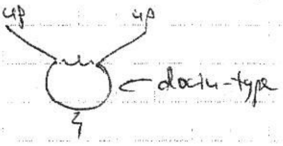
06/01/04 Ⓘ

down-type quarks

<u>B-meson, K-meson decays</u> → $B = \bar{b}q \quad K = \bar{s}q$

$i = u, c, t$

$m_t \gg m_c > m_u$



$x_t > 1$

$x_{u,c} \ll 1$

GIM mechanism not so effective, top mass leads to sizeable matching conditions also for non-current-current op.

<u>D-meson decays</u> → $D = \bar{c}q$ ← up-type → B-$\bar{B}$ mixing, K-$\bar{K}$ mixing } exp. observable

$i = d, s, b \longrightarrow x_{d,s,b} \ll 1$

GIM mechanism very effective → all non-current-current operators have <u>very</u> small matching conditions → very small effects e.g. D-$\bar{D}$ mixing extremely small! (in SM)

<u>Comments:</u> all operators give effects $\propto G_F$!

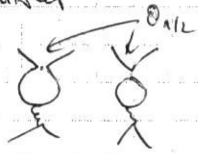
<u>Current-current:</u>

$C_2(M_W) \sim \alpha_s(M_W)$
$C_1(M_W) = 1 + \ldots$
} mixing for $\mu < M_W$ due to QCD effects

<u>QCD penguins:</u> $C_{3,4,5,6}(M_W) \sim \alpha_s(M_W)\, E(x_t)$ ← m_u, m_c can be neglected + GIM mechanism

- mixing between penguins and current-current operators which have 2 quarks of same flavor

e.g.
$Q_1 = (\bar{c}_i b_j)_{V-A} (\bar{s}_j c_i)_{V-A}$
$Q_2 = (\bar{c} b)_{V-A} (\bar{s} c)_{V-A}$
} can mix into penguins (i.e. RGE's for penguin coeffs. dependent on coeff.'s of current-current ops.)



<u>Electroweak penguins:</u> $C_{7,\ldots,10}(M_W) \sim \alpha(M_W)\, \tilde{E}(x_t)$

- leading QCD-QED effects
- current-current & QCD penguins mix into electroweak penguins

<u>Magnetic penguins:</u>

- important for e.g. radiative decays ($B \to X_s \gamma$)
- current-current & QCD penguins mix into magnetic penguins

contains $\bar{s}$ — no strange quark

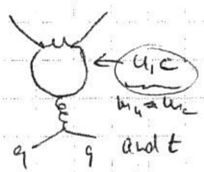
→ back to $K \to \pi\pi$ → $\Delta S = 1$ transition

- use unitarity of CKM matrix to eliminate dependence on $\lambda_c = V_{cs}^* V_{cd}$
- need current-current & QCD penguin operators (electroweak penguins smaller effects → neglected here) because 2 quarks in current-current with same flavor
- one has to evolve the theory down from M_W to scales $\mu < m_c$ — Buchalla, Buras et al.

$M_W > \mu > m_c$ → note: G_F defined differently by factor 4 in Buras et al.

$$H_{eff}(\Delta S=1) = \frac{4G_F}{\sqrt{2}} V_{us}^* V_{ud} \Big[\sum_{i=1,2} C_i(\mu) Q_i + \tau \sum_{i=3,6} v_i(\mu) Q_i + (1-\tau) \sum_{i=1,2} \bar{C}_i(\mu) Q_i^c \Big]$$

$$\tau = -\frac{V_{ts}^* V_{td}}{V_{us}^* V_{ud}} = \frac{V_{us}^* V_{ud} + V_{cs}^* V_{cd}}{V_{us}^* V_{ud}}$$ → from matching conditions for QCD penguins



$$1-\tau = -\frac{V_{cs}^* V_{cd}}{V_{us}^* V_{ud}}$$

$$O_1^c = (s_i \bar{c}_j)_{V-A} (\bar{c}_j d_i)_{V-A}$$
$$O_2^c = (s\bar{c})_{V-A} (\bar{c} d)_{V-A}$$

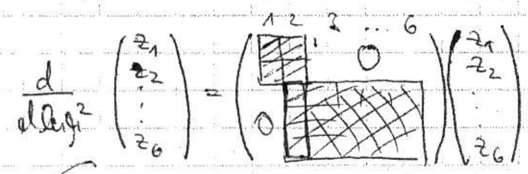
} needs to be included for $\mu > m_c$ due to mixing in the ~~RG~~ running from $M_W \to m_c$ (even if they ~~are~~ not relevant for the process of interest) — do not seem to be

- flavor threshold at $\mu = m_b$: switch from $n_f = 5$ to $n_f = 4$

$\mu < m_c$ → integrate out the $Q_{1,2}^c$ operators & $n_f = 4$ to $n_f = 3$

$$H_{eff}(\Delta S=1) = \frac{4G_F}{\sqrt{2}} V_{us}^* V_{ud} \sum_{i=1}^{6} \big[z_i(\mu) + \tau y_i(\mu) \big] Q_i$$

u.



coefficients of operators $Q_{1,2}$ are numerically dominant

$z_1(1\,\text{GeV}) = -0.6 \qquad y_1 = y_2 = 0$
$z_2(\quad) = 1.3$

$z_3 - z_6,\ y_3 - y_6 < 0.1$ → penguins can be neglected as long as precision $\gtrsim O(10\%)$ is sufficient

- one-loop diagrams with insertions of z_2 (operator Q_2) have UV-divergences with an operator structure of $Q_{3,\dots,6}$
- there are no diagrams with insertions of $z_{3,\dots,6}$ (operators $Q_{3,\dots,6}$) that have UV-divergences with an operator structure of $Q_{1,2}$
- mixing of $Q_{1,2}$ as in the pure current-current case