Problems

Sheet 3, 24.11.2005

Problem 1: Matching with Massive Electrons

Consider the standard QED describing the dynamics of electrons and photons. As discussed in class, for photon momenta q^μ much smaller than m_e one can integrate out the electrons and work with a low-energy effective theory containing only the photons.

a) Take the one-loop photon vacuum polarization diagram with dimensional regularization discussed in class in the $\overline{\mathrm{MS}}$ renormalization scheme and expand it for small q^2/m_e^2. Explain why the first term in the expansion motivates matching onto the effective theory at the scale μ of order m_e rather than, let's say, at $\mu \sim 100$ TeV. The usual convention is to match at $\mu = m_e$.

b) Take the dimension-6 photonic operator in the effective theory discussed in class and show that its Feynman rule can reproduce the momentum-dependence of the second term in the expansion of the one-loop vacuum polarization function. Fix the Wilson coefficient $c_1(\mu = m_e)$ of the dimension-6 operator such that it reproduces the second term exactly. Determine the Wilson coefficients of the effective theory up to dimension-6 (c_0, c_1) for matching at a scale $\mu \neq m_e$.

c) You need to check whether what you did is unambiguous by showing that there is no other photonic dimension-6 operator that is gauge-invariant and might do the same job. Write down a few other possibilities and use integration by parts and the equation of motion in the effective theory $\partial^\mu F_{\mu\nu} = 0$ to show that other possibilities either vanish or reduce to the one you already used in part b).

d) Look up how the electron and photon fields transform under C, P and T and show that QED is C-,P- and T-invariant. Show why these symmetries forbid dimension-6 operators with three field strengths from ever appearing. (This feature is also known as “Furry's Theorem”.)

e) At dimension-8, operators are generated which describe light-by-light scattering ($\gamma\gamma \to \gamma\gamma$). Write down QED one-loop diagrams that can match on these operators and determine the power of α contained in their Wilson coefficients. Use this information and simple dimensional analysis in the effective theory (e.g. take all numbers that arise of order one, but keep momenta, masses and couplings) to obtain a numerical estimate for the cross section $\gamma\gamma \to \gamma\gamma$ for 10 keV photons. Does the reaction happen at a large rate? Compare to QED cross sections you might have computed before in other lectures.

Problem 2: Gauge coupling unification in (SUSY) SU(5) GUT

We discussed SU(5) gauge coupling unification in class. Assuming a SU(5) gauge symmetry exists at very high energy scales with a gauge coupling g_{GUT} one can argue that the symmetry is broken (e.g. by some Higgs mechanism) down to SU(3)×SU(2)×U(1) at some scale $\mu = M_{\rm GUT}$. If $M_{\rm GUT}$ is very much larger than the electroweak scale, one should integrate out all the heavy fields and switch to an effective theory that has the symmetries SU(3)×SU(2)×U(1). (Since this is all a model, one can just assume that all the heavy particles have similar mass of order $M_{\rm GUT}$.) When doing the matching computations at the GUT scale one finds that the breaking pattern requires that the gauge couplings for the unbroken symmetries, g_s, g_2, g_1, are functions of the original gauge couplings,

$$g_s = g_{GUT}\,, \qquad g_2 = g_{GUT}\,, \qquad g_1 = \sqrt{\frac{3}{5}}\, g_{GUT}\,.$$

[s=SU(3),2=SU(2),1=U(1)] One can test whether this unification idea is consistent with low energy data using the measured results for the $\overline{\rm MS}$ couplings obtained from experiments at LEP ($\mu = M_Z = 91.2$ GeV),

$$(\sin^2\theta_W)(M_Z) = 0.232\,, \qquad \alpha(M_Z) = (128.9)^{-1}\,, \qquad \alpha_s(M_Z) = 0.118\pm 0.003\,.$$

The error in the electromagnetic coupling and the Weinberg angle is at the level of 0.1 %.

a) Assume that the effective low energy theory below M_{GUT} is the Standard Model. Determine the values for the $\overline{\mathrm{MS}}$ couplings $\alpha_s = g_s^2/(4\pi)$, $\alpha_2 = g_2^2/(4\pi)$ and $\alpha_1 = 5g_1^2/(12\pi)$ at the scale M_Z. The LL RGE's for these couplings have the form

$$\frac{d}{d\ln\mu^2}\alpha_i = -\frac{\alpha_i^2}{4\pi}b_i$$

with $b_1 = -2/3n_f - 1/10n_h$, $b_2 = 22/3 - 2/3n_f - 1/6n_h$, $b_3 = 11 - 2/3n_f$, where n_f is the number of quarks and n_h the number of Higgs doublets. (You can treat the top quark as a light quark in this context. Think about why this is still a valid approximation in this case.) Compute the LL solution for the running couplings above M_Z taking the values at M_Z as an input. Is the SU(5) unification scenario consistent with low energy data?

b) Assume that the effective theory below M_{GUT} is not the Standard Model, but the minimal supersymmetric Standard Model (MSSM). Since no supersymmetric (SUSY) partner of any Standard Model has ever been seen, this scenario is only possible if all SUSY partners are much heavier than any of the Standard Model particles. Let us assume that all SUSY partners have similar masses of order M_{SUSY}. So for scales below M_{SUSY} one can integrate out the SUSY partners finally arriving at the Standard Model as the effective theory for scales below M_{SUSY}. The matching conditions for the gauge couplings at $\mu = M_{\mathrm{SUSY}}$ are just as discussed in class. The MSSM anomalous dimensions have the form $b_1 = -n_f - 3/10n_h$, $b_2 = 6 - n_f - 1/2n_h$, $b_3 = 9 - n_f$, where n_f is the number of quarks (in the Standard Model) and n_h the number of Higgs doublets. (In the MSSM one has two Higgs doublets!) Compute the LL solution for the running couplings above M_{SUSY} taking the values at M_Z as an input. (Note that this can be done in very compact form.) Can you find scales M_{SUSY} and M_{GUT} such that unification is realized at the scale M_{GUT}? Note that you should account for the fact that the low energy data for the couplings have experimental uncertainties. Is the SUSY SU(5) unification scenario consistent with low energy data? Which scales for M_{SUSY} are the ones most favored by the analysis?

c) Think about the conditions that needed to be satisfied to make the LL analysis carried out above valid. What does the analysis tell you? Discuss the physical implications.