Problems

Sheet 3, 17.11.2005

Problem 1: Change of Renormalization Prescription

The $\overline{\mathrm{MS}}$ subtraction scheme belongs to the class of "mass-independent" renormalization prescriptions. Another such scheme is the MS subtraction scheme. So, consider the coupling in some theory defined in two mass-independent renormalization prescriptions, λ and $\bar{\lambda}$. Their relation can be expressed in terms of a perturbative series,

$$\bar{\lambda}(\mu) = \lambda(\mu) + a_1\lambda(\mu)^2 + a_2\lambda(\mu)^2 + a_3\lambda(\mu)^3 + \dots .$$

Determine the coefficients a_i for the couplings in the MS (λ) and the $\overline{\mathrm{MS}}$ ($\bar{\lambda}$) schemes for the ϕ^4-theory. Prove that the first two coefficients of the β-functions in both schemes,

$$\begin{aligned} \bar{\beta}_{\bar{\lambda}} &= \frac{d\bar{\lambda}}{d\ln\mu^2} = \bar{A}_1\bar{\lambda}^2 + \bar{A}_2\bar{\lambda}^3 + \bar{A}_3\bar{\lambda}^4 + \dots , \\ \beta_\lambda &= \frac{d\lambda}{d\ln\mu^2} = A_1\lambda^2 + A_2\lambda^3 + A_3\lambda^4 + \dots \end{aligned} \tag{1}$$

are the same. Do the results also apply to QED and QCD?

Problem 2: Renormalization Group Equations at Higher Order

a) The renormalization constants of ϕ^4-theory at two-loop order in the $\overline{\mathrm{MS}}$ scheme read

$$
\begin{aligned}
Z_\lambda &= 1 + \frac{3\lambda}{(32\pi^2)\epsilon} + \frac{\lambda^2}{(32\pi^2)^2}\left[\frac{9}{\epsilon^2} - \frac{17}{3\epsilon}\right] + \dots\,, \\
Z_m &= 1 + \frac{\lambda}{(32\pi^2)\epsilon} + \frac{\lambda^2}{(32\pi^2)^2}\left[\frac{2}{\epsilon^2} - \frac{5}{6\epsilon}\right] + \dots\,, \\
Z_\phi &= 1 - \frac{\lambda^2}{(32\pi^2)^2}\frac{1}{6\epsilon} + \dots\,.
\end{aligned}
$$

Determine the two-loop renormalization group equations.

Solve the renormalization group equations. To keep the solution elementary it can be helpful to recall that you anyway need to only consider the case $\lambda \ll 1$.

b) Use the relation between the renormalized and the bare coupling, $\lambda(\mu) = \tilde{\mu}^{-2\epsilon} Z_\lambda^{-1} \lambda_0$, to derive the general expression for $d\lambda/d\ln\mu^2$ in terms of Z_λ in $d = 4 - 2\epsilon$ dimensions. Use the fact that $d\lambda/d\ln\mu^2$ does not diverge for $\epsilon \to 0$ to show that the coefficient of the $1/\epsilon^2$ term in Z_λ can be determined from the coefficient of the $1/\epsilon$ term. For this make the ansatz

$$Z_\lambda = 1 + \sum_{n=1}^{\infty} \frac{z_\lambda^{(n)}(\lambda)}{\epsilon^n}$$

and use the fact that $z_\lambda^{(n)}$ is of order λ^n, i.e. $z_\lambda^{(n)} = a_{n,1}\lambda^n + a_{n,2}\lambda^{n+1} + \dots$. Cross-check the result with the two-loop result of Z_λ in ϕ^4-theory. Explain that in fact all $z_\lambda^{(n)}$ with $n \geq 2$ can be determined from $z_\lambda^{(1)}$. For the ambitious: derive the λ^3/ϵ^3 and λ^3/ϵ^2 three-loop contributions of Z_λ.

Problem 3: Asymptotic Expansion & Power Counting

Consider the following one-dimensional integral

$$f(a) \equiv \int_{-\infty}^{\infty} dk \, \frac{|\arctan(k)|}{(k^2+a^2)^2} \, .$$

You can think of this integral as being a simplified version of a loop-Feynman diagram where the denominator corresponds to a propagator structure. It is your task to compute the expansion for small $a \ll 1$. Naive expansion in a before integration does not work because of an IR singularity. You might want to compute the integral exactly and then expand the result for small a, but this is very difficult. Instead use the two methods below. Use Mathematica or Maple for the computations.

a) (**Cutoff Method**)
In the limit $a \ll 1$ the integral is governed by the two regions $k \sim 1$ ("hard") and $k \sim a \ll 1$ ("soft"). That's the reason why naive expansion does not work. Separate the soft and the hard regions by introducing a cutoff Λ with $a \ll \Lambda \ll 1$ which splits the integral into two parts, $\int_{|k|<\Lambda}$ and $\int_{|k|>\Lambda}$. Carry out the Taylor expansions that now become possible in the two regions and do the integrations. Expand the individual results of the integration using that $a \ll \Lambda \ll 1$ and add back the results. In this way determine the expansion of $f(a)$ neglecting term at order a or higher. You might check your result numerically.

Since Λ has been introduced by hand the result should be independent of Λ at any order in the Λ expansions. Which problem emerges?

b) (**Dimensional Regularization**)
You can use dim reg to do a similar computation. While the cutoff method is probably quite intuitive and easy to understand for you, using dim reg involves a number of subtle issues you have to get used to. So, first continue the integral to $\bar{D} = 1 - 2\epsilon$ dimensions,

$$\int_{-\infty}^{+\infty} dk \to \tilde{\mu}^{2\epsilon} \int d^{\bar{D}}k = \frac{\Omega(\bar{D})}{\tilde{\mu}^{-2\epsilon}} \int_0^{\infty} dk k^{-2\epsilon} \, ,$$

where $\Omega(\bar{D}) = (2\,\pi^{\frac{\bar{D}}{2}})/(\Gamma(\frac{\bar{D}}{2}))$ is the $\bar{D}$-dimensional angular integral and $\tilde{\mu} = \mu(e^{\gamma_E + \ln 4\pi})^{1/2}$. ($\gamma_E$ is the Euler number, which will arise when you later expand the Γ functions for small ϵ.)

b$_1$) Expand the integrand for the soft regime ($a, k \ll 1$) as described in a), integrate the terms in $\bar{D}$ dimensions and expand for $\epsilon \to 0$. Remember what you have learned in class about doing integrations in d dimensions. Have a look at the terms you obtained in the expansion for small k before integration and observe the order in a they contribute. Establish power counting rules for the soft regime that tell you (before integration!) to which order each term contributes. Determine all terms up to order a

b$_2$) Expand the integrand for the hard regime ($a \ll 1$) as described in a), carry out the integral in $\bar{D}$ dimensions and expand for $\epsilon \to 0$. Establish the power counting rules in the hard regime and determine all terms up to order a

b$_3$) Now add the contributions you got from the expansions in the soft and the hard regime. The result is the expansion of $f(a)$ for small a and should agree with your result from a). (If you are motivated, compute also the order a and a^2 terms.) Think about how this could have worked out. Which way of computing the expansion do you find more attractive?

c) Repeat a) and b) for the integral ranging from -1 to 1 instead of $-\infty$ to ∞. Redo only the parts in the computation that have to be modified.