Problems

Sheet 1, 03.11.2005

Problem 1: Muon Decay

In the Fermi theory the amplitude for muon decay $\mu \to e^- \nu_\mu \bar{\nu}_e$ reads

$$\mathcal{M}_{Fermi} = -i\frac{4G_F}{\sqrt{2}}\,[\bar{u}_{\nu_\mu}\gamma_\mu P_L u_{\mu^-}]\,[\bar{u}_{e^-}\gamma^\mu P_L v_{\nu_e}],$$

where $P_{L/R} = \frac{1}{2}(1 \mp \gamma^5)$. Derive the Fermi constant G_F within the Standard Model and determine the μ decay width ($m_e = 0$).
Find the μ width (from the PDG) and determine G_F (with errors).

Problem 2

Think back how the SM is constructed and which factors are relevant for the muon decay. What is the result for the Fermi constant in a theory in which the left-handed electron and muon fields are in triplets (with all right-handed fields in singlets) under $SU(2) \times U(1)$ as follows:

$$\psi_{eL} = \begin{pmatrix} E_L^+ \\ \nu_{eL} \\ e_L^- \end{pmatrix} \qquad \psi_{\mu L} = \begin{pmatrix} M_L^+ \\ \nu_{\mu L} \\ \mu_L^- \end{pmatrix}$$

where E^+ and M^+ are heavy (unobserved) lepton fields?

The $SU(2)$ generators of the triplet representation are:

$$T_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad T_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad T_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Note that $[T_a, T_b] = i\epsilon_{abc}T_c$.

Problem 3: Neutrino-Electron Elastic Scattering

In the $SU(2) \times U(1)$ theory, both $W^\pm$ and Z^0 exchange contribute to the elastic scattering process $\bar{\nu}_e e^- \to \bar{\nu}_e e^-$. For momentum transfers very small compared to M_W, show that the amplitude for this process can be written as

$$\mathcal{M}_0 = -i\frac{G_F}{\sqrt{2}}\, \bar{\nu}_e \gamma_\mu (1-\gamma_5)\nu_e \; \bar{e}^- \gamma^\mu (G_V - G_A\gamma_5) e^- \ .$$

Find G_V and G_A.

Problem 4: Fierz Transformations

Find the constants C_j in the following relations:

(a) $\quad [\gamma^\mu P_\pm]_{ij} [\gamma_\mu P_\pm]_{kl} = C_1 [\gamma^\mu P_\pm]_{il} [\gamma_\mu P_\pm]_{kj}$

(b) $\quad [\gamma^\mu P_\pm]_{ij} [\gamma_\mu P_\mp]_{kl} = C_2 [P_\mp]_{il} [P_\pm]_{kj}$ $\quad (b')$ Note that $[\sigma^{\mu\nu} P_\mp]_{il} [\sigma_{\mu\nu} P_\pm]_{kj} = 0$

(c) $\quad [P_\pm]_{ij} [P_\pm]_{kl} = C_3 [P_\pm]_{il} [P_\pm]_{kj} + C_4 [\sigma^{\mu\nu} P_\pm]_{il} [\sigma_{\mu\nu} P_\pm]_{kj}$

(d) $\quad [\sigma^{\mu\nu} P_\pm]_{ij} [\sigma_{\mu\nu} P_\pm]_{kl} = C_5 [P_\pm]_{il} [P_\pm]_{kj} + C_6 [\sigma^{\mu\nu} P_\pm]_{il} [\sigma_{\mu\nu} P_\pm]_{kj}$

These six numbers are all there is to Fierz transformations.

Hint: For (a) and (b), multiply by γ^ν_{lk} and sum over l and k. For (c) and (d), it's easiest to avoid taking traces of $\sigma^{\mu\nu}$, so multiply by δ_{lk} and sum over l and k to get one equation. Multiply by δ_{jk} and sum over j and k to get another. To prove (b'), multiply by γ^λ_{lk} and sum over l and k.