Homework 5 (May 26, 2004)

Problem 1: Gauge coupling unification in (SUSY) SU(5) GUT

We discussed SU(5) gauge coupling unification in class. Assuming a SU(5) gauge symmetry exists at very high energy scales with a gauge coupling g_{GUT} one can argue that the symmetry is broken (e.g. by some Higgs mechanism) down to SU(3)×SU(2)×U(1) at some scale $\mu = M_{\rm GUT}$. If $M_{\rm GUT}$ is very much larger than the electroweak scale, one should integrate out all the heavy fields and switch to an effective theory that has the symmetries SU(3)×SU(2)×U(1). (Since this is all a model, one can just assume that all the heavy particles have similar mass of order $M_{\rm GUT}$.) When doing the matching computations at the GUT scale one finds that the breaking pattern requires that the gauge couplings for the unbroken symmetries, g_s, g_2, g_1, are functions of the original gauge couplings,

$$g_s = g_{GUT}\,, \qquad g_2 = g_{GUT}\,, \qquad g_1 = \sqrt{\frac{3}{5}}\, g_{GUT}\,.$$

(The conventions for the gauge couplings used in the lectures on the Standard Model are used here.) One can test whether this unification idea is consistent with low energy data using the measured results for the $\overline{\rm MS}$ couplings obtained from experiments at LEP ($\mu = M_Z = 91.2$ GeV),

$$(\sin^2\theta_W)(M_Z) = 0.232\,, \qquad \alpha(M_Z) = (128.9)^{-1}\,, \qquad \alpha_s(M_Z) = 0.118 \pm 0.003\,.$$

The error in the electromagnetic coupling and the Weinberg angle is at the level of 0.1 %.

a) Assume that the effective low energy theory below $M_{\rm GUT}$ is the Standard Model. Determine the values for the $\overline{\rm MS}$ couplings $\alpha_s = g_s^2/(4\pi)$, $\alpha_2 = g_2^2/(4\pi)$ and $\alpha_1 = 5g_1^2/(12\pi)$ at the scale M_Z. The LL RGE's for these couplings have the form

$$\frac{d}{d\ln\mu^2}\alpha_i = -\frac{\alpha_i^2}{4\pi} b_i$$

with $b_1 = -2/3n_f - 1/10n_h$, $b_2 = 22/3 - 2/3n_f - 1/6n_h$, $b_3 = 11 - 2/3n_f$, where n_f is the number of quarks and n_h the number of Higgs doublets. (You can treat the top quark as a light quark in this context. Think about why this is a valid approximation.) Compute the LL solution for the running couplings above M_Z taking the values at M_Z as an input. Is the SU(5) unification scenario consistent with low energy data?

b) Assume that the effective theory below $M_{\rm GUT}$ is not the Standard Model, but the minimal supersymmetric Standard Model (MSSM). Since no supersymmetric (SUSY) partner of any Standard Model has ever been seen this scenario is only possible if all SUSY partners are much heavier than any Standard Model particle. Let us assume that all SUSY partners have similar masses or order $M_{\rm SUSY}$. So for scales below $M_{\rm SUSY}$ one integrate out the SUSY partners finally arriving at the Standard Model as the effective theory for scales below $M_{\rm SUSY}$. The matching conditions for the gauge couplings at $\mu = M_{\rm SUSY}$ are as discussed in class.

The MSSM anomalous dimensions have the form $b_1 = -n_f - 3/10n_h$, $b_2 = 6 - n_f - 1/2n_h$, $b_3 = 9 - n_f$, where n_f is the number of quarks (in the Standard Model) and n_h the number of Higgs doublets. (In the MSSM one has two Higgs doublets!) Compute the LL solution for the running couplings above $M_{\rm SUSY}$ taking the values at M_Z as an input. (Note that this can be done in very compact form.) Can you find scales $M_{\rm SUSY}$ and $M_{\rm GUT}$ such that unification is realized at the scale $M_{\rm GUT}$? Note that you should account for the fact that the low energy data for the couplings have experimental uncertainties. Is the SUSY SU(5) unification scenario consistent with low energy data? Which scales for $M_{\rm SUSY}$ are the ones most favored by the analysis?

c) Think about the conditions that needed to be satisfied to make the LL analysis carried out above valid. What does the analysis tell you? Discuss the physical implications.

Problem 2: Semileptonic and non-leptonic b decay rate

a) Calculate the semileptonic decay rate of a free b quark $\Gamma(b \to c e \bar{\nu}_e)$. Use the effective Hamiltonian discussed in class.

b) Use the effective Hamiltonian discussed in class to compute the non-leptonic free quark decay rate $\Gamma(b \to c d \bar{u})$. Neglect all masses except those of the the b and c quarks. Note the difference between the two computations.