Homework 4 (May 10, 2004)

Problem 1: Change of Renormalization Prescription

The $\overline{\rm MS}$ subtraction scheme belongs to the class of "mass-independent" renormalization prescriptions. Another such scheme is the MS subtraction scheme. Consider the coupling in some theory defined in two mass-independent renormalization prescriptions, λ and $\bar{\lambda}$. Their relation can be expressed in terms of a perturbative series,

$$\bar{\lambda}(\mu) = \lambda(\mu) + a_1\lambda(\mu)^2 + a_2\lambda(\mu)^2 + a_3\lambda(\mu)^3 + \ldots .$$

Prove that the first two coefficients of the β-functions in both schemes,

$$\begin{aligned} \bar{\beta}_{\bar{\lambda}} &= \frac{d\bar{\lambda}}{d\ln\mu^2} = \bar{A}_1\bar{\lambda}^2 + \bar{A}_2\bar{\lambda}^3 + \bar{A}_3\bar{\lambda}^4 + \ldots , \\ \beta_{\lambda} &= \frac{d\lambda}{d\ln\mu^2} = A_1\lambda^2 + A_2\lambda^3 + A_3\lambda^4 + \ldots \end{aligned} \tag{1}$$

are the same.

Problem 2: Renormalization Group Equations at Higher Order

a) Consider the renormalization constants of ϕ^4-theory at two-loop order discussed in class and determine the two-loop renormalization group equations. Solve the renormalization group equation for the field ϕ at two-loop order. Do not forget to check out the dimensions of renormalized fields, masses and couplings as we did it in class for QED!

b) Use the relation between the renormalized and the bare coupling, $\lambda(\mu) = \tilde{\mu}^{-2\epsilon} Z_\lambda^{-1}\lambda_0$, to derive the general expression for $d\lambda/d\ln\mu^2$. Use the fact that $d\lambda/d\ln\mu^2$ does not diverge for $\epsilon \to 0$ to show that the coefficient of the $1/\epsilon^2$ term in Z_λ can be determined from the coefficient of the $1/\epsilon$ term. Make the ansatz

$$Z_\lambda = 1 + \sum_{n=1}^{\infty} \frac{z_\lambda^{(n)}(\lambda)}{\epsilon^n}$$

and use the fact that $z_\lambda^{(n)}$ is of order λ^n, i.e. $z_\lambda^{(n)} = a_{n,1}\lambda^n + a_{n,2}\lambda^{n+1} + \ldots$. Cross-check the result with the two-loop result of Z_λ in ϕ^4-theory. Explain that in fact all $z_\lambda^{(n)}$ with $n \geq 2$ can be determined from $z_\lambda^{(1)}$. For the ambitious: derive the λ^3/ϵ^3 and λ^3/ϵ^2 three-loop contributions of Z_λ.

Problem 3: Matching with Massive Electrons

Consider the standard QED describing the dynamics of electrons and photons. As discussed in class, for photon momenta q^μ much smaller than m_e one can integrate out the electrons and work with a low-energy effective theory containing only the photons.

a) Take the one-loop photon vacuum polarization diagram with dimensional regularization discussed in class in the $\overline{\text{MS}}$ renormalization scheme and expand it for small q^2/m_e^2. Explain why the first term in the expansion motivates matching onto the effective theory at the scale μ of order m_e rather than, let's say, at $\mu \sim 100$ TeV. The usual convention is to match at $\mu = m_e$.

b) Take the dimension-6 photonic operator in the effective theory discussed in class and show that its Feynman rule can reproduce the momentum-dependence of the second term in the expansion of the one-loop vacuum polarization function. Fix the Wilson coefficient $c_1(\mu = m_e)$ of the dimension-6 operator such that it reproduces the second term exactly. Determine the Wilson coefficients of the effective theory up to dimension-6 (c_0, c_1) for matching at a scale $\mu \neq m_e$.

c) You need to check whether what you did is unambiguous by showing that there is no other photonic dimension-6 operator that is gauge-invariant and might do the same job. Write down a few other possibilities and use integration by parts and the equation of motion in the effective theory $\partial^\mu F_{\mu\nu} = 0$ to show that other possibilities either vanish or reduce to the one you already used in part b).

d) Show that QED is invariant under each of the transformations C, P, T. Show why these symmetries forbid dimension-6 operators with three field strengths from ever appearing.

e) At dimension-8, operators are generated which describe light-by-light scattering ($\gamma\gamma \to \gamma\gamma$). Write down QED one-loop diagrams that can match on these operators and determine the power of α contained in their Wilson coefficients. Use this information and simple dimensional analysis in the effective theory (e.g. take all numbers that arise of order one, but keep momenta, masses and couplings) to obtain a numerical estimate for the cross section $\gamma\gamma \to \gamma\gamma$ for 10 keV photons. Compare to QED cross sections you might have computed before in other lectures.