Homework 3 (May 5, 2004)

Problem 1: Asymptotic Expansion & Power Counting

Consider the following one-dimensional integral

$$f(a) \equiv \int\limits_{-\infty}^{\infty} dk \, \frac{|\arctan(k)|}{(k^2 + a^2)^2} \, .$$

Think of this integral as being a simplified version of a Feynman diagram where the denominator corresponds to a propagator structure. It is your task to compute the expansion for small $a \ll 1$. Naive expansion in a before integration does not work because of an IR singularity. You might want to compute the integral exactly and then expand the result for small a, but this is very difficult. Instead use the two methods below. Use Mathematica or Maple for the computations.

a) (**Cutoff Method**)
In the limit $a \ll 1$ the integral is governed by the two regions $k \sim 1$ ("hard") and $k \sim a \ll 1$ ("soft"). That's the reason why naive expansion does not work. Separate the soft and the hard regions by introducing a cutoff Λ with $a \ll \Lambda \ll 1$ which splits the integral into two parts, $\int_{|k|<\Lambda}$ and $\int_{|k|>\Lambda}$. Carry out the Taylor expansions that now become possible in the two regions and do the integrations. Expand the individual results of the integration using that $a \ll \Lambda \ll 1$ and add back the results. In this way determine the expansion of $f(a)$ neglecting term at order a or higher. You might check your result numerically.

Since Λ has been introduced by hand the result should be independent of Λ at any order in the Λ expansions. Which problem emerges?

b) (**Dimensional Regularization**)
You can use dim reg to do a similar computation. First continue the integral to $\bar{D} = 1 - 2\epsilon$ dimensions,

$$\int_{-\infty}^{+\infty} dk \to \tilde{\mu}^{2\epsilon} \int d^{\bar{D}}k = \frac{\Omega(\bar{D})}{\tilde{\mu}^{-2\epsilon}} \int_0^\infty dk k^{-2\epsilon} \, ,$$

where $\Omega(\bar{D}) = (2\,\pi^{\frac{\bar{D}}{2}})/(\Gamma(\frac{\bar{D}}{2}))$ is the $\bar{D}$-dimensional angular integral and $\tilde{\mu} = \mu(e^{\gamma_E + \ln 4\pi})^{1/2}$. ($\gamma_E$ is the Euler number, which will arise when you later expand the Γ functions for small ϵ.)

b$_1$) Expand the integrand for the soft regime ($a, k \ll 1$) as described in a), integrate the terms in $\bar{D}$ dimensions and expand for $\epsilon \to 0$. Have a look at the terms you obtained in the expansion for small k before integration and observe the order in a they contribute. What (important) difference to the computation with a cutoff arises? Establish power counting rules for the soft regime that tell you (before integration!) to which order each term contributes. Determine all terms up to order a

b_2) Expand the integrand for the hard regime ($a \ll 1$) as described in a), carry out the integral in $\bar{D}$ dimensions and expand for $\epsilon \to 0$. Establish the power counting rules in the hard regime and determine all terms up to order a

b_3) Now add the contributions you got from the expansions in the soft and the hard regime. The result is the expansion of $f(a)$ for small a and should agree with your result from a). (If you are motivated, compute also the order a and a^2 terms.) Think about how this could have worked out. Which way of computing the expansion do you find more attractive?