O. Steinmann:

ABELIAN HIGGS-KIBBLE MODEL

Start from a complex scalar field $\phi(x)$ and a real vector field $A_\mu(x)$.

Dynamics:

$$\mathcal{L} = -\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} + (\partial_\alpha - igA_\alpha)\phi^* \times$$
$$\times (\partial^\alpha + igA^\alpha)\phi + \mu^2 \phi^* \phi - \lambda (\phi^* \phi)^2$$

with $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$.

$g, \lambda, \mu > 0$. Mass term $\mu^2 \phi^* \phi$ has the „wrong" sign!

$\mathcal{L}$ invariant under gauge transformation

$$\phi(x) \Rightarrow e^{ig\theta(x)} \phi(x), \quad A_\mu(x) \Rightarrow A_\mu(x) - \partial_\mu \theta(x),$$

for $\theta(x)$ real.

The corresponding equations of motion have (classical) solution of lowest energy

$$\phi(x) = \phi^*(x) = \frac{v}{\sqrt{2}} \quad , \quad A_\mu(x) = 0$$

with

$$v = \frac{\mu}{\sqrt{\lambda}} > 0$$

Perturbative quantization : expansion around this solution.

<u>Ansatz:</u>

$$\phi(x) = \frac{v + R(x) + i\,I(x)}{\sqrt{2}}$$

R, I : real scalar fields.

Minimal solution becomes

$$R = I = A_\mu = 0$$

Henceforth: R, I, A_μ are the basic fields of the formalism. ϕ is forgotten.

Gauge transformations transcribable in new fields, but no longer of basic interest.

Except: The „physical" fields $F_{\alpha\beta}(x)$ and

$$\Psi(x) = R(x) + \frac{1}{2v}\left[R^2(x) + \Gamma^2(x)\right]$$

(Higgs field) are gauge invariant

Transcribe $\mathcal{L}$ into new fields:

$$\mathcal{L} = \mathcal{L}_2 + \underbrace{\mathcal{L}_3 + \mathcal{L}_4}_{\mathcal{L}_{int}} ,$$

$\mathcal{L}_i$: terms of order i in basic fields

A constant $\mathcal{L}_0$ has been dropped as irrelevant.

And: Replace λ, μ, as parameters by

$$m = gv = \frac{g\mu}{\sqrt{\lambda}} , \quad M = \sqrt{2}\,\mu .$$

They are the masses of the gauge boson and the Higgs particle, hence measurable.

Then

$$\Psi(x) = R(x) + \frac{g}{2m}\left[R^2(x) + I^2(x)\right]$$

Field equations (embodiment of dynamics):

$$(\Box + m^2) A^\mu - \partial^\mu \partial_\nu A^\nu + m\,\partial^\mu I = -\frac{\delta \mathcal{L}_{int}}{\delta A_\mu}$$

$$=: R^\mu$$

$$-(\Box + M^2) R = -\frac{\delta \mathcal{L}_{int}}{\delta R} =: R_R$$

$$-\Box I - m\,\partial_\nu A^\nu = -\frac{\delta \mathcal{L}_{int}}{\delta I} =: R_I$$

Gauge invariance ⇒ Apply ∂_μ to l.h.s. of A^μ-equation. Gives $-m$ (l.h.s. of I-equation).

Hence these equations possess a solution only if the consistency condition („Ward identity")

$$\boxed{\mathcal{F} := \partial_\mu R^\mu + m R_I = 0}$$

is satisfied. It is satisfied in our case (formally, i.e. nonrenormalized), essentially because the field equations are derived from a Lagrangian.

Important: Explicit verification like this:
$\mathcal{L}_3$-contribution to $\mathcal{F}$ is

$$\mathcal{F}_3 = gI(\Box + M^2)R - gR(\Box I + m\partial_\mu A^\mu)$$

Insertion of R- and I-equations yields term of 3^{rd} order in the fields, which is cancelled by the $\mathcal{L}_4$-contribution $\mathcal{F}_4$ to $\mathcal{F}$: Verification of Ward identity uses field equations! Looks fishy, but is no problem in PT.

Linear dependence of field equations
$\Longrightarrow$ gauge freedom $\Longrightarrow$ question: what is the physical content of the theory?

Answer lies in:

„Unitary" gauge = „Physical gauge"

Defined by gauge condition

$$\boxed{I = 0}$$

Left with 2 fields only : R, A_μ.

$\Rightarrow$ Need only 2 field equations $\Longrightarrow$

Forget I-equation (consequence of A-eq.)

$\Rightarrow$ Field equations:

$$(\Box + m^2)A^\mu - \partial^\mu \partial_\nu A^\nu = \mathcal{R}^\mu$$
$$(= -2gmA^\mu R - g^2 A^\mu R^2)$$

$$-(\Box + M^2) R = \mathcal{R}_R \quad (= \cdots\cdots)$$

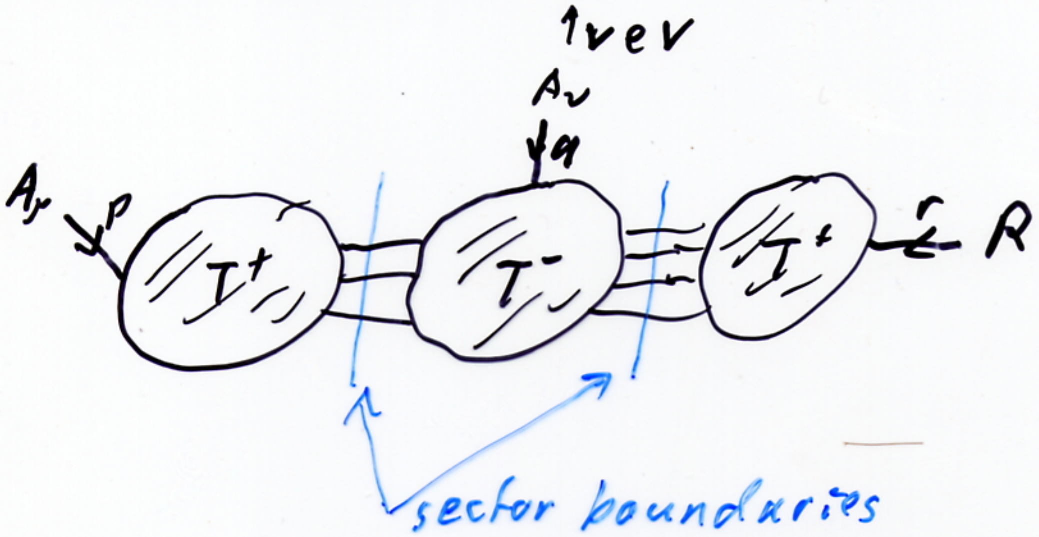
Perturbation theory (my version)

Fundamental objects are not the Green's functions but the Wightman functions. They must satisfy the field equations. $\mathcal{L}$ no longer of fundamental importance.

Result: The W-functions are described by sector graphs (generalized Feynman graphs)

Example: (now in p-space)

$$\langle A_\mu(p)\, A_\nu(q)\, R(r) \rangle_0 = \sum \text{sector graphs}$$



T^+-bubble : usual Feynman graphs, with

sector-propagators : In T^+-sectors

$$\Delta^+_{\mu\nu} = -\frac{i}{2\pi}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{m^2}\right)\frac{1}{p^2-m^2+i\varepsilon}$$

$$\Delta^+_{RR} = \frac{i}{2\pi}\,\frac{1}{p^2-M^2+i\varepsilon} ,$$

vertices as usual derived from $\mathcal{L}_{int}$

In T^--sector : complex-conjugates. (in x-space)

cross-propagators : Solutions of the free field equations, obtained from Δ^+ by

$$\frac{i}{p^2-m^2+i\varepsilon} \Longrightarrow \delta_+^m(p)$$

(also for $m \to M$)

Justification According to these rules the sector propagators are the ~~time~~ (anti-) time-ordered functions of the free W-functions serving as cross propagators.

Ostendorf Theorem: These rules define W-functions satisfying all Wightman properties (Lorentz invariance, locality, spectrum, cluster property), in the case of the unitary gauge even positivity.

Lit.: A. Ostendorf: Ann. I. H. Poincaré 40, 273, 1984
O.S.: Comm. Math. Phys. 152, 627, 1993
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But must also solve the field equations. This is the case, if the sector propagators as defined above are propagators in the sense of the theory of differential eqs.

Means the following:

Transform the field eqs. into p-space. Write them in matrix form

$$L\varphi = R,$$

$$\varphi = \begin{pmatrix} A_\nu \\ R \end{pmatrix}, \quad R = \begin{pmatrix} R_\nu \\ R_R \end{pmatrix}$$

$$L = \begin{array}{cc} & \nu \rightarrow \\ \mu \downarrow & \left(\begin{array}{c|c} -(p^2-m^2)\delta^\mu_\nu + p^\mu p_\nu & 0 \\ \hline 0 & p^2 - M^2 \end{array}\right) \end{array}$$

Then the red eq. is solved by

$$\varphi = P R$$

with

$$P = L^{-1} = \left(\begin{array}{c|c} -\dfrac{\delta^\nu_\varrho - \dfrac{p^\nu p_\varrho}{m^2}}{p^2 - m^2} & 0 \\ \hline 0 & \dfrac{1}{p^2 - M^2} \end{array}\right).$$

This is well defined except at the mass shells $p^2 = m^2$ or $p^2 = M^2$. But there the Ostendorf theorem demands the choice

$$\rho_{ab} = \mp 2\pi i \, r^{\pm}_{ab} \quad \text{in } T^{\pm}\text{-sectors.}$$

This leads to the familiar choice $(p^2 - m^2 \pm i\varepsilon)^{-1}$ etc. Apart from that this propagator condition also fixes the normalization of the free 2-point functions serving as cross propagators. Hence this condition replaces fully the ccr of the canonical formalism.

Note: From these W-functions we can calculate the time-ordered functions of n interacting fields, as being given by the familiar Feynman rules. This is essential for LSZ reduction formalism.

„Physical" gauge?

Physical objects of a theory:

a) Observables: must be gauge invariant

b) Physical states

$\mathcal{V}_U$: state space of the U-gauge

At first: $\mathcal{V}_U$ is generated from the vacuum Ω by applying polynomials in the fields A_μ, R. (Wightman)

But: The same space is generated by the gauge invariant fields $F_{\alpha\beta}$, Ψ

Proof: A_μ and R can be expressed as functions of $F_{\alpha\beta}$ and Ψ:

i) $\Psi = R + \frac{g}{2m} R^2$

solvable for R. Unproblematic in perturbation theory:

$$R = \sum_{\sigma=0}^{\infty} R_\sigma g^\sigma$$

$$\Longrightarrow \quad R_0 = \Psi_0 ,$$

$$R_1 = \Psi_1 - \frac{1}{m} R_0^2 = \Psi_0 - \frac{1}{m} \Psi_0^2$$

etc.

ii) $\partial^\alpha F_{\alpha\beta} = \Box A_\beta - \partial_\beta \partial^\alpha A_\alpha = -m^2 A_\beta + R_\beta (A_\mu, R)$

$$\Longrightarrow \quad m^2 A_\beta = -\partial^\alpha F_{\alpha\beta} + R_\beta (A_\mu, R)$$

Solvable by induction, since R_β contains an explicit factor g.

Hence: $\mathcal{V}_u$ is, in a sense, gauge invariant and thus an appropriate candidate for „physical state space".

Problem: The A-A-propagator

$$\frac{g_{\mu\nu} - \frac{p_\mu p_\nu}{m^2}}{p^2 - m^2 + i\varepsilon}$$

has a bad high-energy behaviour $\Rightarrow$ theory non-renormalizable?

Means: individual graphs in high orders have a very bad, non-renormalizable UV-behavior! Claiming that the theory is nevertheless renormalizable amounts to claiming extensive cancellations between graphs.

Standard procedure: add „gauge-fixing term" $-\frac{1}{2\alpha}(\partial_\mu A^\mu)^2$ to $\mathcal{L}$ $\Rightarrow$ renormalizable theory.
But: misnomer, physical equivalence to HK must be proved!

More natural: If the desired cancellations happen, this must be provable in the correct theory!

Difficult to achieve in U-gauge. Easier in an appropriate different (true!) gauge.

WIGHTMAN GAUGES

A class of covariant, local, gauges, obtained as follows:

Start from local, covariant 2-point functions solving the free field equations of the HK-model. Use them as cross-propagators in the sector graphs of a given W-function. Calculate the corresponding (anti-)time ordered 2-point functions. Use them as sector propagators.
(The vertices are derived as usual from $\mathcal{L}_{int}$)

Then (Osterwalder theorem) these W-functions satisfy the Wightman properties. They define a field theory.

But these W must solve the interacting field equations. This is achieved by adding an additional requirement:

The sector propagators $\tau^{\pm}(p)$ must be connected to propagators in the PDE sense by

$$P_{ab} = \mp i\, \tau^{\pm}(p).$$

(This condition fully replaces the ccr of the canonical formalism).

<u>Def.</u> The matrix P is a propagator matrix if it satisfies

$$LPR = R,$$

with

$$L = \begin{array}{c} \\ \mu\downarrow \\ I \\ R \end{array} \begin{array}{c} \begin{array}{ccc} \nu\rightarrow & I & R \end{array} \\ \left(\begin{array}{c|c|c} -(p^2-m^2)\delta^\mu{}_\nu + p^\mu p_\nu & -imp^\mu & 0 \\ \hline imp_\nu & p^2 & 0 \\ \hline 0 & 0 & p^2-M^2 \end{array}\right) \end{array}$$

and R a 6-vector $\begin{pmatrix} R^\mu \\ R_I \\ R_R \end{pmatrix}$ satisfying the

Ward identity

$$-ip_\mu R^\mu + m R_I = 0$$

<u>Note</u>: L is singular, hence P cannot be defined (like in U-gauge) as $P = L^{-1}$

These conditions fix the propagators of our graph rules up to an arbitrary real covariant function $T(p^2)$.

The case most suited to our purpose is:

$$\langle T\varphi_1(p)\,\varphi_2(q)\rangle_0 =: \tau^+_{12}(p)\,\delta^4(p+q)$$

$$\boxed{\begin{aligned} \tau^+_{\mu\nu}(p) &= \left(-g_{\mu\nu} + \frac{p_\mu p_\nu}{p^2+i\varepsilon}\right)\frac{i}{p^2-m^2+i\varepsilon} \\ \tau^+_{II}(p) &= \frac{-i}{p^2+i\varepsilon} \\ \tau^+_{\mu I}(p) &= -\tau^+_{I\mu}(p) = +\frac{p_\mu}{m(p^2+i\varepsilon)} \\ \tau^+_{RR}(p) &= \frac{i}{p^2-M^2+i\varepsilon} \end{aligned}}$$

Here the A_μ - A_ν has a nice UV-behavior, at the cost of introducing a ghost $\frac{1}{p^2}$.

But the UV-problem has not disappeared, it has merely been shifted to the mixed A-I-propagators. But this shift makes it easier to prove <u>physical</u> renormalizability.

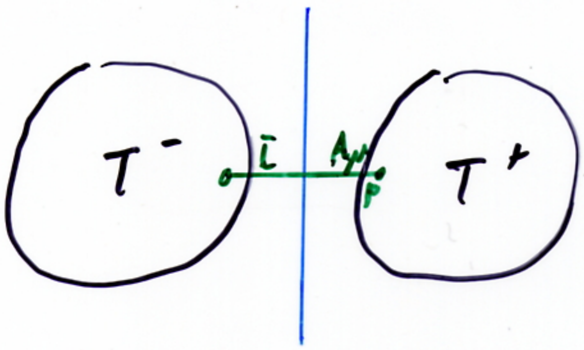
<u>U-gauge:</u> $\mathcal{V}_{ph} = \mathcal{V}_U$ generated from Ω by the gauge-invariant fields $F_{\alpha\beta}, \Psi$

But the physical content of the theory must be gauge independent: $\mathcal{V}_{ph}$ <u>still</u> generated from Ω by $F_{\alpha\beta}, \Psi$. This $\mathcal{V}_{ph}$ is now a proper subspace of the full state space. But is mapped into itself by observables. $\Longrightarrow$

Only the W-functions of $F_{\alpha\beta}, \Psi$ are of physical interest, and only their renormalizability must be proved.

Consider $(\Omega, T^-(\dots)\, T^+(\dots)\Omega)$

Contributing graph:



— : cross line. The A_μ-end is attached to a vertex belonging to a term in R^μ (or to an external vertex). Idea: I-A_μ-propagator contains factor $-ip_\mu$. But:

Ward identity $\Rightarrow$ $-ip_\mu R^\mu = -m R_I$

$\Rightarrow$ can replace A_μ-end of propagator by I-end, the propagator by $\delta_+(p)$, the negative of an I-I-propagator.

Not quite correct: T^+-sector represents a time-ordered product of fields, including R^μ. And in x-space $\leftrightarrow$ $-ip_\mu \Rightarrow \partial_\mu$: contact terms from θ-derivations to be expected.

But: explicit study of graph structure $\Rightarrow$
No contact terms, if external fields of T^+ are only $F_{\alpha\beta}$, Ψ!

Nice feature of this proof: The necessary cancellations between contributing graphs involve only the end vertex of the I-A-line and its next neighbours!

The same procedure is applicable to the T^--end of an A_μ-I crossline.

Result: Can omit the dangerous mixed crosslines and change the sign of I-I-lines:

$$\underline{I \quad A_\mu} + \underline{A_\mu \quad I} + \underline{I \quad I} \to -\underline{I \quad I}$$

without changing the result.

Moreover: The same result holds, if the same replacement has already been effected with the sector propagators inside the two sectors.

$$\Downarrow$$

<u>Theorem.</u> The Wightman functions and related (partially or fully time-ordered) functions of the physical fields $F_{\alpha\beta}$, Ψ are identical in HK and in the theory specified by omitting the mixed A_μ-I-propagators and changing the sign of the I-I-propagators.

Actually, this new, auxiliary theory is the $(\alpha=0)$ version („Landau gauge") of the tradional approach. It is renormalizable, at least in the power-counting sense, but also in a stricter sense!

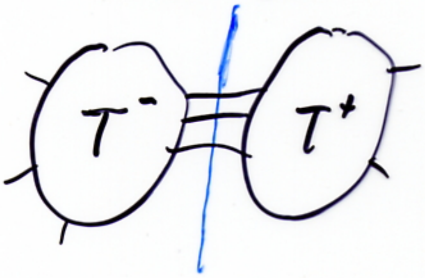
The <u>proof</u> of the Theorem is inductive with respect to the order σ of PT.

a) Theorem true for $\sigma = 1$ or 2. Easy

b) If true for 2-sector functions $(\Omega, T_1^{-}(\ldots) T^{+}(\ldots)\Omega)_\sigma$, then true for n-sector functions $(\Omega, T_1^{\pm}(\ldots) \ldots T_n^{\pm}(\ldots)\Omega)_\sigma$ with the same fields. Because all these functions are in x-space boundary values of the same analytic function.

c) Amputate functions with (p^2-m^2) for $F_{\alpha\beta}(p)$, (p^2-M^2) for $\Psi(p)$. Then Theorem true if true for amputated functions. Because full functions uniquely defined by amputated ones.

d) Theorem true for amputated 2-sector functions of order 6. Because in corresponding two-sector graphs



the sectors are of orders $0 < \tau^\pm < 6$.

$\Rightarrow$ Inductive hypothesis applicable to them. Then cross-lines also changeable (from U-form to auxiliary) by previous argument.