Feynman Graphs in Quantum Dynamics

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joint work with Lászlò Erdős and Horng–Tzer Yau

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Outline

- Definition of the problem and physical motivation
- Main result
- The role of Feynman graphs in the proof

Anderson Model

$$\mathrm{i}\frac{\partial}{\partial t}\psi(t) = H\psi(t), \quad \psi(0) = \psi_0 \qquad \text{with} \qquad H = -\frac{1}{2}\Delta + \lambda V_\omega \quad \text{on } \ell^2(\mathbb{Z}^d)$$

$-\Delta$ discrete Laplacian: $(-\Delta f)(x) = \sum_{\mu=1}^{d} (2f(x) - f(x+e_\mu) - f(x-e_\mu))$.

$$V(x) = \sum_{a\in\mathbb{Z}^d} V_a(x) \qquad V_a(x) = v_a \delta_{x,a}, \qquad v_a \text{ i.i.d. random variables.}$$

Assume that $m_k = \mathbb{E}\left(v_a^k\right)$ satisfies

$$\forall i \leq 2d : m_i < \infty, \qquad m_1 = m_3 = m_5 = 0, \quad m_2 = 1.$$

In this talk, $d = 3$. Our results hold for $d \geq 3$.

Quantum Lorentz Model

$\mathrm{i}\frac{\partial}{\partial t}\psi(t) = H\psi(t), \quad \psi(0) = \psi_0$ with $H = -\frac{1}{2}\Delta + \lambda V_\omega$ on $L^2(\mathbb{R}^d)$

Δ standard Laplacian, $V_\omega(x) = \int_{\mathbb{R}^d} B(x-y)\mathrm{d}\mu_\omega(y)$

B a spherically symmetric Schwarz function with $0 \in$ supp $\hat{B}$

μ_ω a Poisson point process on $\mathbb{R}^d$ with homogeneous unit density and i.i.d. random masses.

$$\mu_\omega = \sum_{\gamma=1}^{\infty} v_\gamma(\omega)\delta_{y_\gamma(\omega)}$$

$\{y_\gamma(\omega)\}$ is Poisson, independent of the weights $\{v_\gamma(\omega)\}$

$m_k := \mathbb{E}_v\, v_\gamma^k$ satisfies

$$\forall i \leq 2d : m_i < \infty, \qquad m_1 = m_3 = m_5 = 0, \quad m_2 = 1.$$

Time evolution

Suppose the initial state is localized, i.e. $\hat{\psi}_0$ is smooth.
How does the solution $\psi(t) = \mathrm{e}^{-\mathrm{i}tH}\psi_0$ behave for large t ?

- $\lambda = 0$: $\hat{\psi}(t,k) = \mathrm{e}^{-\mathrm{i}te(k)}\hat{\psi}_0(k)$,
  with $e(k) = k^2/2$ (QLM) or $e(k) = \sum_{i=1}^{d}(1-\cos k_i)$ (AM).
  $$\langle X^2\rangle_t = \langle \psi(t), X^2\psi(t)\rangle \sim t^2$$
- $\lambda \neq 0$: expect
  $$\langle X^2\rangle_t = \begin{cases} O(t) & \text{diffusive} \\ O(1) & \text{localized} \end{cases}$$
  depending on λ and $\hat{\psi}_0$.

Spectrum of H

- localization $\leftrightarrow$ (dense) point spectrum
- extended states $\leftrightarrow$ absolutely continuous spectrum
- $d = 1$, $\lambda > 0$: localization at all energies
  [Goldsheid, Molchanov, Pastur]
- $d \geq 2$, λ very large $\Rightarrow$ localization
  [Fröhlich–Spencer; Aizenman–Molchanov, $\ldots$]
- $d \geq 2$, λ small, but energy away from spec $-\frac{1}{2}\Delta \Rightarrow$ localization
- $d = \infty$ ($\leftrightarrow$ Cayley tree) $\Rightarrow$ extended states exist for small $\lambda > 0$.
  [Klein; Aizenman-Sims-Warzel, Froese-Hasler-Spitzer]

Major open problem

At this time there is no proof of existence of extended states in $d = 3$.

Simpler case. Randomness with a decaying envelopping function

$V_\omega(x) = \omega_x h(x)$, ω_x i.i.d., h fixed.

Theorem. [Rodnianski & Schlag; Bourgain]

$\eta > \frac{1}{2}$ and $h(x) \sim |x|^{-\eta}$ as $|x| \to \infty$

Then $H = -\Delta + V_\omega$ has absolutely continuous spectrum.

Motivations

- One–electron model of a metal with disorder
  $k \mapsto e(k)$ a band of a periodic Schrödinger operator
  V disorder
  extended vs. localized: metal–insulator transition.

- Caricature of the many–body problem
  true many–body Hamiltonian is

$$\sum_{i=1}^{n} -\frac{1}{2}\Delta_i + \lambda \sum_{i<j} v(x_i - x_j).$$

- Emergence of irreversibility from reversible dynamics

Wigner function

$$W_\psi(x,v) = \int \mathrm{d}y\ \mathrm{e}^{\mathrm{i}vy}\overline{\psi(x+\frac{y}{2})}\,\psi(x-\frac{y}{2})$$

Marginals

$$\int W_\psi(x,v)\mathrm{d}x = |\hat{\psi}(v)|^2 \qquad \int W_\psi(x,v)\mathrm{d}v = |\psi(x)|^2$$

Also, $\hat{W}_\psi(\xi,v) = \int \mathrm{d}x\ \mathrm{e}^{-\mathrm{i}x\xi} W_\psi(x,v) = \overline{\hat{\psi}(v-\xi/2)}\,\hat{\psi}(v+\xi/2)$.

$W_\psi(x,v)$ can get negative, so it is not simply a phase space density. (uncertainty principle) $\rightarrow$ Husimi function

On the lattice, one has to modify the definition of the Wigner transform slightly.

Macroscopic Scales

Ratio of typical atomic to macroscopic length scales: $\varepsilon = 10^{-8}$.

$$(\mathcal{X}, \mathcal{T}) = (\varepsilon x, \varepsilon t)$$

Velocities remain unscaled.

$$W_\psi^\varepsilon(\mathcal{X}, \mathcal{V}) = \varepsilon^{-d} W_\psi \left(\frac{\mathcal{X}}{\varepsilon}, \mathcal{V} \right)$$

The results we discuss in the following are about limits $\varepsilon \to 0$, where ε depends on λ.

Kinetic Scale

$$\eta = \lambda^2, \qquad \mathcal{T} = \eta t, \qquad \mathcal{X} = \eta x$$

Theorem. [Erdös–Yau 2000, Chen 2003]

$$\mathbb{E} W^{\eta}_{\psi(\mathcal{T}\eta^{-1})}(\mathcal{X}, \mathcal{V}) \underset{\eta \to 0}{\longrightarrow} F(\mathcal{X}, \mathcal{V}, \mathcal{T}),$$

F the solution of the *linear Boltzmann equation*

$$\frac{\partial}{\partial \mathcal{T}} F(\mathcal{X}, \mathcal{V}, \mathcal{T}) + (\nabla e)(\mathcal{V}) \cdot \nabla_{\mathcal{X}} F(\mathcal{X}, \mathcal{V}, \mathcal{T})$$
$$= 2\pi \int \mathrm{d}\mathcal{U}\, \delta(e(\mathcal{U}) - e(\mathcal{V})) \left| \hat{B}(\mathcal{U} - \mathcal{V}) \right|^2 \, [F(\mathcal{X}, \mathcal{U}, \mathcal{T}) - F(\mathcal{X}, \mathcal{V}, \mathcal{T})]$$

. .

Many–body Boltzmann equation

conjecture for the right hand side of the Boltzmann equation is, with $F_k = F(\mathcal{X}, k, \mathcal{T})$,

$$
\begin{aligned}
- \quad & 4\pi \int \mathrm{d}k_2\, \mathrm{d}k_3\, \mathrm{d}k_4\; \delta(k_1 + k_2 - k_3 - k_4)\; \delta(E_1 + E_2 - E_3 - E_4) \\
& \left| \hat{v}(k_1 - k_4) - \hat{v}(k_2 - k_3) \right|^2 \\
& \Big[F_{k_1} F_{k_2} (1 - F_{k_3})(1 - F_{k_4}) - F_{k_4} F_{k_3} (1 - F_{k_2})(1 - F_{k_1}) \Big]
\end{aligned}
$$

Diffusive Time Scale

$$\varepsilon = \lambda^{2+\kappa/2}, \qquad X = \varepsilon x, \qquad T = \varepsilon \lambda^{\kappa/2} t = \lambda^{\kappa+2} t$$

This is long compared to the kinetic timescale:

$$\mathcal{X} = \lambda^{-\kappa/2} X, \qquad \mathcal{T} = \lambda^{-\kappa} T$$

Theorem. [ESY]
Let $d = 3$, $\psi_0 \in \ell^2(\mathbb{Z}^3)$ and ψ_t be the solution to the random Schrödinger equation. If $\lambda > 0$ is small and if $\kappa > 0$ is small enough and $\varepsilon = \lambda^{2+\kappa/2}$, then $\mathbb{E} W^{\varepsilon}_{\psi(\lambda^{-2-\kappa} T)}$ converges weakly to the solution f of a heat equation.

More precisely: denote $\Phi(E) = \int \mathrm{d}v \; \delta(E - e(v))$ and

$$\langle F \rangle_E = \Phi(E)^{-1} \int \mathrm{d}v \; F(v) \delta(E - e(v)).$$

Main Theorem

Let $E \in [0,3]$ and $\qquad D_{ij}(E) = \frac{1}{2\pi\Phi(E)} \langle \nabla_i e \, \nabla_j e \rangle_E$

and let f be the solution of the heat equation

$$\begin{aligned} \frac{\partial}{\partial T} f(T,X,E) &= \nabla_X \cdot D(E) \nabla_X \, f(T,X,E) \\ f(0,X,E) &= \delta(X) \, \langle |\hat{\psi}_0|^2 \rangle_E \end{aligned}$$

Let $\mathcal{O}(x,v)$ be a Schwartz function on $\mathbb{R}^d \times \mathbb{T}^d$. Then

$$\lim_{\varepsilon \to 0} \int_{(\varepsilon\mathbb{Z}/2)^d} \mathrm{d}X \int \mathrm{d}v \; \mathcal{O}(X,v) \; \mathbb{E} W^{\varepsilon}_{\psi(\lambda^{-\kappa-2}T)}(X,v)$$
$$= \int_{\mathbb{R}^d} \mathrm{d}X \int \mathrm{d}v \; \mathcal{O}(X,v) \; f(T,X,e(v)).$$

The limit is uniform on $[0, T_0]$ for any $T_0 > 0$.

In fact, if $\hat{\psi}_0 \in C^1$ and λ is small enough, we have

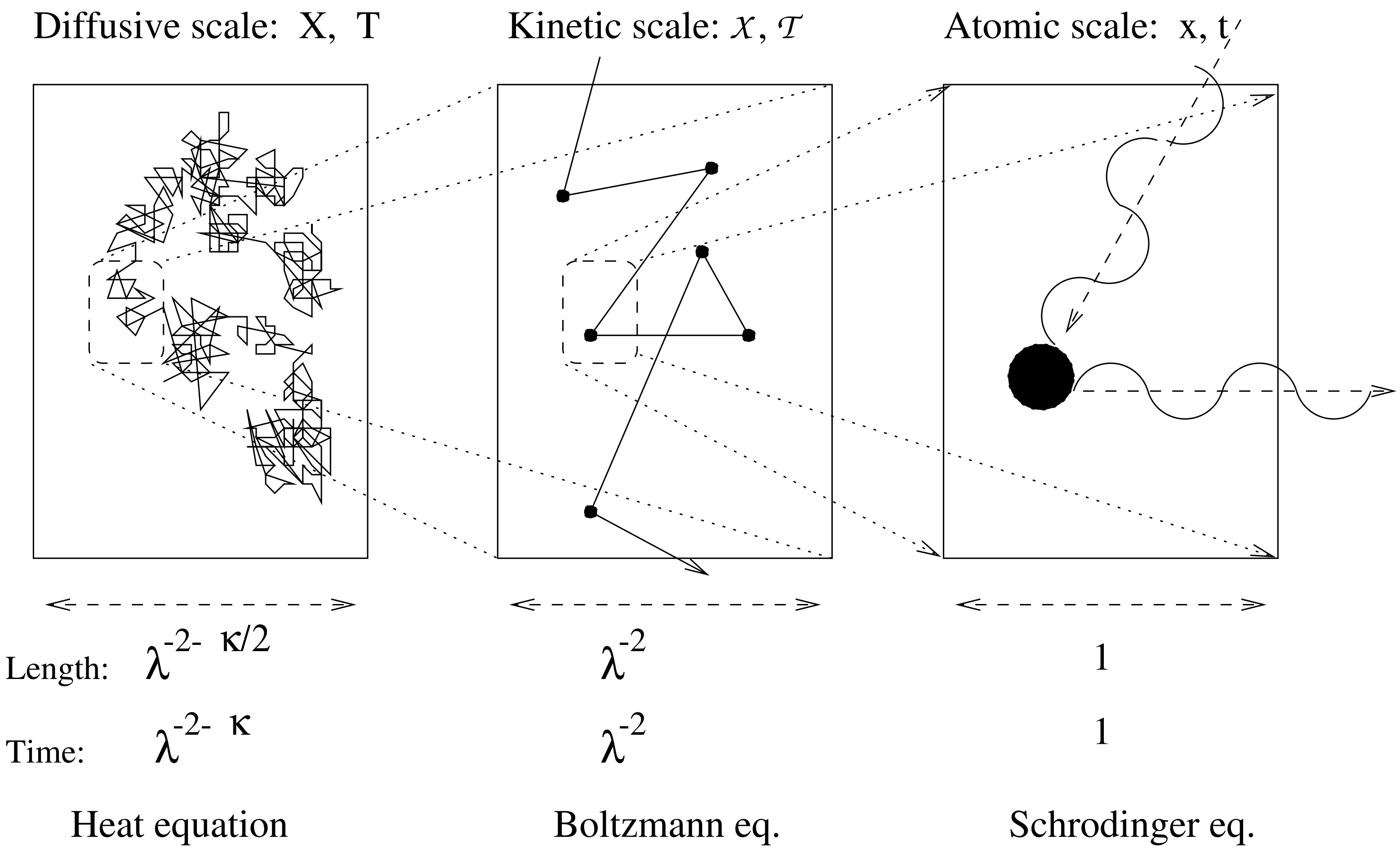
$$\begin{aligned} & \langle \hat{\mathcal{O}},\ \mathbb{E}\hat{W}^{\varepsilon}_{\psi(\varepsilon^{-1}\lambda^{-\kappa/2}T)} \rangle \\ = \quad & \int \mathrm{d}\xi \int \Phi(E)\mathrm{d}E\ \mathrm{e}^{-(2\pi)^2 T \langle \xi,\ D(E)\xi \rangle_E} \langle \hat{\mathcal{O}}(\xi, \cdot) \rangle_E\ \langle \hat{W}_{\psi_0}(\varepsilon\xi, \cdot) \rangle_E \\ + \quad & o(\lambda) \end{aligned}$$

Here

$$\langle \hat{\mathcal{O}},\ \mathbb{E}\hat{W}^{\varepsilon}_{\psi} \rangle = \int \mathrm{d}v \int \mathrm{d}\xi\ \hat{\mathcal{O}}(\xi, v)\ \mathbb{E}W^{\varepsilon}_{\psi}(\xi, v)$$

Remarks

- The Boltzmann equation also gives the same diffusion equation in the long time limit, but it was itself derived from the QM time evolution only for shorter timescales.
- Diffusion in energy space is expected to start at $t = \lambda^{-4}$.
- $\kappa_0 = 1/6000$ for technical reasons; expected restriction of the method is $\kappa < 2$.
- Main extension of previous work is that on this time scale, the effective number of collisions per particle diverges.


Diffusive scale: X, T
Kinetic scale: $\mathcal{X}$, $\mathcal{T}$
Atomic scale: x, t
Length: $\lambda^{-2-\kappa/2}$
λ^{-2}
1
Time: $\lambda^{-2-\kappa}$
λ^{-2}
1
Heat equation
Boltzmann eq.
Schrodinger eq.

Overview of the Proof

- Expansion and collision histories
- Classification of Feynman graphs
- Lowest–order renormalization
- Unitarity and expansions with remainders
- Refined classification of Feynman graphs

Expansion

$H_0 = -\frac{1}{2}\Delta \Rightarrow \psi(t) = \mathrm{e}^{-\mathrm{i}tH}\psi_0 = \sum_{n\geq 0}\psi^{(n)}(t)$,

$$\psi^{(n)}(t) \quad = \quad (-\mathrm{i}\lambda)^n \int \mathrm{d}\mu_{n+1}(s)\mathrm{e}^{-\mathrm{i}s_{n+1}H_0} V \mathrm{e}^{-\mathrm{i}s_n H_0} \ldots V \mathrm{e}^{-\mathrm{i}s_1 H_0}\psi_0$$

$$\mathrm{d}\mu_{n+1}(s) = \int_{[0,\infty)^{n+1}} \mathrm{d}s_0 \ldots \mathrm{d}s_n \; \delta\left(t - \sum_{j=0}^{n} s_j\right)$$

$$V = \sum_{a\in\mathbb{Z}^d} V_a \quad \Rightarrow \quad \psi^{(n)}(t) = \sum_{\mathbf{a}_n} \psi^{(n)}_{\mathbf{a}_n}(t)$$

collision histories $\mathbf{a}_n = (a_1, \ldots, a_n) \in \mathbb{Z}^n$.

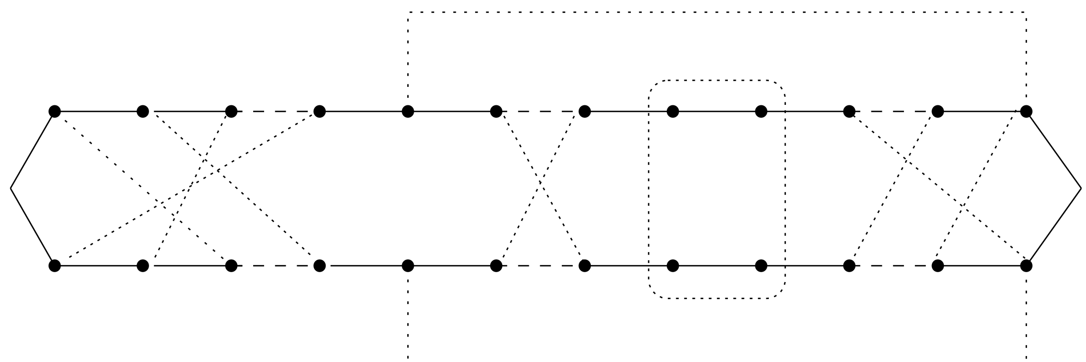
$$\hat{\psi}_n(t, p_n) = (-\mathrm{i})^n \int \prod_{j=0}^{n-1} \text{đ}p_j \int \mathrm{d}\mu_{n+1}(s) \; \prod_{j=0}^{n} \mathrm{e}^{-\mathrm{i}s_j e(p_j)} \; \prod_{j=1}^{n} \hat{V}(p_j - p_{j-1})\hat{\psi}_0(p_0)$$

Disorder average and Graphs

Recall $\hat{W}_\psi(\xi, v) = \overline{\hat{\psi}(v - \xi/2)}\, \hat{\psi}(v + \xi/2)$.

$$\mathbb{E}\left[\hat{W}_{\psi(t)}(\xi, v)\right] = \sum_{n,n'} \sum_{\mathbf{a}_n, \mathbf{a}'_{n'}} \mathbb{E}\left[\hat{\psi}^{(n)}_{\mathbf{a}_n}(t, v - \xi/2)\overline{\hat{\psi}^{n'}_{\mathbf{a}'_{n'}}}(t, v + \xi/2)\right]$$

The result can be represented graphically



A graph contributing to the expansion

Particle lines get propagators $\mathrm{e}^{-\mathrm{i}s_j e(p_j)}$, interaction lines give factors λ^2.

Propagator representation

$\eta > 0$

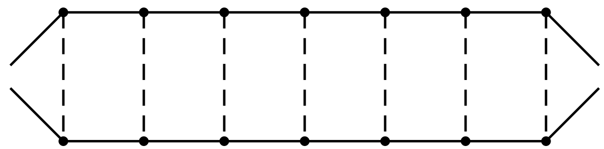
$$
\begin{aligned}
& \int_{[0,\infty)^{n+1}} \mathrm{d}^{n+1}s \; \delta\left(t - \sum_{j=1}^{n+1} s_j\right) \prod_{j=1}^{n+1} \mathrm{e}^{-\mathrm{i}s_j e(p_j)} \\
= \quad & \mathrm{e}^{t\eta} \int_{[0,\infty)^{n+1}} \mathrm{d}^{n+1}s \; \delta\left(t - \sum_{j=1}^{n+1} s_j\right) \prod_{j=1}^{n+1} \mathrm{e}^{-\mathrm{i}s_j (e(p_j) - \mathrm{i}\eta)} \\
= \quad & \mathrm{e}^{t\eta} \int \bar{d}\alpha \; \mathrm{e}^{-\mathrm{i}t\alpha} \int_{[0,\infty)^{n+1}} \mathrm{d}^{n+1}s \; \prod_{j=1}^{n+1} \mathrm{e}^{-\mathrm{i}s_j (\alpha - e(p_j) + \mathrm{i}\eta)} \\
= \quad & \mathrm{i}^{-n} \mathrm{e}^{t\eta} \int \frac{\mathrm{d}\alpha}{2\pi} \; \mathrm{e}^{-\mathrm{i}\alpha t} \prod_{j=1}^{n+1} \frac{1}{\alpha - e(p_j) + \mathrm{i}\eta}
\end{aligned}
$$

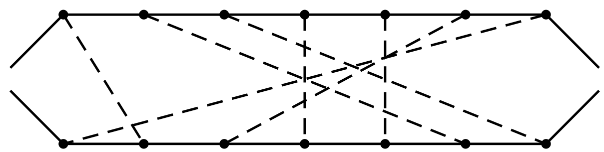
Choose $\eta = t^{-1}$.

Pairings

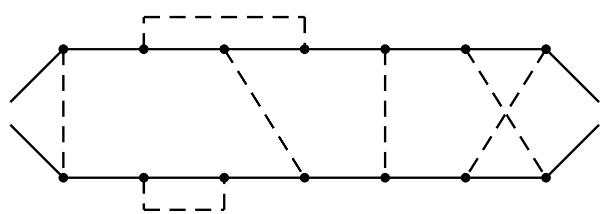
The "up–down" pairings correspond to permutations $\sigma \in \mathcal{P}_n$

$\sigma =$ id: ladder graph





Other pairings are of course possible:



Contributions at the kinetic time scale

a ladder of length n gives $\sim \frac{1}{n!}(\lambda^2 t)^n = \frac{1}{n!}\mathcal{T}^n$.

graphs with *crossings* get inverse powers of t compared to the ladder:

$$\int \mathrm{d}p \, \frac{1}{|\alpha - \omega(p) + \mathrm{i}\eta|} \, \frac{1}{|\beta - \overline{\omega(\pm p + q)} - \mathrm{i}\eta|} \le C \frac{\eta^{-b}}{\| q \| + \eta}$$

($b = 0$ for the continuum; $1/2 \le b \le 3/4$ on the lattice). $\| p \| = |p|$ in the continuum, $\| p \| = \min\{|p - v| : v_i \in \{0, \pm\pi\}\}$ on the lattice. Here $\omega(p) = e(p)$.

However, the number of graphs goes like $n!$, so expanding to infinite order is useful only on very short kinetic timescales

Can one do a finite order expansion? $n! t^{-1} = O(1) \Rightarrow$ roughly, $n \sim \log t$.

Duhamel formula

$$\psi(t) = \mathrm{e}^{-\mathrm{i}tH}\psi_0 = \mathrm{e}^{-\mathrm{i}tH_0}\psi_0 + \int_0^t \mathrm{d}s\; \mathrm{e}^{-\mathrm{i}(t-s)H}\lambda V \mathrm{e}^{-\mathrm{i}sH_0}\psi_0$$

Iteration gives

$$\psi(t) = \sum_{n=0}^{N-1} \psi^{(n)}(t) + \Psi_N(t),$$

$$\Psi_N(t) = (-\mathrm{i}) \int_0^t \mathrm{d}s\; \mathrm{e}^{-\mathrm{i}(t-s)H}\lambda V \psi^{(N-1)}(s)$$

$$\psi^{(n)}(t) = (-\mathrm{i}\lambda)^n \int \mathrm{d}\mu_{n+1}(s) \mathrm{e}^{-\mathrm{i}s_n H_0} V \ldots V\; \mathrm{e}^{-\mathrm{i}s_0 H_0}\psi_0$$

Unitarity of the full time evolution

$$
\begin{aligned}
\|\Psi_N(t)\| &\leq \int_0^t \mathrm{d}s \left\| \mathrm{e}^{-\mathrm{i}(t-s)H} \lambda V \psi^{(N-1)}(s) \right\| \\
&\leq \int_0^t \mathrm{d}s \left\| \lambda V \psi^{(N-1)}(s) \right\|
\end{aligned}
$$

Thus

$$
\|\Psi_N(t)\|^2 \leq t\, |\lambda|^2 \int_0^t \mathrm{d}s \left\| V \psi^{(N-1)}(s) \right\|^2
$$

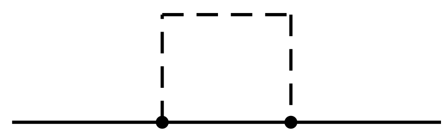
By a Schwarz inequality

$$
\left| \mathbb{E} \left(\langle \hat{\mathcal{O}}, \hat{W}^{\varepsilon}_{\psi_1} \rangle - \langle \hat{\mathcal{O}}, \hat{W}^{\varepsilon}_{\psi_2} \rangle \right) \right| \leq C \int \mathrm{d}\xi \, \sup_v \left| \hat{\mathcal{O}}(\xi, v) \right| \sqrt{\mathbb{E}\|\psi_1\|^2 \, \mathbb{E}\|\psi_1 - \psi_2\|^2}
$$

Thus one can get rid of H at the expense of a factor t, which can be controlled by making a further crossing explicit.

Boltzmann equation

The ladder terms give the gain term in the Boltzmann equation. The lowest order self–energy correction gives the loss term in the Boltzmann equation.
It corresponds to the "gate" graph



Long Time Scale: Renormalization

Ladders $\sim (\lambda^2 t)^n/n!$ blow up on the longer time scale. For $\varepsilon > 0$ set

$$\Theta_\varepsilon(\alpha, r) = \int \mathrm{d}q \frac{|\hat{B}(r-q)|^2}{\alpha - e(q) + \mathrm{i}\varepsilon},$$

$\Theta_\varepsilon(\alpha) = \Theta_\varepsilon(\alpha, r)$ for any r with $e(r) = \alpha$. Let $\Theta(\alpha) = \lim_{\varepsilon \to 0+} \Theta_\varepsilon(\alpha)$ and set

$$\theta(p) = \Theta(e(p))$$

Let $\omega(p) = e(p) + \lambda^2 \theta(p)$ and decompose

$$H = \omega(P) + U, \qquad U = \lambda V - \lambda^2 \theta(P)$$

Iterate Duhamel with $H_0 = \omega(P)$.

H_0 is not selfadjoint because ω has an imaginary part
but $\mathrm{e}^{-\mathrm{i}sH_0}$ is bounded for $s \geq 0$.

Essential features of the new propagator

For $d \geq 3$ there is $c > 0$ such that

$$\text{Im}\,\omega(p) \leq -c\lambda^2 \| p \|^{d-2} \qquad .$$

There is a constant C_0 such that

$$\sup_{\alpha,\beta,r} \int \frac{\lambda^2 \,\mathrm{d}p}{|\alpha - \overline{\omega(p+r)} - i\eta|\;|\beta - \omega(p-r) + i\eta|} \leq 1 + C_0 \lambda^{1-O(\kappa)} \;.$$

Thus with this renormalization, the ladders become of order 1.

Expand up to order $n \sim \lambda^2 t \sim \lambda^{-\kappa}$; ladder term gives the limiting equation.

Use flexibility of the Duhamel formula to do the cancellation of the “gate” terms against the counterterm $-\lambda^2\theta(P)$ in the interaction term.

Key estimate for controlling combinatorics

For a permutation $\sigma \in \mathcal{S}_n$, let $d(\sigma)$ be the degree of the permutation, defined as the number of non–ladder indices. Let Γ_σ be the Feynman graph corresponding to σ. There is $\gamma > 0$ such that for all σ

$$|Val(\Gamma_\sigma)| \leq \lambda^{\gamma d(\sigma)}. \qquad (*)$$

The number of permutations with degree D is

$$\mathcal{N}_{n,D} = |\{\sigma \in \mathcal{S}_n : d(\sigma) = D\}| \leq 2(2n)^D$$

Expanding up to $n = O(\lambda^{-\kappa-\delta})$, $\delta > 0$, we have, if $\gamma - \kappa - \delta > 0$,

$$\sum_{\substack{\sigma \in \mathcal{S}_n \\ d(\sigma) \geq D}} \lambda^{\gamma d(\sigma)} = \sum_{d=D}^{k} \lambda^{\gamma d} \mathcal{N}_{n,d} \leq 2 \sum_{d=D}^{k} (2\lambda)^{d(\gamma-\kappa-\delta)} \leq O(\lambda^{D(\gamma-\kappa-\delta)})$$

$(*)$ is proven using a special integration algorithm for bounding the values of large Feynman graphs.

Conclusion

- Have proven diffusive behaviour on a timescale $T = \lambda^{-\kappa}\mathcal{T}$, $X = \lambda^{-\kappa/2}\mathcal{X}$.
- The number of collisions to follow in the proof is $n \sim \lambda^2 t \sim \lambda^{-\kappa}$, so effectively, need sharp bounds on Feynman graphs of arbitrary size.
- Classification in terms of degrees of permutations allows us to control combinatorics.
- Our results imply that localization is not possible in a region of size $\lambda^{-2-\delta}$, $\delta > 0$.
- Extended states conjecture remains open

Husimi function

$$H_\psi^{\ell_1,\ell_2} = W_\psi \; *_x \; G^{\ell_1/\sqrt{2}} \; *_v \; G^{\ell_2/\sqrt{2}}$$

where

$$G^\delta(z) = (2\pi\delta^2)^{-d/2}\,\mathrm{e}^{-\frac{z^2}{2\delta^2}}$$

Then $H_\psi^{\ell_1,\ell_2} \geq 0$ if$\ell_1\ell_2 \geq 1$, so

$$C_{\psi,\ell}(x,v) = H_\psi^{\ell,\ell^{-1}}(x,v) \geq 0$$

can be viewed as a probability density on phase space, and

$$\int C_{\psi,\ell}(x,v)\mathrm{d}x\,\mathrm{d}v = \|\psi\|_2^2.$$

$C_{\psi,\ell}(x,v) = \langle \psi \mid \Pi_{x,v}^{(\ell)}\psi\rangle$ where $\Pi_{x,v}^{(\ell)}$ is the projection on $\phi_{x,v}^{(\ell)}(z) = G^{\ell/\sqrt{2}}(x-z)\,\mathrm{e}^{\mathrm{i}z\cdot v}$

.

Lattice Pedanteries

For $x \in \frac{1}{2}\mathbb{Z}^d$, $v \in \mathbb{T}^d$, $W_\psi(x,v) = 2^d \sum_{\substack{y,z,\in\mathbb{Z}^d \\ y+z=2x}} \mathrm{e}^{2\pi\mathrm{i}v\cdot(y-z)}\overline{\psi(y)}\psi(z)$

Then for $\xi \in (2\mathbb{T})^d$, $\quad \hat{W}_\psi(\xi,v) = \overline{\hat{\psi}\left(v-\frac{\xi}{2}\right)}\hat{\psi}\left(v+\frac{\xi}{2}\right)$.

(here $\hat{f}(p) = \int_{(\delta\mathbb{Z})^d} e^{-2\pi\mathrm{i}p\cdot x} f(x)\mathrm{d}x = \delta^d \sum_{x\in(\delta\mathbb{Z})^d} e^{-2\pi\mathrm{i}p\cdot x} f(x)$)

$$\int W_\psi(x,v)\mathrm{d}v = \begin{cases} 2^d|\psi(x)|^2 & \text{if } x \in \mathbb{Z}^d \\ 0 & \text{if } x \in (\mathbb{Z}/2)^d \setminus \mathbb{Z}^d \end{cases}$$

$$\int_{(\mathbb{Z}/2)^d} W_\psi(x,v)\mathrm{d}x = 2^{-d} \sum_{x\in(\mathbb{Z}/2)^d} W_\psi(x,v) = |\hat{\psi}(v)|^2 \ ,$$

and in particular

$$\int_{(\mathbb{Z}/2)^d} \int W_\psi(x,v)\mathrm{d}v\mathrm{d}x = \|\psi\|^2 \ .$$

.

Initial conditions for kinetic scaling

$$\psi_0^\eta(x) = \eta^{-d/2} h(\eta x) \mathrm{e}^{\mathrm{i}\frac{S(x\eta)}{\eta}} \qquad h, S \in \mathcal{S}(\mathbb{R}^d)$$

$$\lim_{\eta \to 0} W^\eta_{\psi_0^\eta}(\mathcal{X}, \mathcal{V}) = |h(\mathcal{X})|^2 \, \delta(\mathcal{V} - \nabla S(\mathcal{X})) = F_0(\mathcal{X}, \mathcal{V})$$

.

Contribution of graphs with up-down pairing

The contribution of a permutation $\sigma \in \mathcal{S}_n$ to $\langle \hat{\mathcal{O}}, \hat{W}^{\varepsilon}_{\psi} \rangle$ is

$$\begin{aligned} Val(\Gamma_\sigma) \quad = \quad & \lambda^{2n}\, \mathrm{e}^{2t\eta} \int \frac{\mathrm{d}\alpha\, \mathrm{d}\beta}{(2\pi)^2}\, \mathrm{e}^{\mathrm{i}(\beta-\alpha)t} \\ & \int \mathrm{d}\xi \int \prod_{j=0}^{n} \mathrm{d}p_j \int \prod_{k=0}^{n} \mathrm{d}q_k\; \hat{\mathcal{O}}(\xi, p_n) \hat{W}^{\varepsilon}_{\psi_0}(\xi, p_0) \\ & \prod_{j=0}^{n} \frac{1}{\beta - \overline{\omega(q_j - \frac{\varepsilon\xi}{2})} - \mathrm{i}\eta}\, \frac{1}{\alpha - \omega(q_j - \frac{\varepsilon\xi}{2}) - \mathrm{i}\eta} \\ & \prod_{j=1}^{n} \delta\left(p_j - p_{j-1} - (q_{\sigma(j)} - q_{\sigma(j)-1})\right) \end{aligned}$$

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